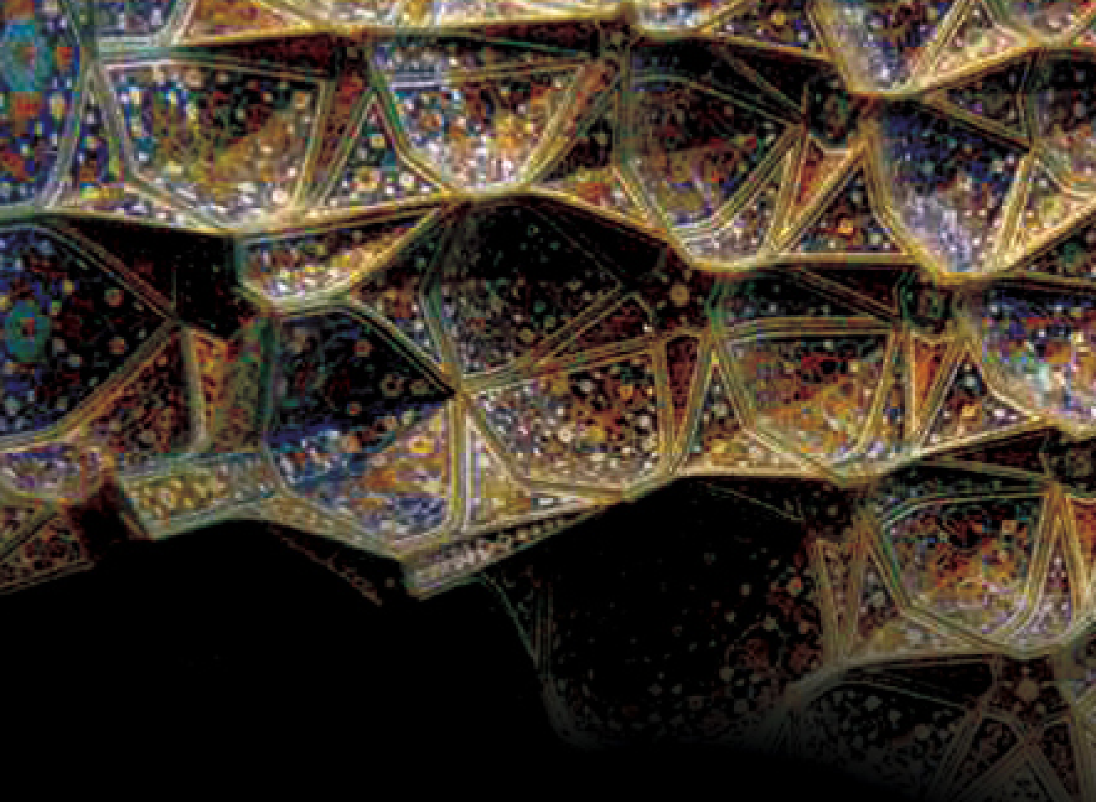

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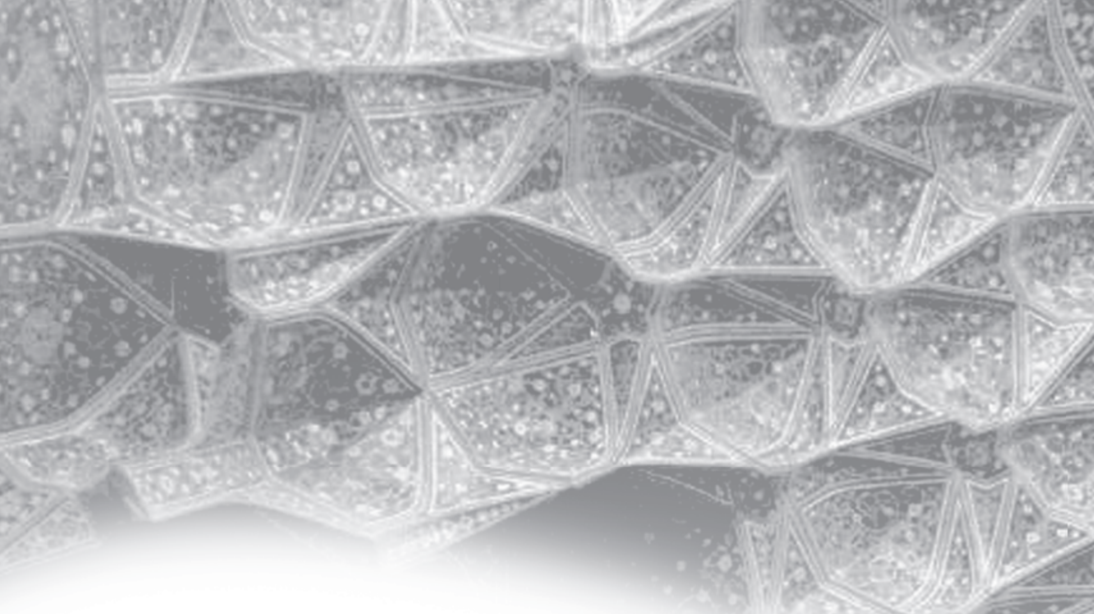

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NONCOMMUTATIVE GEOMETRY

Editors

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Preface

This volume collects the lecture series given at the *International Workshop on Noncommutative Geometry* that took place at the Institute for Studies in Theoretical Physics and Mathematics (IPM) in Tehran in 2005. The courses were addressed to graduate students and postdocs in both mathematics and physics. For this reason, the lectures provided different "points of access" to the field that are natural starting points for people with a background in physics, or in algebra and algebraic geometry, or in differential and Riemannian geometry. By topic the lectures can be roughly divided in three groups, which can be read independently, according to the specific interests of the readers.

A first combination of lectures deals with "operator algebras and differential noncommutative geometry". The first contribution in this group is written by Masoud Khalkhali and introduces the basic tools of operator algebras, topological K-theory, cyclic cohomology, and Hopf cyclic theory. The second set of lectures in this group is by Alain Connes and Matilde Marcolli and is aimed at presenting an overview of the various different research directions in the field and a description of recent developments. These lectures are based on presenting many different sources of examples of noncommuatative spaces, from physical systems, foliations, bad quotients, fractals, dimensional regularization, spacetime and particle physics, deformations of spectral geometries and of algebras, quantum groups, and noncommutative spherical manifolds. In addition to these examples, recent interactions between noncommutative geometry and number theory are discussed.

Another set of lectures can be grouped together under the theme of "categories and algebraic noncommutative geometry". Among these, the lectures by Behrang Noohi give an introduction to the language of categories as an important tool for noncommutative geometry. These lectures cover basic material on abelian categories, categories of sheaves, Morita equivalences, complexes and derived categories, derived functors, triangulated categories, t-structures. The lecture series by Snigdhayan Mahanta gives a brief introduction to algebraic noncommutative geometry in the sense of Artin, Tate and van den Bergh. Mahanta's lectures as well as the lecture series by Jorge Plazas also give an overview of recent results of Polishchuk on algebraic models of noncommutative tori based on the category of holomorphic bundles on noncommutative tori embedded as a t-structure in the derived category of coherent sheaves on an elliptic curve, as well as later developments along these lines. The Polishchuk construction presents a good example of a setting where a very nontrivial interplay between the differential and the algebraic approaches to noncommutative geometry takes place.

The volume also collects a set of lectures that are more directly focused on applications of noncommutative geometry to physics. The contributions to this part include a series of lectures by Richard Szabo on noncommutative field theories arising as states of D-branes in Type II superstring theories. Another set of lectures by Harald Grosse and Raimer Wulkenhaar discusses the renormalization of noncommutative quantum field theories. The lecture series by Giovanni Landi and Walter van Suijlekom gives an overview of the theory of instantons on noncommutative manifolds.

The lectures are meant to be accessible to advanced graduate students in either mathematics or theoretical physics. The volume stresses the interplay of analysis, differential and algebraic geometry, categorical constructions and physics in the context of noncommutative geometry.

We thank the IPM for the wonderful hospitality during the NCG 2005 activity and especially "Shahin" Mohammad Sheikh-Jabbari and Mehrdad Shahshahani for helping us co-organize the workshop and Gholamreza Khosrovshahi for the support generously provided by the IPM School of Mathematics. We also thank the ICTP for providing support for the activity. We thank all the participants and lecturers for making this a very successful event. We thank Arthur Greenspoon for a very careful proofreading of the volume.

Masoud Khalkhali
Matilde Marcolli

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A WALK IN THE NONCOMMUTATIVE GARDEN

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Contents

1. Introduction

If you cleave the hearth of one drop of water
a hundred pure oceans emerge from it.
(Mahmud Shabistari, *Gulshan-i-raz*)

We have decided to contribute to the volume of the IPM lectures on noncommutative geometry a text that collects a list of examples of noncommutative spaces. As the quote of the Sufi poet above suggests, it is often better to approach a new subject by analyzing specific examples rather than presenting the general theory. We hope that the diversity of examples the readers will encounter in this text will suffice to convince them of the fact that noncommutative geometry is a very rich field in rapid evolution, full of interesting and yet unexplored landscapes. Many of the examples collected here have not yet been fully explored from the point of view of the general guidelines we propose in Section 2 and the main point of this text is to provide a great number of open questions. The reader should interpret this survey as a suggestion of possible interesting problems to investigate, both in the settings described here, as well as in other examples that are available but did not fit in this list, and in the many more that still await to be discovered. Besides the existing books on NCG such as [55], [153], [56], [133], [143] [109] [158], two new books are being written: one by the two authors of this paper [80], and one by Connes and Moscovici [90].

2. Handling Noncommutative Spaces in the Wild: Basic Tools

We are going to see in many examples how one obtains the algebra of coordinates $\mathcal{A}$ of a noncommutative space X. Here we think of $\mathcal{A}$ as being the algebra of "smooth functions", which will usually be a dense subalgebra of a C^*-algebra $\bar{\mathcal{A}}$.

Here are some basic steps that one can perform in order to acquire a good understanding of a given noncommutative space X with algebra of coordinates $\mathcal{A}$.

(1) Resolve the diagonal of $\mathcal{A}$ and compute the cyclic cohomology.
(2) Find a geometric model of X up to homotopy.
(3) Construct the spectral geometry $(\mathcal{A}, \mathcal{H}, D)$.
(4) Compute the time evolution and analyze the thermodynamics.

(1) The first step means finding a resolution of the $\mathcal{A}$-bimodule $\mathcal{A}$ by projective $\mathcal{A}$-bimodules making it possible to compute the Hochschild homology of $\mathcal{A}$ effectively. In general, such resolutions will be of Koszul type and the typical example is the resolution of the diagonal for the algebra $C^\infty(X)$ of smooth functions on a compact manifold as in the C^∞ version [60] of the Hochschild, Kostant, Rosenberg theorem (*cf.* [124]). This makes it possible to know what is the analogue of differential forms and of de Rham currents on the space X and to take the next step of computing the cyclic homology and cyclic cohomology of $\mathcal{A}$, which are the natural replacements for the de Rham theory. For foliation algebras this was done long ago (*cf.* [58], [34], [97]). It ties in with the natural double complex of transverse currents.

It is not always easy to perform this step of finding a resolution and computing Hochschild and cyclic (co)homology. For instance, in the case of algebras given by generators and relations this uses the whole theory of Koszul duality, which has been successfully extended to N-homogeneous algebras (*cf.* [104], [105], [106], [107], [108], [22]).

One specific example in which it would be very interesting to resolve the diagonal is the modular Hecke algebras (Section 23). In essence, finding a resolution of the diagonal in the algebra of modular forms of arbitrary level, equivariant with respect to the action of the group $\mathrm{GL}_2(\mathbb{A}_f)$ of finite adèles, would yield formulas for the compatibility of Hecke operators with the algebra structure. This is a basic and hard problem of the theory of modular forms.

Cyclic cohomology (and homology) is a well developed theory which was first designed to handle the leaf spaces of foliations as well as group rings of discrete groups (*cf.* [133]). The theory admits a purely algebraic version which is at center stage in "algebraic" noncommutative geometry, but it is crucial in the analytic set-up to construct cyclic cocycles with good compatibility properties with the topology of the algebra. For instance, when the domain of definition of the cocycle is a dense subalgebra stable under holomorphic functional calculus, it automatically gives an invariant of the K-theory of the underlying C^*-algebra (*cf.* [58]).

(2) The essence of the second step is that many noncommutative spaces defined as "bad quotients" (*cf.* Section 4) can be desingularized, provided one is ready to work up to homotopy. Thus for instance if the space X is defined as the quotient

$$X = Y/\sim$$

of an "ordinary" space Y by an equivalence relation $\sim$ one can often find a description of the same space X as a quotient

$$X = Z/\sim$$

where the equivalence classes are now *contractible* spaces. The homotopy type of Z is then uniquely determined and serves as a substitute for that of X (see [15]).

For instance, if the equivalence relation on Y comes from the free action of a torsion-free discrete group Γ, the space Z is simply a product over Γ of the form

$$Z = Y \times_\Gamma E\Gamma \,,$$

where $E\Gamma$ is a contractible space on which Γ acts freely and properly.

The main point of this second step is that it gives a starting point for computing the K-theory of the space X, *i.e.* of the C^*-algebra $A = \bar{\mathcal{A}}$ playing the role of the algebra of continuous functions on X. Indeed, for each element of the K-homology of the classifying space Z, there is a general construction of an index problem for "families parameterized by X" that yields an assembly map (*cf.* [15])

$$\mu \;:\; K_*(Z) \to K_*(A)\,. \tag{2.1}$$

This Baum–Connes map is an isomorphism in a lot of cases (with suitable care taken of torsion, *cf.* [16]) including all connected locally compact groups, all amenable groupoids and all hyperbolic discrete groups. It thus gives a computable guess for the K-theory of X.

The next step is not only to really compute $K(A)$ but also to get a good model for the "vector bundles" on X, *i.e.* the finite projective modules over A. This step should then be combined with the above first step to compute the Chern character using connections, curvature, and eventually computing moduli spaces of Yang–Mills connections as was done for instance for the NC-torus in [91].

(3) The third step makes it possible to pass from the soft part of differential geometry to the harder "Riemannian" metric aspect. The sought for spectral geometry $(\mathcal{A}, \mathcal{H}, D)$ has three essential features:

- The K-homology class of $(\mathcal{A}, \mathcal{H}, D)$.
- The smooth structure.
- The metric.

One should always look for a spectral triple whose K-homology class is as non-trivial as possible. Ideally it should extend to a class for the double algebra $\mathcal{A} \otimes \mathcal{A}^o$ and then be a generator for Poincaré duality. In general this is too much to ask for, since many interesting spaces do not fulfill Poincaré duality. The main tool for determining the stable homotopy class of the spectral triple is Kasparov's bivariant KK-theory. Thus it is quite important to already have taken step 2 and to look for classes whose pairing with K-theory is as non-trivial as can be. For the smooth structure, there is often a natural guess for a subalgebra $A^\infty \subset A$ of the C^*-algebra $A = \bar{\mathcal{A}}$ that will play the role of the algebra of smooth functions. It should in general contain the original algebra $\mathcal{A}$ but should have the further property of being stable under the holomorphic functional calculus. This ensures that the inclusion $A^\infty \subset A$ is an isomorphism in K-theory and makes it possible to complete the classification of smooth vector bundles.

The role of the unbounded operator D for the smooth structure is that it defines the geodesic flow by the formula

$$F_t(a) = e^{it|D|}\, a\, e^{-it|D|}\,, \quad \forall\, a \in A^\infty$$

and one expects that smoothness is governed by the smoothness of the operator-valued map $\mathbb{R} \ni t \mapsto F_t(a)$. The main result of the general theory is the local index formula of [84], which provides the analogue of the Pontrjagin classes of smooth manifolds in the noncommutative framework.

The problem of determining D from the knowledge of the K-homology class is very similar to the choice of a connection on a bundle. There are general results that assert the existence of an unbounded selfadjoint D with bounded commutators with $\mathcal{A}$ from estimates on the commutators with the phase F. The strongest is obtained (*cf.* [56], p. 391) just assuming that the $[F, a]$ are in an ideal called $\mathrm{Li}\,\mathcal{H}$ and it ensures the existence of a theta-summable spectral triple which is what one needs to get started.

It is not always possible to find a finitely-summable spectral triple, first because of growth conditions on the algebra [61], but also since the finitely-summable condition is very analogous to type II in the theory of factors. In very general cases, like the noncommutative space coming from foliations, one can however go from type III to type II by passing to the total space of the space of transverse metrics and then using the theory of hypoelliptic operators [85].

Another way to attack the problem of determining D is to consider the larger algebra generated by $\mathcal{A}$ and D, write *a priori* relations between $\mathcal{A}$ and D and then look for irreducible representations that fall in the correct stable homotopy class. Ideally one should minimize the spectral action functional [45] in this homotopy class, thus coming close to gravity. In practise one should use anything available and the example of the NC-space given by the quantum group $SU_q(2)$ shows that things can be quite subtle [100].

Once the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has been determined, the basic steps are the following, one should compute

- The dimension spectrum $\Sigma \subset \mathbb{C}$.
- The local index formula.
- The inner fluctuations, scalar curvature, and spectral action.

(4) Often a noncommutative space comes with a measure class, which in turn determines a time evolution σ_t, namely a 1-parameter family of automorphisms of the C^*-algebra $A = \bar{\mathcal{A}}$. In the type II situation one can apply the discussion of step 3 above and, in the finite dimensional case, use the operator D to represent functionals in the measure class in the form

$$\varphi(a) = ⨍ a\,|D|^{-p}\,, \quad \forall\, a \in \mathcal{A}$$

where $⨍$ is the noncommutative integral, *i.e.* the Dixmier trace and p is the "dimension".

In the general case one should expect to be in the type III situation in which the time evolution σ_t is highly non-trivial. We shall see some examples, for instance in Section 22. Given the data $(\bar{\mathcal{A}}, \sigma_t)$ it is natural to regard it as a quantum statistical mechanical system, with $\bar{\mathcal{A}}$ as algebra of observables and σ_t as time evolution. One can then look for equilibrium states for the system, for a given value β of the thermodynamic parameter (inverse temperature).

If the algebra $\bar{\mathcal{A}}$ is concretely realized as an algebra of bounded operators on a Hilbert space $\mathcal{H}$, then one can consider the Hamiltonian H, namely the (unbounded) operator on $\mathcal{H}$ that is the infinitesimal generator of the time evolution. If the operator $\exp(-\beta H)$ is trace class, then one has equilibrium states for the system $(\bar{\mathcal{A}}, \sigma_t)$ written in the usual Gibbs form

$$\varphi_\beta(a) = \frac{\mathrm{Tr}(a \exp(-\beta H))}{\mathrm{Tr}(\exp(-\beta H))}\,,$$

where $Z(\beta) = \mathrm{Tr}(\exp(-\beta H))$ is the partition function of the system. The notion of equilibrium state continues to make sense when $\exp(-\beta H)$ is not necessarily trace class, and is given by the more subtle notion of KMS (Kubo–Martin–Schwinger) states.

These are states on $\bar{\mathcal{A}}$, namely continuous functionals $\varphi : \bar{\mathcal{A}} \to \mathbb{C}$ with $\varphi(1) = 1$ and $\varphi(a^*a) \geq 0$, satisfying the KMS_β condition that, for all $a, b \in \bar{\mathcal{A}}$ there exists a function $F_{a,b}(z)$ which is holomorphic on the strip $0 < \Im(z) < \beta$, continuous and bounded on the closed strip and such that, for all $t \in \mathbb{R}$,

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a). \tag{2.2}$$

KMS states at zero temperature can be defined as weak limits as $\beta \to \infty$ of KMS_β states. One can construct using KMS states very refined invariants of noncommutative spaces. For a fixed β, the KMS_β states form a simplex, hence one can consider only the extremal KMS_β states $\mathcal{E}_\beta$, from which one recovers all the others by convex combinations. An extremal KMS_β state is always factorial and the type of the factor is an invariant

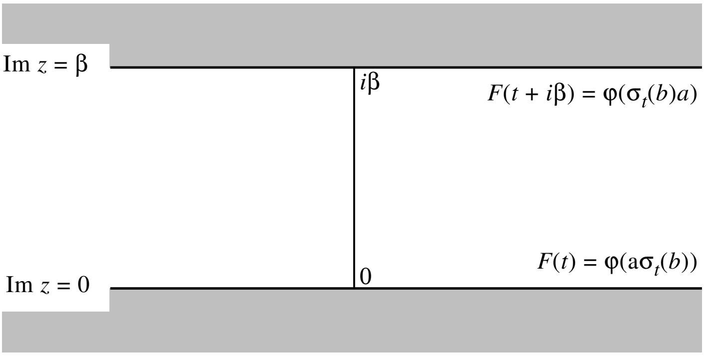


Figure 1. The KMS condition.

of the state. The simplest situation is type I. One can show under minimal hypotheses ([69]) that extremal KMS_β states continue to survive when one lowers the temperature, *i.e.* one increases β. Thus, in essence, when cooling down the system this tends to become more and more "classical" and in the 0-temperature limit $\mathcal{E}_\beta$ gives a good replacement of the notion of classical points for a noncommutative space. We shall see in Section 24 how, in examples related to arithmetic, the "classical points" described by the zero temperature KMS states of certain quantum statistical mechanical systems recover classical arithmetic varieties. The extremal KMS states at zero temperature, evaluated on suitable arithmetic elements in the noncommutative algebra, can be shown in significant cases to have an interesting Galois action, related to interesting questions in number theory (*cf.* [27], [81], [77]).

In joint work with Consani, we showed in [69] how to define an analog in characteristic zero of the action of the Frobenius on the étale cohomology by a process involving the above thermodynamics. One key feature is that the analogue of the Frobenius is the "dual" of the above time evolution σ_t. The process involves cyclic homology and its three basic steps are ([69])

- Cooling.
- Distillation.
- Dual action of $\mathbb{R}_+^*$ on the cyclic homology of the distilled space.

When applied to the simplest system (the Bost–Connes system of [27]) this yields a cohomological interpretation of the spectral realization of the zeros of the Riemann zeta function ([66], [69]).

3. Phase Spaces of Microscopic Systems

What can be regarded historically as the first example of a noncommutative space is the Heisenberg formulation of the observational Ritz–Rydberg law of spectrocopy. In fact, quantum mechanics showed that indeed the parameter space, or phase space of the mechanical system given by a single atom, fails to be a manifold. It is important to convince oneself of this fact and to understand that this conclusion is indeed dictated by the *experimental findings of spectroscopy.*

At the beginning of the 20th century a wealth of experimental data was being collected on the spectra of various chemical elements. These spectra obey experimentally discovered laws, the most notable being the Ritz–Rydberg combination principle. The principle can be stated as follows: spectral lines are indexed by pairs of labels. The statement of the principle then is that certain pairs of spectral lines, when expressed in terms of frequencies, do add up to give another line in the spectrum. Moreover, this happens precisely when the labels are of the form i, j and j, k.

In the seminal paper [123] of 1925, Heisenberg considers the classical prediction for the radiation emitted by a moving electron in a field, where the observable dipole moment can be computed, with the motion of the electron given in Fourier expansion. The classical model would predict (in his notation) frequencies distributed according to the law

$$\nu(n,\alpha) = \alpha\nu(n) = \alpha\frac{1}{h}\frac{dW}{dn}\,. \tag{3.1}$$

When comparing the frequencies obtained in this classical model with the data, Heisenberg noticed that the classical law (3.1) did not match the phenomenon observed.

The spectral rays provide a 'picture' of an atom: if atoms were classical systems, then the picture formed by the spectral lines would be (in our modern mathematical language) a *group*, which is what (3.1) predicts. That is, the classical model predicts that the observed frequencies should simply add, obeying a group law, or, in Heisenberg's notation, that

$$\nu(n,\alpha) + \nu(n,\beta) = \nu(n,\alpha+\beta)\,. \tag{3.2}$$

Correspondingly, observables would form the convolution algebra of a group.

What the spectral lines were instead providing was the picture of a *groupoid*. Heisenberg realized that the classical law of (3.1) (3.2) would have to be replaced by the quantum–mechanical

$$\nu(n, n-\alpha) = \frac{1}{h}\left(W(n) - W(n-\alpha)\right). \tag{3.3}$$

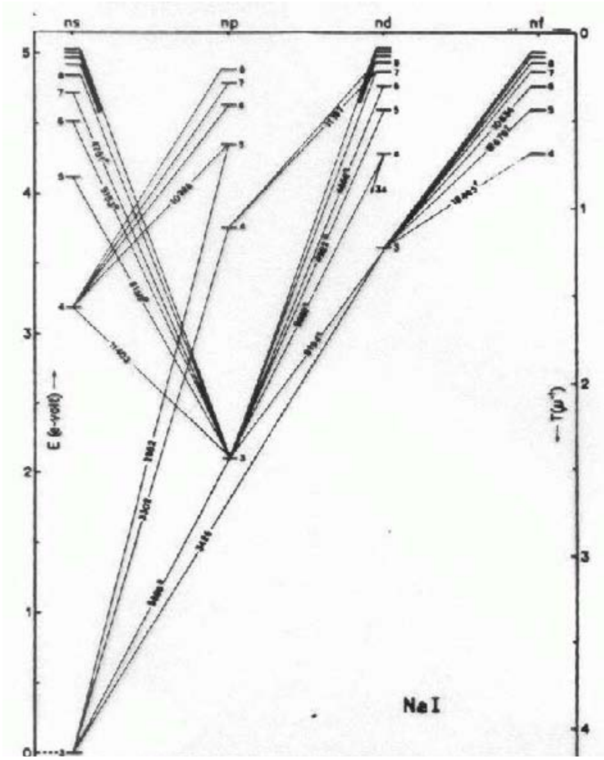


Figure 2. Spectral lines and the Ritz–Rydberg law.

This replaces the group law with that of a groupoid, replacing the classical (3.2) by the quantum-mechanical

$$\nu(n, n-\alpha) + \nu(n-\alpha, n-\alpha-\beta) = \nu(n, n-\alpha-\beta)\,. \tag{3.4}$$

Similarly, the classical Fourier modes $\mathfrak{U}_\alpha(n)e^{i\omega(n)\alpha t}$ were replaced by $\mathfrak{U}(n, n-\alpha)e^{i\omega(n,n-\alpha)t}$.

The analysis of the emission spectrum given by Heisenberg was in very good agreement with the Ritz–Rydberg law, or combination principle, for spectral lines in emission or absorption spectra.

In the same paper, Heisenberg also extends his redefinition of the multiplication law for the Fourier coefficients to coordinates and momenta, by introducing transition amplitudes that satisfy similar product rules. This is, in Born's words, "his most audacious step": in fact, it is precisely this step that brings noncommutative geometry on the scene.

It was Born who realized that what Heisenberg described in his paper corresponded to replacing classical coordinates with coordinates which no longer commute, but which obey the laws of matrix multiplication. In his own words reported in [207],

> After having sent Heisenberg's paper to the Zeitschrift für Physik for publication, I began to ponder about his symbolic multiplication, and was soon so involved in it that I thought the whole day and could hardly sleep at night. For I felt there was something fundamental behind it ... And one morning ... I suddenly saw light: Heisenberg's symbolic multiplication was nothing but the matrix calculus.

Thus, spectral lines are parameterized by two indices $L_{\alpha\beta}$ satisfying a cocycle relation $L_{\alpha\beta} + L_{\beta\gamma} = L_{\alpha\gamma}$, and a coboundary relation expresses each spectral line as a difference $L_{\alpha\beta} = \nu_\alpha - \nu_\beta$. In other words, the Ritz–Rydberg law gives the groupoid law (3.4), or equivalently,

$$(i,j) \bullet (j,k) = (i,k)$$

and the convolution algebra of the group is replaced by observables satisfying the matrix product

$$(AB)_{ik} = \sum_j A_{ij} B_{jk} \,,$$

for which in general commutativity is lost:

$$AB \neq BA \,.$$

The Hamiltonian H is a matrix with the frequencies on the diagonal, and observables obey the evolution equation

$$\frac{d}{dt} A = i[H, A] \,.$$

Out of Heisenberg's paper, and Born's interpretation of the same in terms of matrix calculus, emerged the statement of Heisenberg's uncertainty principle in the form of a commutation relation of matrices

$$[P, Q] = \frac{h}{2\pi i} I \,.$$

The matrix calculus and the uncertainty principle were formulated in the subsequent paper of Born and Jordan [25], also published in 1925. This viewpoint on quantum mechanics was later somewhat obscured by the advent of the Schrödinger equation. The Schrödinger approach shifted the emphasis back to the more traditional technique of solving partial differential equations, while the more modern viewpoint of Heisenberg implied a much more serious change of paradigm, affecting our most basic understanding of the notion of space. Heisenberg's approach can be regarded as the historic origin of noncommutative geometry.

4. Noncommutative Quotients

A large source of examples of noncommutative spaces is given by quotients of equivalence relations. One starts with an ordinary commutative space X (*e.g.* a smooth manifold or more generally a locally compact Hausdorff topological space). This can be described via its algebra of functions $C(X)$, and abelian C^*-algebra. Suppose then that we are interested in taking a quotient $Y = X/\sim$ of X with respect to an equivalence relation. In general, one should not expect the quotient to be nice. Even when X is a smooth manifold, the quotient Y need not even be a Hausdorff space. In general, one would like to still be able to characterize Y through its ring of functions. One usually defines $C(Y)$ to be functions on X that are invariant under the equivalence relation,

$$C(Y) = \{f \in C(X) : f(a) = f(b), \forall\, a \sim b\}\,. \tag{4.1}$$

Clearly, for a "bad" equivalence relation one typically gets this way only constant functions, $C(Y) = \mathbb{C}$.

There is a better way to associate to the quotient space Y a ring of functions which is nontrivial for any equivalence relation. This requires dropping the commutativity requirement. One can then consider functions of two variables f_{ab} defined on the graph of the equivalence relation, with a product which is no longer the commutative pointwise product, but the noncommutative convolution product dictated by the groupoid of the equivalence relation. In general the elements in the algebra of functions

$$\text{“}C(Y)\text{”} = \{(f_{ab}) : a \sim b\} \tag{4.2}$$

act as bounded operators on the Hilbert space L^2 of the equivalence class. This also guarantees the convergence in the operator norm of the convolution product

$$\sum_{a\sim b\sim c} f_{ab} g_{bc}\,.$$

We give a few examples to illustrate the difference between the traditional construction and that of noncommutative geometry.

Example 4.1. Consider the space $Y = \{x_0, x_1\}$ with the equivalence relation $x_0 \sim x_1$. From the first point of view the algebra of functions on the quotient is $\mathbb{C}$, while from the second point of view it is $\mathcal{B} = M_2(\mathbb{C})$, that is,

$$\mathcal{B} = \left\{ f = \begin{pmatrix} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{pmatrix} \right\}. \tag{4.3}$$

These two algebras are not the same, though in this case they are Morita equivalent.

Notice that, when one computes the spectrum of the algebra (4.3), it turns out that it is composed of only one point, so the two points a and b have been identified. This first trivial example represents the typical situation where the quotient space is "nice": the two constructions give Morita equivalent algebras. In this sense, Morita equivalent algebras are regarded as "the same" (or better isomorphic) spaces in noncommutative geometry.

Example 4.2. Consider the space $Y = [0,1] \times \{0,1\}$ with the equivalence relation $(x,0) \sim (x,1)$ for $x \in (0,1)$. Then from the first viewpoint the algebra of functions is again given just by the constant functions $\mathbb{C}$, but in the second case we obtain

$$\{f \in C([0,1]) \otimes M_2(\mathbb{C}) : f(0) \text{ and } f(1) \text{ diagonal}\}. \tag{4.4}$$

In this case, these algebras are not Morita equivalent. This can be seen by computing their K-theory. This means that the approach of noncommutative spaces produces something genuinely new, as soon as the quotient space ceases to be "nice" in the classical sense.

In general, the first kind of construction of functions on the quotient space is cohomological in nature: one seeks for functions satisfying certain equations or constraints. Usually there are very few solutions. The second approach, instead, typically produces a very large class of functions.

5. Spaces of Leaves of Foliations

There is a very rich collection of examples of noncommutative spaces given by the leaf spaces of foliations. The connection thus obtained between noncommutative geometry and the geometric theory of foliations is very far-reaching, for instance through the role of Gelfand–Fuchs cohomology, of the Godbillon–Vey invariant and of the passage from type III to type II using the transverse frame bundle. It is this class of examples that triggered the initial development of cyclic cohomology (*cf.* [133], Section 4), of the local index formula in noncommutative geometry as well as the theory of characteristic classes for Hopf algebra actions.

The construction of the algebra associated to a foliation is a special case of the construction of Section 4 but both the presence of holonomy and the case when the graph of the foliation is non-Hausdorff require special care, so we shall recall the basic steps below.

Let V be a smooth manifold and TV its tangent bundle, so that for each $x \in V$, $T_x V$ is the tangent space of V at x. A smooth subbundle F of TV is called *integrable* if one of the following equivalent conditions is satisfied:

(a) Every $x \in V$ is contained in a submanifold W of V such that
$$T_y(W) = F_y \qquad \forall\, y \in W\,.$$

(b) Every $x \in V$ is in the domain $U \subset V$ of a submersion $p : U \to \mathbb{R}^q$ ($q = \operatorname{codim} F$) with
$$F_y = \operatorname{Ker}(p_*)_y \qquad \forall\, y \in U\,.$$

(c) $C^\infty(F) = \{X \in C^\infty(TV)\,,\ X_x \in F_x \quad \forall\, x \in V\}$ is a Lie algebra.

(d) The ideal $J(F)$ of smooth exterior differential forms which vanish on F is stable under exterior differentiation.

Any 1-dimensional subbundle F of TV is integrable, but for $\dim F \geq 2$ the condition is non-trivial. For instance, if $P \xrightarrow{p} B$ is a principal H-bundle (with compact structure group H), then the bundle of horizontal vectors for a given connection is integrable iff this connection is flat.

A foliation of V is given by an integrable subbundle F of TV. The leaves of the foliation (V, F) are the maximal connected submanifolds L of V with $T_x(L) = F_x$, $\forall\, x \in L$, and the partition of V into leaves
$$V = \cup L_\alpha\,, \quad \alpha \in X$$
is characterized geometrically by its "local triviality": every point $x \in V$ has a neighborhood U and a system of local coordinates $(x^j)_{j=1,\dots,\dim V}$ called *foliation charts*, so that the partition of U in connected components of leaves corresponds to the partition of
$$\mathbb{R}^{\dim V} = \mathbb{R}^{\dim F} \times \mathbb{R}^{\operatorname{codim} F}$$
into the parallel affine subspaces $\mathbb{R}^{\dim F} \times \mathrm{pt}$. These are the leaves of the restriction of F and are called *plaques*.

The set $X = V/F$ of leaves of a foliation (V, F) is in most cases a noncommutative space. In other words, even though as a set it has the cardinality of the continuum, it is in general not so at the effective level and it is in general impossible to construct a countable set of measurable functions on V that form a complete set of invariants for the equivalence relation coming from the partition of V into leaves $V = \cup L_\alpha$. Even in the simple cases in which the set $X = V/F$ of leaves is classical it helps to introduce the associated algebraic tools in order to get a feeling for their role in the general singular case.

To each foliation (V, F) is canonically associated a C^*-algebra $C^*(V, F)$ which encodes the topology of the space of leaves. The construction is basically the same as the general one for quotient spaces of Section 4, but there are interesting nuances coming from the presence of holonomy in the foliation context. To take this into account one first constructs a manifold G, $\dim G = \dim V + \dim F$, called the graph (or holonomy groupoid) of the foliation, which refines the equivalence relation coming from the partition of V in leaves $V = \cup L_\alpha$. This construction is due to Thom, Pradines and Winkelnkemper, see [210].

An element γ of G is given by two points $x = s(\gamma)$, $y = r(\gamma)$ of V together with an equivalence class of smooth paths: $\gamma(t) \in V$, $t \in [0,1]$; $\gamma(0) = x$, $\gamma(1) = y$, tangent to the bundle F (*i.e.* with $\dot\gamma(t) \in F_{\gamma(t)}$, $\forall\, t \in \mathbb{R}$) up to the following equivalence: γ_1 and γ_2 are equivalent iff the *holonomy* of the path $\gamma_2 \circ \gamma_1^{-1}$ at the point x is the *identity*. The graph G has an obvious composition law. For $\gamma, \gamma' \in G$, the composition $\gamma \circ \gamma'$ makes sense if $s(\gamma) = r(\gamma')$. If the leaf L which contains both x and y has no holonomy, then the class in G of the path $\gamma(t)$ only depends on the pair (y, x). The condition of trivial holonomy is generic in the topological sense of dense G_δ's. In general, if one fixes $x = s(\gamma)$, the map from $G_x = \{\gamma, s(\gamma) = x\}$ to the leaf L through x, given by $\gamma \in G_x \mapsto y = r(\gamma)$, is the holonomy covering of L.

Both maps r and s from the manifold G to V are smooth submersions and the map (r, s) to $V \times V$ is an immersion whose image in $V \times V$ is the (often singular) subset

$$\{(y, x) \in V \times V : \ y \text{ and } x \text{ are on the same leaf}\}. \tag{5.1}$$

In first approximation one can think of elements of $C^*(V, F)$ as continuous matrices $k(x, y)$, where (x, y) varies in the set (5.1). We now describe this C^*-algebra in full details. We assume, for notational convenience, that the manifold G is Hausdorff. Since this fails to be the case in very interesting examples, we also explain briefly how to remove this hypothesis.

The basic elements of $C^*(V, F)$ are smooth half-densities $f \in C_c^\infty(G, \Omega^{1/2})$ with compact support on G. The bundle $\Omega_G^{1/2}$ of half-densities on G is obtained as follows. One first defines a line bundle $\Omega_V^{1/2}$ on V. For $x \in V$ one lets $\Omega_x^{1/2}$ be the one-dimensional complex vector space of maps from the exterior power $\wedge^k F_x$, $k = \dim F$, to $\mathbb{C}$ such that

$$\rho\,(\lambda\, v) = |\lambda|^{1/2}\, \rho\,(v) \qquad \forall\, v \in \wedge^k F_x\,, \quad \forall\, \lambda \in \mathbb{R}\,.$$

Then, for $\gamma \in G$, one can identify $\Omega_\gamma^{1/2}$ with the one-dimensional complex vector space $\Omega_y^{1/2} \otimes \Omega_x^{1/2}$, where $\gamma : x \to y$. In other words

$$\Omega_G^{1/2} = r^*(\Omega_V^{1/2}) \otimes s^*(\Omega_V^{1/2})\,.$$

Of course the bundle $\Omega_V^{1/2}$ is trivial on V, and we could choose once and for all a trivialization ν turning elements of $C_c^\infty(G, \Omega^{1/2})$ into functions. Let us however stress that the use of half densities makes all the constructions completely canonical.

For $f, g \in C_c^\infty(G, \Omega^{1/2})$, the convolution product $f * g$ is defined by the equality

$$(f * g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)\, g(\gamma_2)\,.$$

This makes sense because, for fixed $\gamma : x \to y$ and fixing $v_x \in \wedge^k F_x$ and $v_y \in \wedge^k F_y$, the product $f(\gamma_1)\, g(\gamma_1^{-1}\gamma)$ defines a 1-density on $G^y = \{\gamma_1 \in G,\, r(\gamma_1) = y\}$, which is smooth with compact support (it vanishes if $\gamma_1 \notin$ support f), and hence can be integrated over G^y to give a scalar, namely $(f * g)(\gamma)$ evaluated on v_x, v_y.

The $*$ operation is defined by $f^*(\gamma) = \overline{f(\gamma^{-1})}$, *i.e.* if $\gamma : x \to y$ and $v_x \in \wedge^k F_x$, $v_y \in \wedge^k F_y$ then $f^*(\gamma)$ evaluated on v_x, v_y is equal to $\overline{f(\gamma^{-1})}$ evaluated on v_y, v_x. We thus get a $*$-algebra $C_c^\infty(G, \Omega^{1/2})$. For each leaf L of (V, F) one has a natural representation of this $*$-algebra on the L^2 space of the holonomy covering $\tilde{L}$ of L. Fixing a base point $x \in L$, one identifies $\tilde{L}$ with $G_x = \{\gamma, s(\gamma) = x\}$ and defines

$$(\pi_x(f)\, \xi)\, (\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)\, \xi(\gamma_2) \qquad \forall\, \xi \in L^2(G_x)\,,$$

where ξ is a square integrable half-density on G_x. Given $\gamma : x \to y$ one has a natural isometry of $L^2(G_x)$ on $L^2(G_y)$ which transforms the representation π_x in π_y.

By definition $C^*(V, F)$ is the C^* algebra completion of $C_c^\infty(G, \Omega^{1/2})$ with the norm

$$\|f\| = \sup_{x \in V}\, \|\pi_x(f)\|\,.$$

Note that $C^*(V, F)$ is always norm separable and admits a natural smooth subalgebra, namely the algebra $C_c^\infty(V, F) = C_c^\infty(G, \Omega^{1/2})$ of smooth compactly supported half-densities.

If the leaf L has trivial holonomy then the representation π_x, $x \in L$, is irreducible. In general, its commutant is generated by the action of the (discrete) holonomy group G_x^x on $L^2(G_x)$. If the foliation comes from a submersion $p : V \to B$, then its graph G is $\{(x, y) \in V \times V,\, p(x) = p(y)\}$ which is a submanifold of $V \times V$, and $C^*(V, F)$ is identical to the algebra of the continuous field of Hilbert spaces $L^2(p^{-1}\{x\})_{x \in B}$. Thus (unless $\dim F = 0$) it is isomorphic to the tensor product of $C_0(B)$ with the elementary C^*-algebra of compact operators. If the foliation comes from an action of a Lie group H in such a way that the graph is identical to $V \times H$ (this is not always true even for flows) then $C^*(V, F)$ is identical to the reduced crossed product of $C_0(V)$ by H. Moreover the construction of $C^*(V, F)$ is local in the following sense.

If $V' \subset V$ is an open set and F' is the restriction of F to V', then the graph G' of (V', F') is an open set in the graph G of (V, F), and the inclusion

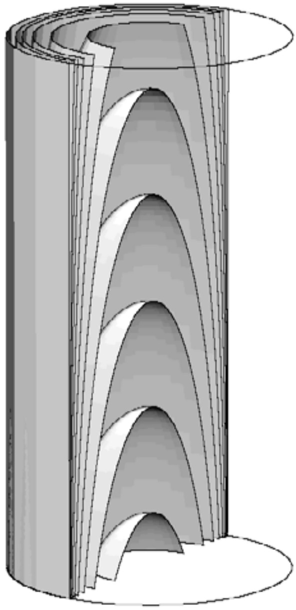


Figure 3. The Reeb foliation.

$C_c^\infty(G', \Omega^{1/2}) \subset C_c^\infty(G, \Omega^{1/2})$ extends to an isometric $*$-homomorphism of $C^*(V', F')$ in $C^*(V, F)$. The proof is straightforward and also applies in the case of non-Hausdorff graph.

Let us now briefly explain how the construction of the C^*-algebra $C^*(V, F)$ has to be done in the case when the graph of the foliation is not Hausdorff. This case is rather rare, since it never occurs if the foliation is real analytic. However, it does occur in cases which are topologically interesting for foliations, such as the Reeb foliation of the 3-sphere, which are constructed by patching together foliations of manifolds with boundaries (V_i, F_i) where the boundary ∂V_i is a leaf of F_i. In fact, most of the constructions done in geometry to produce smooth foliations of given codimension on a given manifold give a non-Hausdorff graph. The C^*-algebra $C^*(V, F)$ turns out in this case to be obtained as a fibered product of the $C^*(V_i, F_i)$.

In the general non-Hausdorff case the graph G of (V, F), being non-Hausdorff, may have only very few continuous fonctions with compact support. However, being a manifold, we can give a local chart $U \xrightarrow{\chi} \mathbb{R}^{\dim G}$. Take a smooth function $\varphi \in C_c^\infty(\mathbb{R}^{\dim G})$, $\operatorname{Supp} \varphi \subset \chi(U)$ and consider the function on G equal to $\varphi \circ \chi$ on U and to 0 outside U. If G were Hausdorff, this would generate all of $C_c^\infty(G)$ by taking linear combinations, and in general we take this linear span as the *definition* of $C_c^\infty(G)$. Note that we do not get *continuous* functions, since there may well be a sequence $\gamma_n \in U$ with two limits, one in $\mathrm{Supp}\,\varphi\circ\chi$, one in the complement of U. The above definition of $C_c^\infty(G)$ obviously extends to get $C_c^\infty(G,\Omega^{1/2})$, the space of smooth $\frac{1}{2}$ densities on G. One then shows that the convolution $\varphi_1 * \varphi_2$ of $\varphi_1, \varphi_2 \in C_c^\infty(G,\Omega^{1/2})$ is in $C_c^\infty(G,\Omega^{1/2})$.

Then we proceed exactly as in the Hausdorff case, and construct the representation π_x of the $*$-algebra $C_c^\infty(G,\Omega^{1/2})$ in the Hilbert space $L^2(G_x)$. We note that, though G is not Hausdorff, each G_x *is* Hausdorff, being the holonomy covering of the leaf through x.

For each $\varphi \in C_c^\infty(G,\Omega^{1/2})$ and $x \in V$, $\pi_x(\varphi)$ is an ordinary smoothing operator, bounded in $L^2(G_x)$.

Exactly as in the Hausdorff case $C^*(V,F)$ is defined as the C^* completion of $C_c^\infty(G,\Omega^{1/2})$ with norm $\sup_{x\in V}\,\|\pi_x(\varphi)\|$.

The obtained functor from foliations to C^*-algebras makes it possible first of all to translate from basic geometric properties to algebraic ones and the simplest examples of foliations already exhibit remarkable C^*-algebras. For instance the horocycle foliation of the unit sphere bundle of a Riemann surface of genus > 1 gives a simple C^*-algebra without idempotents. The Kronecker foliation gives rise to the noncommutative torus, which we describe in more detail in Section 6.

In the type II situation, *i.e.* in the presence of a holonomy invariant transverse measure Λ, the basic result of the theory is the longitudinal index theorem which computes the L^2-index of differential operators D on the foliated manifold (V,F) which are elliptic in the longitudinal direction (*i.e.* D restricts to the leaves L as elliptic operators D_L). One starts with a pair of smooth vector bundles E_1, E_2 on V together with a differential operator D on V from sections of E_1 to sections of E_2 such that:

(1) D restricts to leaves, *i.e.* $(D\xi)_x$ only depends on the restriction of ξ to a neighborhood of x in the leaf of x (*i.e.* D only uses partial differentiation in the leaf direction).
(2) D is elliptic when restricted to any leaf.

Theorem 5.1. [54] (a) *There exists a Borel transversal B (resp. B') such that the bundle $(\ell^2(L\cap B))_{L\in V/F}$ is measurably isomorphic to the bundle $(\mathrm{Ker}\,D_L)_{L\in V/F}$ (resp. to $(\mathrm{Ker}\,D_L^*)_{L\in V/F}$).*
(b) *The scalar $\Lambda(B) < \infty$ is independent of the choice of B and is denoted $\dim_\Lambda(\mathrm{Ker}\,(D))$.*
(c) $\dim_\Lambda(\mathrm{Ker}\,(D)) - \dim_\Lambda(\mathrm{Ker}\,(D^*)) = \varepsilon\,\langle \mathrm{ch}\,\sigma_D\,\mathrm{Td}\,(F_{\mathbb{C}}), [C]\rangle$
($\varepsilon = (-1)\,\frac{k(k+1)}{2}$, $k = \dim F$, $\mathrm{Td}\,(F_{\mathbb{C}})$ = *Todd genus*, σ_D = *symbol of* D).

Here $[C] \in H_k(V, \mathbb{C})$ is the homology class of the Ruelle–Sullivan current, a closed de Rham current of dimension $k = \dim F$ which encodes the transverse measure Λ by integration of a k-dimensional differential form ω on V along the plaques of foliation charts.

In particular the Betti numbers β_j of a measured foliation were defined in [54] and give the L^2-dimension of the space of L^2-harmonic forms along the leaves. More precisely, one has the following result.

(a) For each $j = 0, 1, 2, \ldots, \dim F$, there exists a Borel transversal B_j such that the bundle $(H^j(L, \mathbb{C}))_{L \in V/F}$ of j-th square integrable harmonic forms on L is measurably isomorphic to $(\ell^2(L \cap B))_{L \in V/F}$.
(b) The scalar $\beta_j = \Lambda(B_j)$ is finite, independent of the choice of B_j, of the choice of the Euclidean structure on F.
(c) One has $\Sigma\, (-1)^j\, \beta_j = \chi(F, \Lambda)$.

Here the Euler characteristic is simply given by the pairing of the Ruelle-Sullivan current with the Euler class $e(F)$ of the oriented bundle F on V.

Extending ideas of Cheeger and Gromov [48] in the case of discrete groups, D. Gaboriau has shown in a remarkable recent work (*cf.* [110], [111]) that the Betti numbers $\beta_j(F, \Lambda)$ of a foliation with contractible leaves are invariants of the measured equivalence relation $\mathcal{R} = \{(x, y) \,|\, y \in \mathrm{leaf}(x)\}$.

In the general case one cannot expect to have a holonomy invariant transverse measure and in fact the simplest foliations are of type III from the measure theoretic point of view. Obtaining an analogue in general of Theorem 5.1 was the basic motivation for the construction of the assembly map (the second step of Section 2). Let us now briefly state the longitudinal index theorem.

Let D be as above an elliptic differential operator along the leaves of the foliation (V, F). Since D is elliptic it has an inverse modulo $C^*(V, F)$, hence it gives an element $\mathrm{Ind}_a(D)$ of $K_0(C^*(V, F))$ which is the *analytic index* of D. The *topological index* is obtained as follows. Let i be an auxiliary imbedding of the manifold V in $\mathbb{R}^{2n}$. Let N be the total space of the normal bundle to the leaves: $N_x = (i_*(F_x))^\perp \subset \mathbb{R}^{2n}$. Let us foliate $\tilde{V} = V \times \mathbb{R}^{2n}$ by $\tilde{F}$, $\tilde{F}_{(x,t)} = F_x \times \{0\}$, so that the leaves of $(\tilde{V}, \tilde{F})$ are just $\tilde{L} = L \times \{t\}$, where L is a leaf of (V, F) and $t \in \mathbb{R}^{2n}$. The map $(x, \xi) \mapsto (x, i(x) + \xi)$ turns an open neighborhood of the 0-section in N into an open transversal T of the foliation $(\tilde{V}, \tilde{F})$. For a suitable open neighborhood Ω of T in $\tilde{V}$, the C^*-algebra $C^*(\Omega, \tilde{F})$ of the restriction of $\tilde{F}$ to Ω is (Morita) equivalent to $C_0(T)$, hence the inclusion $C^*(\Omega, \tilde{F}) \subset C^*(\tilde{V}, \tilde{F})$ yields a K-theory map: $K^0(N) \to K_0(C^*(\tilde{V}, \tilde{F}))$. Since $C^*(\tilde{V}, \tilde{F}) = C^*(V, F) \otimes C_0(\mathbb{R}^{2n})$, one has, by Bott periodicity, the equality $K_0(C^*(\tilde{V}, \tilde{F})) = K_0(C^*(V, F))$.

Using the Thom isomorphism, $K^0(F^*)$ is identified with $K^0(N)$ so that one gets by the above construction the topological index:

$$\mathrm{Ind}_t : K^0(F^*) \to K_0(C^*(V,F))\,.$$

The *longitudinal index theorem* [92] is the equality

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_t([\sigma_D])\,, \tag{5.2}$$

where σ_D is the longitudinal symbol of D and $[\sigma_D]$ is its class in $K^0(F^*)$.

Since the group $K_0(C^*(V,F))$ is still fairly hard to compute one needs computable invariants of its elements and this is where cyclic cohomology enters the scene. In fact its early development was already fully completed in 1981 for that precise goal (*cf.* [133]). The role of the trace on $C^*(V,F)$ associated to the transverse measure Λ is now played by cyclic cocycles on a dense subalgebra of $C^*(V,F)$. The hard analytic problem is to show that these cocycles have enough semicontinuity properties to define invariants of $K_0(C^*(V,F))$. This was achieved for some of them in [58] and makes it possible to formulate corollaries whose statements are independent of the general theory, such as the following.

Theorem 5.2. [58] *Let M be a compact oriented manifold and assume that the $\hat{A}$-genus $\hat{A}(M)$ is non-zero (since M is not assumed to be a spin manifold $\hat{A}(M)$ need not be an integer). Let then F be an integrable Spin sub-bundle of TM. There exists no metric on F for which the scalar curvature (of the leaves) is strictly positive ($\geq \varepsilon > 0$) on M.*

There is a very rich interplay between the theory of foliations and their characteristic classes and operator algebras even at the purely measure theoretic level, *i.e.* the classification of factors.

In a remarkable series of papers (see [126] for references), J. Heitsch and S. Hurder have analyzed the interplay between the vanishing of the Godbillon–Vey invariant of a compact foliated manifold (V,F) and the type of the von Neumann algebra of the foliation. Their work culminates in the following beautiful result of S. Hurder ([126]). If the von Neumann algebra is *semi-finite*, then the Godbillon–Vey invariant *vanishes.* We have shown, in fact, that cyclic cohomology yields a stronger result, proving that, if $\mathrm{GV} \neq 0$, then the central decomposition of M necessarily contains factors M whose virtual modular spectrum is of finite covolume in $\mathbb{R}_+^*$.

Theorem 5.3. [58] *Let (V,F) be an oriented, transversally oriented, compact, foliated manifold,* ($\operatorname{codim} F = 1$). *Let M be the associated von Neumann algebra, and* $\mathrm{Mod}(M)$ *be its flow of weights. Then, if the Godbillon–Vey class of (V,F) is different from* 0*, there exists an invariant probability measure for the flow* $\mathrm{Mod}(M)$.

One actually constructs an invariant measure for the flow $\mathrm{Mod}(M)$, exploiting the following remarkable property of the natural cyclic 1-cocycle τ on the algebra $\mathcal{A}$ of the transverse 1-jet bundle for the foliation. When viewed as a linear map δ from $\mathcal{A}$ to its dual, δ is an unbounded derivation, which is *closable*, and whose domain extends to the center Z of the von Neumann algebra generated by $\mathcal{A}$. Moreover, δ vanishes on this center, whose elements $h \in Z$ can then be used to obtain new cyclic cocycles τ_h on $\mathcal{A}$. The pairing

$$L(h) = \langle \tau_h, \mu(x) \rangle$$

with the K-theory classes $\mu(x)$ obtained from the assembly map μ, which we had constructed with P. Baum [15], then gives a measure on Z, whose invariance under the flow of weights follows from the discreteness of the K-group. To show that it is non-zero, one uses an index formula that evaluates the cyclic cocycles, associated as above to the Gelfand-Fuchs classes, on the range of the assembly map μ.

The central question in the analysis of the noncommutative leaf space of a foliation is step 3) (of Section 2), namely the metric aspect which entails in particular constructing a spectral triple describing the transverse geometry. The reason why the problem is really difficult is that it essentially amounts to doing "metric" geometry on manifolds in a way which is "background independent", to use the terminology of physicists, *i.e.* which is *invariant* under diffeomorphisms rather than covariant as in traditional Riemannian geometry. Indeed, the transverse space of a foliation is a manifold endowed with the action of a large pseudo group of partial diffeomorphisms implementing the holonomy. Thus, in particular no invariant metric exists in the general case and the situation is very similar to trying to develop gravity without making use of any particular "background" metric, which automatically destroys the invariance under the action of diffeomeorphisms (*cf.* [89]). Using the theory of hypoelliptic differential operators and the basic technique of reduction from type III to type II, a general construction of a spectral triple was done by Connes–Moscovici in [84]. The remaining problem of the computation of the local index formula in cyclic cohomology was solved in [85] and led in particular to the discovery of new symmetries given by an action of a Hopf algebra which only depends upon the transverse dimension of the foliation.

This also led to the development of the noncommutative analogue of the Chern–Weil theory of characteristic classes [86] in the general context of Hopf algebra actions on noncommutative spaces and cyclic cohomology, a subject which is undergoing rapid progress, in particular thanks to the recent works of M. Khalkhali and collaborators [134], [135], [136], [120].

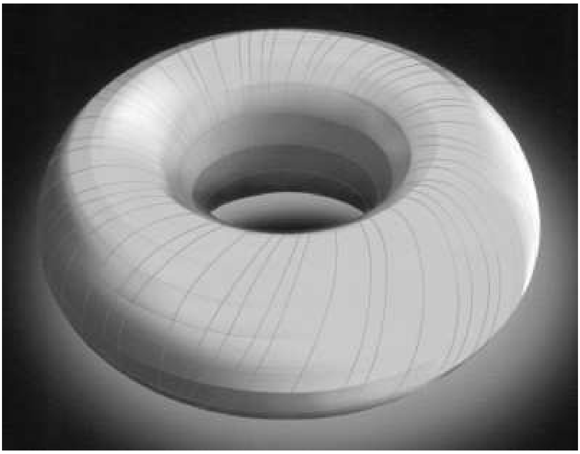


Figure 4. The Kronecker foliation and the noncommutative torus.

6. The Noncommutative Tori

This is perhaps considered as the prototype example of a noncommutative space, since it illustrates very clearly the properties and structures of noncommutative geometries. Noncommutative tori played a key role in the early developments of the theory in the 1980's (*cf.* [59]), giving rise to noncommutative analogues of vector bundles, connections, curvature, etc.

One can regard noncommutative tori as a special case of noncommutative spaces arising from foliations. In this case, one considers certain vector fields on the ordinary 2-dimensional real torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. In fact, one considers on T^2 the Kronecker foliation $dx = \theta dy$, where θ is a given real number. We are especially interested in the case where θ is irrational. That is, we consider the space of solutions of the differential equation,

$$dx = \theta dy, \quad \forall\, x, y \in \mathbb{R}/\mathbb{Z}\,, \tag{6.1}$$

where $\theta \in (0, 1)$ is a fixed irrational number. In other words, we are considering the space of leaves of the Kronecker foliation on the torus (*cf.* Figure 4).

We can choose a transversal T to the foliation, given by

$$T = \{y = 0\}, \qquad T \cong S^1 \cong \mathbb{R}/\mathbb{Z}\,.$$

Two points of the transversal which differ by an integer multiple of θ give rise to the same leaf. We want to describe the further quotient

$$S^1/\theta\mathbb{Z} \tag{6.2}$$

by the equivalence relation which identifies any two points on the orbits of the irrational rotation

$$R_\theta x = x + \theta \mod 1\,. \tag{6.3}$$

We can regard the circle S^1 and the quotient space (6.2) at various levels of regularity (smooth, topological, measurable). This corresponds to different algebras of functions on the circle,

$$C^\infty(S^1) \subset C(S^1) \subset L^\infty(S^1)\,. \tag{6.4}$$

When passing to the quotient (6.2), if we just consider invariant functions we obtain a very poor algebra of functions, since, even at the measurable level, we would only have constant functions. If instead we consider the noncommutative algebra of functions obtained by the general recipe of "noncommutative quotients" (functions on the graph of the equivalence relation with the convolution product), we obtain a very interesting and highly non–trivial algebra of functions describing the space of leaves of the foliation. This is given (in the topological category) by the "irrational rotation algebra", *i.e.* the C^*-algebra

$$\mathcal{A}_\theta := \{(a_{ij})\,,\quad i,j \in T \ \text{ in the same leaf}\}\,. \tag{6.5}$$

Namely, elements in the algebra $\mathcal{A}_\theta$ associated to the transversal $T \simeq S^1$ are just matrices (a_{ij}) where the indices are arbitrary pairs of elements i,j of S^1 belonging to the same leaf.

The algebraic rules are the same as for ordinary matrices. In the above situation, since the equivalence is given by a group action, the construction coincides with the crossed product. For instance, in the topological category, $\mathcal{A}_\theta$ is identified with the crossed product C^*-algebra

$$\mathcal{A}_\theta = C(S^1) \rtimes_{R_\theta} \mathbb{Z}\,. \tag{6.6}$$

The algebra (6.5) has two natural generators:

$$U = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6.7}$$

and

$$V = \begin{cases} e^{2\pi i a} & n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

In fact, an element $b = (a_{ij})$ of $\mathcal{A}_\theta$ can be written as power series

$$b = \sum_{n\in\mathbb{Z}} b_n U^n\,, \tag{6.9}$$

where each b_n is an element of the algebra (6.4), with the multiplication rule given by

$$U h U^{-1} = h \circ R_\theta^{-1}\,. \tag{6.10}$$

The algebra (6.4) is generated by the function V on S^1,

$$V(\alpha) = \exp(2\pi i\alpha) \quad \forall\, \alpha \in S^1 \tag{6.11}$$

and it follows that $\mathcal{A}_\theta$ is generated by two unitaries (U, V) with presentation given by the relation

$$VU = \lambda\, UV, \quad \text{with} \;\; \lambda = \exp(2\pi i\theta)\,. \tag{6.12}$$

If we work in the smooth category, then a generic element b in (6.4) is given by a power series

$$b = \sum_{\mathbb{Z}^2} b_{nm} U^n V^m \;\; \in \mathcal{S}(\mathbb{Z}^2)\,, \tag{6.13}$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the Schwartz space of sequences of rapid decay on $\mathbb{Z}^2$. We refer to the algebra of smooth functions (6.13) as $\mathcal{C}^\infty(\mathbb{T}^2_\theta)$, where we think of $\mathbb{T}^2_\theta$ as the (smooth) noncommutative torus.

Notice that in the definition (6.5) it is not necessary to restrict to the condition that i, j lie on the transversal T. It is possible to also form an algebra

$$\mathcal{B}_\theta = \{(a_{ij}) \quad i, j \in T^2 \;\; \text{in the same leaf}\}\,, \tag{6.14}$$

where now the parameter of integration is no longer discrete. This ought to correspond to the same noncommutative space, and in fact the algebras are related by

$$\mathcal{B}_\theta = \mathcal{A}_\theta \otimes \mathcal{K}\,,$$

where $\mathcal{K}$ is the algebra of all compact operators.

The tangent space to the ordinary torus T^2 is spanned by the tangent directions $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. By choosing coordinates U, V, with $U = e^{2\pi i x}$ and $V = e^{2\pi i y}$, the tangent vectors are given by $\frac{\partial}{\partial x} = 2\pi i U \frac{\partial}{\partial U}$ and $\frac{\partial}{\partial y} = 2\pi i V \frac{\partial}{\partial V}$. These have analogs in terms of derivations of the algebra of the noncommutative torus. The two commuting vector fields which span the tangent space for an ordinary (commutative) 2-torus correspond algebraically to two commuting derivations of the algebra of smooth functions.

These derivations continue to make sense when we replace the generators U and V of $C^\infty(\mathbb{T}^2)$ by the generators of the algebra $\mathcal{C}^\infty(\mathbb{T}^2_\theta)$, which no longer commute, as shown in (6.12). The derivations are still given by the same formulas as in the commutative case,

$$\delta_1 = 2\pi i U \,\frac{\partial}{\partial U}\,, \qquad \delta_2 = 2\pi i V \,\frac{\partial}{\partial V} \tag{6.15}$$

so that $\delta_1 \left(\sum b_{nm} U^n V^m\right) = 2\pi i \sum n b_{nm} U^n V^m$, and similarly for δ_2.

The operators (6.15) are commuting derivations of the algebra $\mathcal{C}^\infty(\mathbb{T}^2_\theta)$.

In fact, it is straightforward to verify that δ_1 and δ_2 satisfy

$$\delta_1\delta_2 = \delta_2\delta_1 \tag{6.16}$$

and

$$\delta_j(bb') = \delta_j(b)b' + b\delta_j(b') \quad \forall\, b, b' \in \mathcal{A}_\theta\,. \tag{6.17}$$

Just as in the classical case of a (commutative) manifold, what ensures that the derivations considered are enough to span the whole tangent space is the condition of ellipticity for the Laplacian

$$\Delta = \delta_1^2 + \delta_2^2\,.$$

In Fourier modes the Laplacian is of the form $n^2 + m^2$, hence Δ^{-1} is a compact operator.

The geometry of the Kronecker foliation is closely related to the structure of the algebra. In fact, a choice of a *closed transversal* T of the foliation corresponds canonically to a *finite projective module* over the algebra $\mathcal{A}_\theta$.

In fact, the main result on finite projective modules over the noncommutative tori $\mathbb{T}^2_\theta$ is the following classification, which is obtained by combining the results of [177], [59], [186].

Theorem 6.1. *Finite projective modules over $\mathcal{A}_\theta$ are classified up to isomorphism by a pair of integers (p, q) such that $p + q\theta \geq 0$. For a choice of such pair, the corresponding module $\mathcal{H}^\theta_{p,q}$ is obtained from the transversal $T_{p,q}$ given by the closed geodesic of the torus T^2 specified by (p, q), via the following construction. Elements of the module associated to the transversal $T_{p,q}$ are rectangular matrices, $(\xi_{i,j})$ with $(i, j) \in T \times S^1$, and with i and j belonging to the same leaf. The right action of $(a_{i,j}) \in \mathcal{A}_\theta$ is by matrix multiplication.*

For instance, from the transversal $x = 0$ one obtains the following right module over $\mathcal{A}_\theta$. The underlying linear space is the usual Schwartz space

$$\mathcal{S}(\mathbb{R}) = \{\xi : \xi(s) \in \mathbb{C}, \quad \forall\, s \in \mathbb{R}\} \tag{6.18}$$

of complex-valued smooth functions on $\mathbb{R}$, all of whose derivatives are of rapid decay. The right module structure is given by the action of the generators U, V:

$$(\xi U)(s) = \xi(s + \theta)\,, \qquad (\xi V)(s) = e^{2\pi i s}\xi(s) \quad \forall\, s \in \mathbb{R}\,. \tag{6.19}$$

One of course checks that the relation (6.12) is satisfied, and that, as a right module over $\mathcal{A}_\theta$, the space $\mathcal{S}(\mathbb{R})$ is *finitely generated* and *projective* (*i.e.* it complements to a free module).

Finitely generated projective modules play an important role in noncommutative geometry, as they replace *vector bundles* in the commutative setting. In fact, in ordinary commutative geometry, one can equivalently describe vector bundles through their sections, which in turn form a finite projective module over the algebra of smooth functions. The notion of finite projective module continues to make sense in the noncommutative setting, and provides this way a good notion of "noncommutative vector bundles".

Suppose given a vector bundle E, described algebraically through its space of smooth sections $C^\infty(X, E)$. One can compute the dimension of E by computing the trace of the identity endomorphism. In terms of the space of smooth sections, hence of finite projective modules, it is possible to recover the dimension of the vector bundle as a limit

$$\dim_{\mathcal{A}}(\mathcal{E}) = \lim_{N\to\infty} \frac{1}{N} \left(\# \text{ Generators of } \underbrace{\mathcal{E} \oplus \cdots \oplus \mathcal{E}}_{N \text{ times}} \right). \tag{6.20}$$

This method applies to the noncommutative setting. In the case of noncommutative tori, one finds that the Schwartz space $\mathcal{S}(\mathbb{R})$ has dimension the real number

$$\dim_{\mathcal{B}}(\mathcal{S}) = \theta\,. \tag{6.21}$$

One similarly finds values $p + q\theta$ for the more general case of Theorem 6.1.

The appearance of a real-valued dimension is related to the *density* of transversals in the leaves, that is, the limit of

$$\frac{\# B_R \cap S}{\text{size of } B_R},$$

for a ball B_R of radius R in the leaf. In this sense, the dimension θ of the Schwartz space measures the relative densities of the two transversals $S = \{x = 0\}$ and $T = \{y = 0\}$.

In general, the appearance of non-integral dimension is a basic feature of von Neumann algebras of type II. The dimension of a vector bundle is the only invariant that remains when one looks from the measure theoretic point of view, using the algebra of measurable functions $L^\infty(S^1)$ in (6.4). The von Neumann algebra which describes the quotient space from the measure theoretic point of view is the crossed product

$$R = L^\infty(S^1) \rtimes_{R_\theta} \mathbb{Z}\,. \tag{6.22}$$

This is the well known hyperfinite factor of type II_1. In particular the classification of finite projective modules $\mathcal{E}$ over R is given by a positive real number, the Murray–von Neumann dimension

$$\dim_R(\mathcal{E}) \in \mathbb{R}_+\,. \tag{6.23}$$

The simplest way to describe the phenomenon of Morita equivalence for noncommutative tori is in terms of the Kronecker foliation, where it corresponds to reparameterizing the leaf space in terms of a different closed transversal. Thus, Morita equivalence of the algebras $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ for θ and θ' in the same $\mathrm{PGL}(2,\mathbb{Z})$ orbit becomes simply a statement that the leaf space of the original foliation is independent of the transversal used to parameterize it. For instance, Morita equivalence between $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$ corresponds to changing the parameterization of the space of leaves from the transversal $T = \{y = 0\}$ to the transversal $S = \{x = 0\}$.

More generally, an explicit construction of bimodules $\mathcal{M}_{\theta,\theta'}$ was obtained in [59]. These are given by the Schwartz space $\mathcal{S}(\mathbb{R}\times\mathbb{Z}/c)$, with the right action of $\mathcal{A}_\theta$ given by

$$Uf\,(x,u) = f\left(x - \frac{c\theta+d}{c}, u-1\right),$$

$$Vf\,(x,u) = \exp(2\pi i(x-ud/c))f(x,u)$$

and the left action of $\mathcal{A}_{\theta'}$

$$U'f\,(x,u) = f\left(x - \frac{1}{c}, u-a\right),$$

$$V'f\,(x,u) = \exp\left(2\pi i\left(\frac{x}{c\theta+d} - \frac{u}{c}\right)\right) f(x,u)\,.$$

The bimodule $\mathcal{M}_{\theta,\theta'}$ realizes the Morita equivalences between $\mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ for

$$\theta' = \frac{a\theta+b}{c\theta+d} = g\theta$$

with $g\in \mathrm{PGL}(2,\mathbb{Z})$, *cf.* [59], [185].

7. Duals of Discrete Groups

Noncommutative geometry provides naturally a generalization of Pontrjagin duality for discrete groups. While the Pontrjagin dual $\hat{\Gamma}$ of a finitely generated discrete abelian group is a compact abelian group, the dual of a more general finitely generated discrete group is a noncommutative space.

To see this, recall that the usual Pontrjagin duality assigns to a finitely generated discrete abelian group Γ its group of characters $\hat{\Gamma} = \mathrm{Hom}(\Gamma, U(1))$. The duality is given by Fourier transform $e^{i\langle k,\gamma\rangle}$, for $\gamma\in\Gamma$ and $k\in\hat{\Gamma}$.

In particular, Fourier transform gives an identification between the algebra of functions on $\hat{\Gamma}$ and the (reduced) C^*-algebra of the group Γ,

$$C(\hat{\Gamma}) \cong C^*_r(\Gamma)\,, \tag{7.1}$$

where the reduced C^*-algebra $C^*_r(\Gamma)$ is the C^*-algebra generated by Γ in the regular representation on $\ell^2(\Gamma)$.

When Γ is non-abelian Pontrjagin duality no longer applies in the classical sense. However, the left hand side of (7.1) still makes sense and it behaves "like" the algebra of functions on the dual group. One can then say that, for a non-abelian group, the Pontrjagin dual $\hat{\Gamma}$ still exists as a noncommutative space whose algebra of coordinates is the C^*-algebra $C^*_r(\Gamma)$.

As an example that illustrates this general philosophy we give a different version of Example 4.2.

Example 7.1. The algebra (4.4) of Example 4.2 is the group ring of the dihedral group $\mathbb{Z} \rtimes \mathbb{Z}/2 \cong \mathbb{Z}/2 * \mathbb{Z}/2$.

In fact, first notice that a representation of the group $\mathbb{Z}/2 * \mathbb{Z}/2$ (free product of two copies of the group with two elements) is the same as a pair of subspaces in the Hilbert space, $E, F \subset \mathcal{H}$. The corresponding operators are $U = I - 2P_E$, $V = I - 2P_F$, with P_E, P_F the projections. The operators U, V represent reflections, since $U = U^*$, $U^2 = I$, $V = V^*$, $V^2 = I$. The group $\Gamma = \mathbb{Z}/2 * \mathbb{Z}/2$ realized as words in the generators U and V can equivalently be described as the semi-direct product $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}/2$, by setting $X = UV$, with the action $UXU^{-1} = X^{-1}$. The regular representation of Γ is analyzed using Mackey's theory for semi-direct products. One considers first representations of the normal subgroup, and then orbits of the action of $\mathbb{Z}/2$. The irreducible representations of the normal subgroup $\mathbb{Z}$ are labeled by $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and given by $X^n \mapsto z^n$. The action of $\mathbb{Z}/2$ is the involution given by conjugation $z \mapsto \bar{z}$. The quotient of S^1 by the $\mathbb{Z}/2$ action is identified with the interval $[-1, 1]$ by the map $z \mapsto \Re(z)$. For points inside the interval the corresponding irreducible representation of Γ is two-dimensional. At each of the two endpoints ± 1 one gets two inequivalent irreducible representations of Γ. Thus we recover the picture of Example 4.2 and an isomorphism $C^*(\Gamma) \sim A$ where A is the algebra (4.4).

The first two basic steps of the general theory are known for arbitrary discrete groups Γ, namely:

(1) The resolution of the diagonal and computation of the cyclic cohomology are provided by the geometric model (due to Burghelea [36]) given by the free loop space of the classifying space $B\Gamma$.

(2) The assembly map (BC-map) of [15] from the K-homology of the classifying space $B\Gamma$ to the K-theory of the reduced C^*-algebra $C_r^*(\Gamma)$ is refined in [16] to take care of torsion in the group Γ and gives a pretty good approximation to the K-theory of $C_r^*(\Gamma)$ (*cf.* [196] and references therein).

In the presence of a natural smooth subalgebra of $C_r^*(\Gamma)$ containing the group ring and stable under holomorphic functional calculus, the combination of the two steps described above makes it possible to prove an index theorem which is an higher dimensional form of Atiyah's L^2-index theorem for coverings. This gave the first proof of the Novikov conjecture for hyperbolic groups ([83]). Since then the analysis of dense smooth subalgebras has played a key role, in particular in the ground-breaking work of Vincent Lafforgue. See [15], [128], [142], [196], [197].

The next step, *i.e.* the construction of a spectral geometry, is directly related to geometric group theory. In general one cannot expect to get a finite dimensional spectral triple since the growth properties of the group

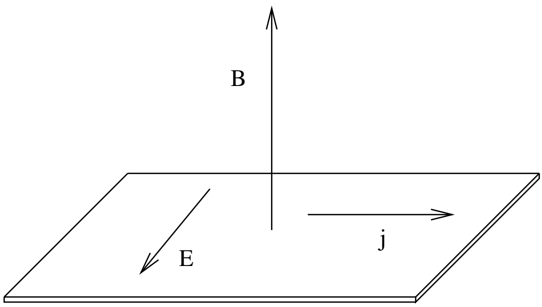


Figure 5. The quantum Hall effect experiment.

give (except for groups of polynomial growth) a basic obstruction (*cf.* [61]). A general construction of a theta summable spectral triple was given in [56], Section IV.9. Basically the transition from finitely summable spectral triples to the theta summable ones is the transition from finite dimensional geometry to the infinite dimensional case. In the theta summable case the Chern character is no longer a finite dimensional cyclic cocycle and one needs to extend cyclic cohomology using cocycles with infinite support in the (b, B) bicomplex fulfilling a subtle growth condition. The general theory of *entire* cyclic cohomology was developed in [62]. It is in general quite difficult to compute the Chern character in the theta summable case and one had to wait quite a long time until it was done for the basic example of discrete subgroups of semi-simple Lie groups. This has been achieved in a remarkable recent paper of M. Puschnigg [179] in the case of real rank one.

The fourth step, *i.e.* the thermodynamics might seem irrelevant in the type II context of discrete groups. However a small variant of the construction of the group ring, namely the Hecke algebra associated to an almost normal inclusion of discrete groups (in the sense considered in [27]) suffices to meet the type III world. One of the open fields is to extend the above steps (1), (2) and (3) in the general context of almost normal inclusions of discrete groups, and to perform the thermodynamical analysis in the spirit of [69] in that context.

8. Brillouin Zone and the Quantum Hall Effect

An important application to physics of the theory of noncommutative tori was the development of a rigorous mathematical model for the Integer Quantum Hall Effect (IQHE) obtained by Bellissard and collaborators [18], [19], [56].

The classical Hall effect is a physical phenomenon first observed in the 19th century [121]. A very thin metal sample is immersed in a constant uniform strong magnetic field orthogonal to the surface of the sample. By forcing a constant current to flow through the sample, the flow of charge carriers in the metal is subject to a Lorentz force perpendicular to the current and the magnetic field. The equation for the equilibrium of forces in the sample is of the form

$$Ne\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = 0\,, \tag{8.1}$$

where $\mathbf{E}$ is the electric field, e and N the charge and number of the charge carriers in the metal, $\mathbf{B}$ the magnetic field, and $\mathbf{j}$ the current.

The equation (8.1) defines a linear relation: the ratio of the intensity of the Hall current to the intensity of the electric field defines the Hall conductance,

$$\sigma_H = \frac{Ne\delta}{B}\,, \tag{8.2}$$

with $B = |\mathbf{B}|$ the intensity of the magnetic field and δ the sample width. The dimensionless quantity

$$\nu_H = \frac{N\delta h}{Be} = \sigma_H R_H \tag{8.3}$$

is called the filling factor, while the quantity $R_H = h/e^2$ is the Hall resistance. The filling factor measures the fraction of Landau level filled by conducting electrons in the sample. Thus, classically, the Hall conductance, measured in units of e^2/h, equals the filling factor.

In 1980, about a century after the classical Hall effect was observed, von Klitzing's experiment showed that, lowering the temperature below 1 K, the relation of Hall conductance to filling factor shows plateaux at integer values, [138]. The integer values of the Hall conductance are observed with a surprising experimental accuracy of the order of 10^{-8}. This phenomenon of quantization of the Hall conductance is known as the Integer Quantum Hall Effect (IQHE).

Laughlin first suggested that IQHE should be of a geometric origin [145]. A detailed mathematical model of the IQHE, which accounts for all the important features of the experiment (quantization, localization, insensitivity to the presence of disorder, vanishing of direct conductance at plateaux levels) improving over the earlier Laughlin model, was developed by Bellissard and collaborators [18], [19].

Bellissard's approach to the IQHE is based on noncommutative geometry. The quantization of the Hall conductance at integer values is indeed geometric in nature: it resembles another well known "quantization" phenomenon that happens in the more familiar setting of the geometry of compact 2-dimensional manifolds, namely the Gauss–Bonnet theorem, where the integral of the curvature is an integer multiple of 2π, a property that is stable under deformations. In the same spirit, the values of the Hall conductance are related to the evaluation of a certain characteristic class, or, in other words, to an index theorem for a Fredholm operator.

More precisely, in the physical model one makes the simplifying assumption that the IQHE can be described by non-interacting particles. The Hamiltonian then describes the motion of a single electron subject to the magnetic field and an additional potential representing the lattice of ions in the conductor. In a perfect crystal and in the absence of a magnetic field, there is a group of translational symmetries. This corresponds to a group of unitary operators $U(a)$, $a \in G$, where G is the locally compact group of symmetries. Turning on the magnetic field breaks this symmetry, in the sense that translates of the Hamiltonian $H_a = U(a)HU(a)^{-1}$ no longer commute with the Hamiltonian H. Since there is no preferred choice of one translate over the others, the algebra of observables must include all translates of the Hamiltonian, or better their resolvents, namely the bounded operators

$$R_a(z) = U(a)(zI - H)^{-1}U(a)^{-1} . \tag{8.4}$$

For a particle of (effective) mass m and charge e confined to the plane, subject to a magnetic field of vector potential $\mathbf{A}$ and to a bounded potential V, the Hamiltonian is of the form

$$H = \frac{1}{2m} \sum_{j=1,2} (p_j - eA_j)^2 + V = H_0 + V , \tag{8.5}$$

where the unperturbed part H_0 is invariant under the magnetic translations, namely the unitary representation of the translation group $\mathbb{R}^2$ given by

$$U(a)\psi(x) = \exp\left(\frac{-ieB}{2\hbar}\omega(x,a)\right)\psi(x-a) ,$$

with ω the standard symplectic form in the plane. The hull (strong closure) of the translates (8.4) yields a topological space, whose homeomorphism type is independent of the point z in the resolvent of H. This provides a noncommutative version of the Brillouin zone.

Recall that the Brillouin zones of a crystals are fundamental domains for the reciprocal lattice $\Gamma^\sharp$ obtained via the following inductive procedure. The *Bragg hyperplanes* of a crystal are the hyperplanes along which a pattern of diffraction of maximal intensity is observed when a beam of radiation (X-rays for instance) is shone at the crystal. The N-th Brillouin zone consists of all the points in (the dual) $\mathbb{R}^d$ such that the line from that point to the origin crosses exactly $(n-1)$ Bragg hyperplanes of the crystal.

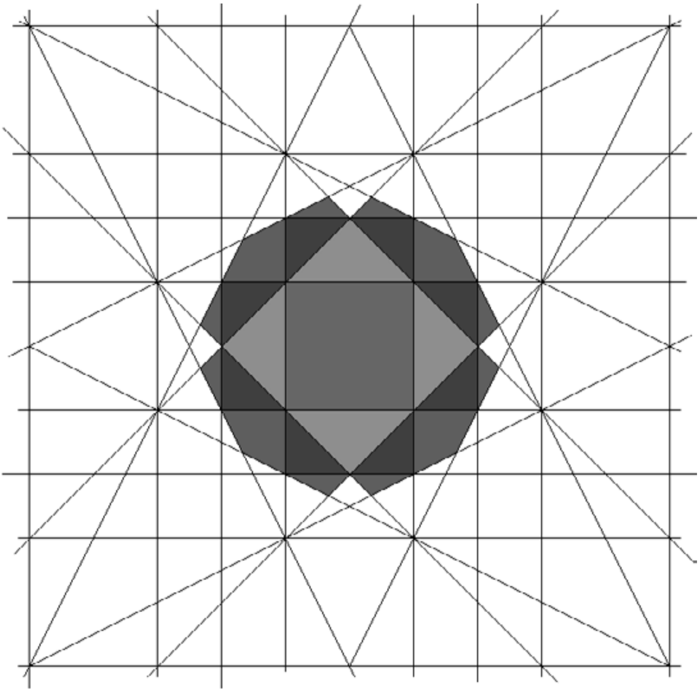


Figure 6. Brillouin zones for a 2-dimensional crystal.

More precisely, in our case, if e_1 and e_2 are generators of the periodic lattice, we obtain a commutation relation

$$U(e_1)U(e_2) = e^{2\pi i\theta}U(e_2)U(e_1)\,,$$

where θ is the flux of the magnetic field through a fundamental domain for the lattice, in dimensionless units, hence the noncommutative Brillouin zone is described by a noncommutative torus.

This can also be seen easily in a discrete model, where the Hamiltonian is given by an operator

$$(H_a\, f)(m,n) = e^{-ia_1 n} f(m+1,n) + e^{ia_1 n} f(m-1,n) \\ + e^{-ia_2 m} f(m,n+1) + e^{ia_2 m} f(m,n-1)\,, \qquad (8.6)$$

for $f \in L^2(\mathbb{Z}^2)$. This is a discrete version of the magnetic Laplacian. Notice then that (8.6) can be written in the form

$$H_a = U + V + U^* + V^*\,,$$

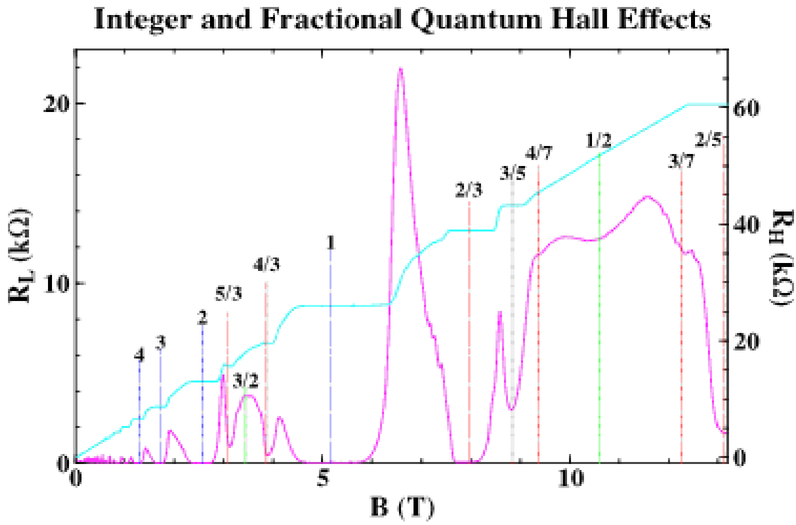


Figure 7. Observed fractions in the quantum Hall effect.

for

$$(U\,f)(m,n) = e^{-ia_2 m} f(m, n+1)\,, \qquad (V\,f)(m,n) = e^{-ia_1 n} f(m+1, n)\,.$$

These clearly satisfy the commutation relation (6.12) of $\mathbb{T}^2_\theta$ with $\theta = a_2 - a_1$.

In the zero-temperature limit, the Hall conductance satisfies the Kubo formula

$$\sigma_H = \frac{1}{2\pi i R_H} \tau(P_\mu[\delta_1 P_\mu, \delta_2 P_\mu])\,, \tag{8.7}$$

where P_μ is the spectral projection of the Hamiltonian on energies smaller than or equal to the Fermi level E_μ, τ is the trace on $\mathcal{A}_\theta$ given by

$$\tau\left(\sum a_{n,m} U^n V^m\right) = a_{0,0}\,. \tag{8.8}$$

and δ_1, δ_2 are as in (6.15). Here we assume that the Fermi level μ is in a gap in the spectrum of the Hamiltonian. Then the spectral projections P_μ belong to the C^*-algebra of observables.

The Kubo formula (8.7) can be derived from purely physical considerations, such as transport theory and the quantum adiabatic limit.

The main result is then the fact that the integrality of the conductance observed in the Integer Quantum Hall Effect is explained topologically, that is, in terms of the integrality of the *cyclic cocycle* $\tau(a^0(\delta_1 a^1 \delta_2 a^2 - \delta_2 a^1 \delta_1 a^2))$ (*cf.* [59]).

The fractional QHE was discovered by Størmer and Tsui in 1982. The setup is as in the quantum Hall effect: in a high quality semiconductor interface, which will be modelled by an infinite 2-dimensional surface, with low carrier concentration and extremely low temperatures $\sim 10mK$, in the presence of a very strong magnetic field, the experiment shows that the graph of $\frac{h}{e^2}\sigma_H$ against the filling factor ν exhibits plateaux at certain fractional values.

The independent electron approximation that, in the case of the integer quantum Hall effect, reduces the problem to a single electron wavefunction is no longer viable in this case and one has to incorporate the Coulomb interaction between the electrons in a many-electron theory. Nonetheless, it is possible to use a crude approximation, whereby one alters the underlying geometry to account for an average effect of the multi-electron interactions. One can obtain in this way a model of the fractional quantum Hall effect via noncommutative geometry (*cf.* [161]–[163]), where one uses hyperbolic geometry to simulate the interactions.

The noncommutative geometry approach to the quantum Hall effect described above was extended to hyperbolic geometry in [37]. The analog of the operator (8.6) is given by the Harper operator on the Cayley graph of a finitely generated discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$. Given $\sigma : \Gamma \times \Gamma \to U(1)$ satisfying $\sigma(\gamma, 1) = \sigma(1, \gamma) = 1$ and

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3)\,,$$

one considers the right σ-regular representation on $\ell^2(\Gamma)$ of the form

$$R^\sigma_\gamma \psi(\gamma') = \psi(\gamma'\gamma)\sigma(\gamma', \gamma) \tag{8.9}$$

satisfying

$$R^\sigma_\gamma R^\sigma_{\gamma'} = \sigma(\gamma, \gamma') R^\sigma_{\gamma\gamma'}\,.$$

For $\{\gamma_i\}_{i=1}^r$ a symmetric set of generators of Γ, the Harper operator is of the form

$$\mathcal{R}_\sigma = \sum_{i=1}^r R^\sigma_{\gamma_i}\,, \tag{8.10}$$

and the operator $r - \mathcal{R}_\sigma$ is the discrete analog of the magnetic Laplacian (*cf.* [204]).

The idea is that, by the effect of the strong interaction with the other electrons, a single electron "sees" the surrounding geometry as hyperbolic, with lattice sites that appear (as a multiple image effect) as the points in a lattice $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$. Thus, one considers the general form of such a lattice

$$\Gamma = \Gamma(g; \nu_1, \dots, \nu_n)\,, \tag{8.11}$$

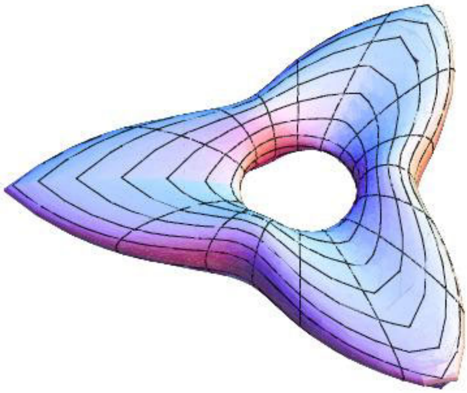


Figure 8. Hyperbolic orbifolds.

with generators a_i, b_i, c_j, with $i = 1, \ldots, g$ and $j = 1, \ldots, n$ and a presentation of the form

$$\Gamma(g; \nu_1, \ldots, \nu_n) = \langle a_i, b_i, c_j \;\Big|\; \prod_{i=1}^{g} [a_i, b_i] c_1 \cdots c_n = 1, \;\; c_j^{\nu_j} = 1 \rangle . \tag{8.12}$$

The quotient of the action of Γ by isometries on $\mathbb{H}$,

$$\Sigma(g; \nu_1, \ldots, \nu_n) := \Gamma \backslash \mathbb{H} , \tag{8.13}$$

is a hyperbolic orbifold.

Let P_E denote the spectral projection associated to the Fermi level, *i.e.* $P_E = \chi_{(-\infty, E]}(H)$. Then, in the zero temperature limit, the Hall conductance is given by

$$\sigma_E = \mathrm{tr}_K(P_E, P_E, P_E) , \tag{8.14}$$

where tr_K denotes the conductance 2-cocycle. It is a cyclic 2-cocycle on the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$ of the form

$$\mathrm{tr}_K(f_0, f_1, f_2) = \sum_{j=1}^{g} \mathrm{tr}(f_0(\delta_j(f_1)\delta_{j+g}(f_2) - \delta_{j+g}(f_1)\delta_j(f_2))) , \tag{8.15}$$

where the δ_j are derivations associated to the 1-cocycles a_j associated to a symplectic basis $\{a_j, b_j\}_{j=1,\ldots,g}$ of $H^1(\Gamma, \mathbb{R})$ (*cf.* [163]).

Within this model, one obtains the fractional values of the Hall conductance as integer multiples of orbifold Euler characteristics

$$\chi_{orb}(\Sigma(g; \nu_1, \ldots, \nu_n)) = 2 - 2g + \nu - n \in \mathbb{Q} . \tag{8.16}$$

In fact, one shows (*cf.* [162], [163]) that the conductance 2-cocycle is cohomologous to another cocycle, the area 2-cocycle, for which one can compute the values on K-theory (hence the value of (8.14)) by applying a twisted version of the Connes–Moscovici higher index theorem [83].

While in the case of the integer quantum Hall effect the noncommutative geometry model is completely satisfactory and explains all the physical properties of the system, in the fractional case the orbifold model can be considered as a first rough approximation to the quantum field theory that governs the fractional quantum Hall effect. For instance, the geometry of 2-dimensional hyperbolic orbifolds is related to Chern–Simons theory through the moduli spaces of vortex equations. This remains an interesting open question.

9. Tilings

In general, by a *tiling* $\mathcal{T}$ in $\mathbb{R}^d$ one means the following. One considers a finite collection $\{\tau_1, \ldots, \tau_N\}$ of closed bounded subsets of $\mathbb{R}^d$ homeomorphic to the unit ball. These are called the *prototiles*. One usually assumes that the prototiles are polytopes in $\mathbb{R}^d$ with a single d-dimensional cell which is the interior of the prototile, but this assumption can be relaxed. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is then a covering of $\mathbb{R}^d$ by sets with disjoint interior, each of which is a *tile*, *i.e.* a translate of one of the prototiles.

Given a tiling $\mathcal{T}$ of $\mathbb{R}^d$ one can form its orbit closure under translations. The metric on tilings is defined by saying that two tilings are close if they almost agree on a large ball centered at the origin in $\mathbb{R}^d$ (for more details and equivalent definitions see *e.g.* [4], [20]).

Tilings can be periodic or aperiodic. There are many familiar examples of periodic tilings, while the best known examples of aperiodic tilings are the Penrose tilings [175]. Similar types of aperiodic tilings have been widely studied in the physics of quasicrystals (*cf. e.g.* [13], [20]).

It was understood very early on in the development of noncommutative geometry (*cf.* [61] and pp. 5–7, pp. 88–93, and pp. 175–178 of [56]) that Penrose tilings provide an interesting class of noncommutative spaces.

In fact, one can consider on the set Ω of tilings $\mathcal{T}$ with given prototiles $\{\tau_1, \ldots, \tau_N\}$ the equivalence relation given by the action of $\mathbb{R}^d$ by translations, *i.e.* one identifies tilings that can be obtained from one another by translations. In the case of aperiodic tilings, this yields the type of quotient construction described in Section 4, which leads naturally to noncommutative spaces. An explicit description of this noncommutative space for the case of Penrose tilings can be found in §II.3 of [56].

To simplify the picture slightly, we can consider the similar problem (dually) with arrangements of points in $\mathbb{R}^d$ instead of tilings. This is the formulation used in the theory of aperiodic solids and quasicrystals (*cf.*

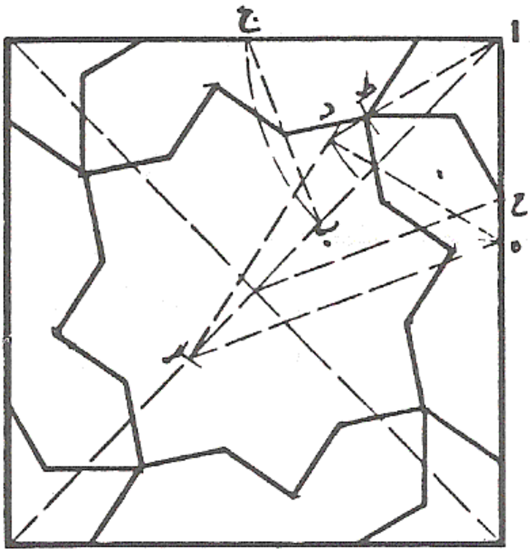


Figure 9. Prototiles for tilings.

[20]). Then, instead of tilings $\mathcal{T}$, we consider discrete subsets of points $\mathcal{L} \subset \mathbb{R}^d$. Such an $\mathcal{L}$ is a Delaunay set if there are radii $r, R > 0$ such that every open ball of radius r meets $\mathcal{L}$ in at most one point and every closed ball of radius R meets $\mathcal{L}$ in at least one point. One can describe $\mathcal{L}$ by the counting measure

$$\mu_{\mathcal{L}}(f) = \sum_{x \in \mathcal{L}} f(x)\,,$$

and one can take the orbit closure Ω of the action of $\mathbb{R}^d$ by translations

$$\mu_{\mathcal{L}} \mapsto T_{-a}\mu_{\mathcal{L}} = \mu_{\mathcal{L}} \circ T_a, \qquad \text{for } a \in \mathbb{R}^d,$$

in the space $\mathcal{M}(\mathbb{R}^d)$ of Radon measures with the weak* topology. The hull of $\mathcal{L}$ is the dynamical system (Ω, T), where T denotes the action of $\mathbb{R}^d$ by translations.

This dynamical system determines a corresponding noncommutative space, describing the quotient of Ω by translations, namely the crossed product C^*-algebra

$$\mathcal{A} = C(\Omega) \rtimes_T \mathbb{R}^d\,. \tag{9.1}$$

In fact, one can also consider the groupoid with set of units the transversal

$$X = \{\omega \in \Omega : 0 \in \mathrm{Support}(\omega)\}\,, \tag{9.2}$$

arrows of the form $(\omega, a) \in \Omega \times \mathbb{R}^d$, with source and range maps $s(\omega, a) = T_{-a}\omega$, $r(\omega, a) = \omega$ and $(\omega, a) \circ (T_{-a}\omega, b) = (\omega, a + b)$ (*cf.* [20]). This defines

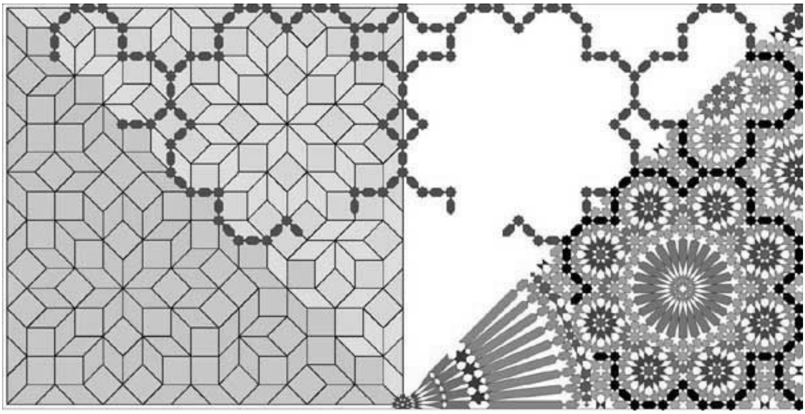


Figure 10. Quasiperiodic tilings and zellijs.

a locally compact groupoid $\mathcal{G}(\mathcal{L}, X)$. The C^*-algebras $C^*(\mathcal{G}(\mathcal{L}, X))$ and $C(\Omega) \rtimes_T \mathbb{R}^d$ are Morita equivalent.

In the case where $\mathcal{L}$ is a periodic arrangement of points with cocompact symmetry group $\Gamma \subset \mathbb{R}^d$, the space Ω is an ordinary commutative space, which is topologically a torus $\Omega = \mathbb{R}^d/\Gamma$. The C^*-algebra $\mathcal{A}$ is in this case isomorphic to $C(\hat{\Gamma}) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators and $\hat{\Gamma}$ is the Pontrjagin dual of the abelian group $\Gamma \cong \mathbb{Z}^d$, isomorphic to T^d, obtained by taking the dual of $\mathbb{R}^d$ modulo the reciprocal lattice

$$\Gamma^\sharp = \{k \in \mathbb{R}^d : \langle k, \gamma \rangle \in 2\pi\mathbb{Z}, \quad \forall\, \gamma \in \Gamma\}\,. \tag{9.3}$$

Thus, in physical language, $\hat{\Gamma}$ is identified with the Brillouin zone $B = \mathbb{R}^d/\Gamma^\sharp$ of the periodic crystal $\mathcal{L}$ (*cf.* Section 8). In this periodic case, the transversal $X = \mathcal{L}/\Gamma$ is a finite set of points. The groupoid C^*-algebra $C^*(\mathcal{G}(\mathcal{L}, X))$ is in this case isomorphic to $C(\hat{\Gamma}) \otimes M_k(\mathbb{C})$, where k is the cardinality of the transversal X. Thus, the periodic case falls back into the realm of commutative spaces, while the aperiodic patterns give rise to truly noncommutative spaces, which are highly nontrivial and interesting.

One of the richest sources of interesting tilings are the *zellijs* and *muqarnas* widely used in ancient architecture. Also collectively defined as “arabesques”, not only do these patterns exhibit highly nontrivial geometries, but they reflect the intricate interplay between philosophy, mathematics, and aesthetics (*cf.* [9], [35]). Some of the best studies on zellijs and muqarnas concentrate on 2-dimensional *periodic* patterns. For instance we find in [9], p. 43:

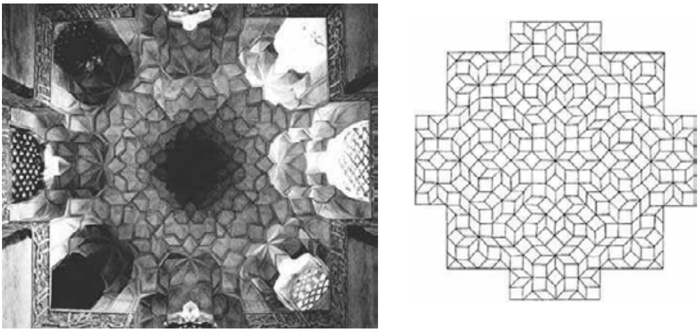


Figure 11. Quasiperiodic tilings and muqarnas.

> "As Nature is based on rhythm, so the arabesque is rhythmic in concept. It reflects movement marked by the regular recurrence of features, elements, phenomena; hence it has periodicity."

It seems from this viewpoint that only the theory of periodic tilings (*i.e.* commutative geometry) should be relevant in this context. However, more recent studies (*cf.* [35], [40], [41], [171]) suggest that the design of zellijs and muqarnas was not limited to 2-dimensional crystallographic groups, but, especially during the Timurid period, it involved also *aperiodic* patterns with fivefold symmetry, analogous to those observed in quasicrystals. This is no accident and was certainly the result of a highly developed geometric theory: already in the historic textbook of Abu'l-Wafa' al-Buzjani (940–998) on geometric constructions [208] there is explicit mention of meetings and discussions where mathematicians were directly involved alongside artisans in the design of arabesque patterns.

The appearance of aperiodic tilings is documented in the anonymous Persian manuscript [5] "On interlocking similar and congruent figures", which dates back to the 11th-13th century. Some of these aperiodic aspects of zellijs and muqarnas were studied by Bulatov in the book [35], which also contains Vil′danova's Russian translation of the ancient Persian text. For a more recent study of quasiperiodic tilings in Persian architecture see [149].

10. Noncommutative Spaces from Dynamical Systems

We will look at some examples of noncommutative spaces associated to a discrete dynamical system T, for instance given by a self-mapping of a Cantor set. Such noncommutative spaces have been extensively studied in a series of papers (*cf.* [113] and [198] for a survey) where C. Skau and his coworkers have obtained remarkable results on the classification of minimal actions of $\mathbb{Z}$ on Cantor sets using the K-theory of the associated C^*-algebra.

It was found recently (*cf.* [94], [95], §4 of [158] and §8 of [157]) that the mapping torus of such systems can be used to model the "dual graph" of the fibers at the archimedean primes of arithmetic surfaces, in Arakelov geometry, in the particular case in which the dynamical system T is a subshift of finite type encoding the action of a Schottky group $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ on its limit set $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$. In fact, the results of [94] were motivated by earlier results of Manin [154] that provided a geometric model for such dual graphs in terms of hyperbolic geometry and Schottky uniformizations.

More generally, given an alphabet with letters $\{\ell_1, \ldots, \ell_N\}$, the space $\mathcal{S}_A^+$ of a subshift of finite type consists of all right-infinite *admissible* sequences

$$w = a_0 a_1 a_2 \ldots a_n \ldots \tag{10.1}$$

in the letters of the alphabet. Namely, $a_i \in \{\ell_1, \ldots, \ell_N\}$ subject to an admissibility condition specified by an $N \times N$ matrix A with entries in $\{0, 1\}$. Two letters ℓ_i and ℓ_j in the alphabet can appear as consecutive digits a_k, a_{k+1} in the word w if and only if the entry A_{ij} of the admissibility matrix A is equal to 1. One defines similarly the space $\mathcal{S}_A$ as the set of doubly-infinite admissible sequences

$$w = \ldots a_{-m} \ldots a_{-2} a_{-1} a_0 a_1 a_2 \ldots a_n \ldots . \tag{10.2}$$

The sets $\mathcal{S}_A^+$ and $\mathcal{S}_A$ have a natural choice of topology. In fact, on $\mathcal{S}_A$ we can put the topology generated by the sets $W^s(x, \ell) = \{y \in \mathcal{S}_A | x_k = y_k, k \geq \ell\}$, and the $W^u(x, \ell) = \{y \in \mathcal{S}_A | x_k = y_k, k \leq \ell\}$ for $x \in \mathcal{S}_A$ and $\ell \in \mathbb{Z}$. This induces a topology with analogous properties on $\mathcal{S}_A^+$ by realizing it as a subset of $\mathcal{S}_A$, for instance, by extending each sequence to the left as a constant sequence. One then considers on $\mathcal{S}_A$ (or on $\mathcal{S}_A^+$) the action of the two-sided (resp. one-sided) shift T defined by $(Tw)_k = a_{k+1}$, where the a_k are the digits of the word w. Namely, the one-sided shift on $\mathcal{S}_A^+$ is of the form

$$T(a_0 a_1 a_2 \ldots a_\ell \ldots) = a_1 a_2 \ldots a_\ell \ldots \tag{10.3}$$

while the two-sided shift on $\mathcal{S}_A$ acts as

$$\begin{aligned} &T(\ \ldots \quad a_{-m} \quad \ldots a_{-1}\ a_0\ a_1 \ldots \quad a_\ell \quad \ldots\ ) \\ &\quad = \ldots a_{-m+1} \ldots \ a_0 \ \ a_1\ a_2 \ldots a_{\ell+1} \ldots . \end{aligned} \tag{10.4}$$

Typically, spaces $\mathcal{S}_A^+$ and $\mathcal{S}_A$ are topologically Cantor sets. The one-sided shift T of (10.3) is a continuous surjective map on $\mathcal{S}_A^+$, while the two-sided shift T of (10.4) is a homeomorphism of $\mathcal{S}_A$.

For example, let Γ be a free group on g generators $\{\gamma_1, \ldots, \gamma_g\}$. Consider the alphabet $\{\gamma_1, \ldots, \gamma_g, \gamma_1^{-1}, \ldots, \gamma_g^{-1}\}$. Then one can consider the right-infinite or doubly-infinite words in these letters, without cancellations, that is, subject to the admissibility rule that $a_{k+1} \neq a_k^{-1}$. This defines a subshift of finite type where the matrix A is the symmetric $2g \times 2g$ matrix with $A_{ij} = 0$ for $|i-j| = g$ and $A_{ij} = 1$ otherwise. Suppose that Γ is a Schottky group of genus g, *i.e.* a finitely generated discrete subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$, isomorphic to a free group on g generators, where all nontrivial elements are hyperbolic. Then the points in $\mathcal{S}_A^+$ parameterize points in the limit set $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ (the set of accumulation points of orbits of Γ). The points in $\mathcal{S}_A$ parametrize geodesics in the three-dimensional real hyperbolic space $\mathbb{H}^3$ with ends at points on the limit set Λ_Γ.

The pair $(\mathcal{S}_A, T)$ is a typical example of an interesting class of dynamical systems, namely it is a *Smale space*. This means that locally $\mathcal{S}_A$ can be decomposed as the product of expanding and contracting directions for T. Namely, the following properties are satisfied.

- For every point $x \in \mathcal{S}_A$ there exist subsets $W^s(x)$ and $W^u(x)$ of $\mathcal{S}_A$, such that $W^s(x) \times W^u(x)$ is homeomorphic to a neighborhood of x.
- The map T is contracting on $W^s(x)$ and expanding on $W^u(x)$, and $W^s(Tx)$ and $T(W^s(x))$ agree in some neighborhood of x, and so do $W^u(Tx)$ and $T(W^u(x))$.

A construction of Ruelle shows that one can associate different C^*-algebras to Smale spaces (*cf.* [191], [180], [181]). For Smale spaces like $(\mathcal{S}_A, T)$ there are four basic possibilities: the crossed product algebra $C(\mathcal{S}_A) \rtimes_T \mathbb{Z}$ and the C^*-algebras $C^*(\mathcal{G}^s) \rtimes_T \mathbb{Z}$, $C^*(\mathcal{G}^u) \rtimes_T \mathbb{Z}$, $C^*(\mathcal{G}^a) \rtimes_T \mathbb{Z}$ obtained by considering the action of the shift T on the groupoid C^*-algebra associated to the groupoids $\mathcal{G}^s$, $\mathcal{G}^u$, $\mathcal{G}^a$ of the stable, unstable, and asymptotic equivalence relations on $(\mathcal{S}_A, T)$.

The first choice, $C(\mathcal{S}_A) \rtimes_T \mathbb{Z}$, is closely related to the continuous dynamical system given by the mapping torus of T, while a choice like $C^*(\mathcal{G}^u) \rtimes_T \mathbb{Z}$ is related to the "bad quotient" of $\mathcal{S}_A^+$ by the action of T. In the example of the Schottky group this corresponds to the action of Γ on its limit set.

One can consider the suspension flow $\mathcal{S}_T$ of a dynamical system T, that is, the mapping torus of the dynamical system $(\mathcal{S}_A, T)$, which is defined as

$$\mathcal{S}_T := \mathcal{S}_A \times [0,1]/(x,0) \sim (Tx,1)\,. \tag{10.5}$$

The first cohomology group of $\mathcal{S}_T$ is the "ordered cohomology" of the dynamical system T, in the sense of [30], [174]. There is an identification of $H^1(\mathcal{S}_T, \mathbb{Z})$ with the K_0-group of the crossed product C^*-algebra for the action of T on $\mathcal{S}_A$,

$$H^1(\mathcal{S}_T, \mathbb{Z}) \cong K_0(C(\mathcal{S}_A) \rtimes_T \mathbb{Z})\,. \tag{10.6}$$

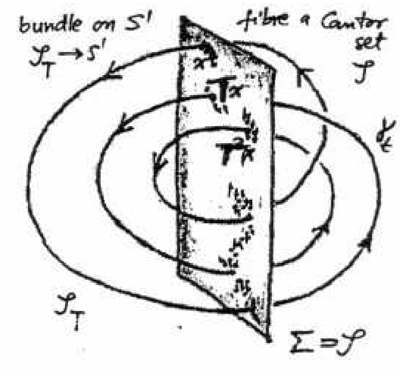


Figure 12. Mapping torus.

This can be seen from the Pimsner–Voiculescu exact sequence (*cf.* [177]) for the K-theory of a crossed product by $\mathbb{Z}$, which in this case reduces to

$$0 \to K_1(C(\mathcal{S}) \rtimes_T \mathbb{Z}) \to C(\mathcal{S}, \mathbb{Z}) \stackrel{I-T_*}{\to} C(\mathcal{S}, \mathbb{Z}) \to K_0(C(\mathcal{S}) \rtimes_T \mathbb{Z}) \to 0\,, \tag{10.7}$$

It can also be seen in terms of the Thom isomorphism of [57], [58].

In fact, as we discussed in Section 2, one of the fundamental construction of noncommutative geometry (*cf.* [58]) is that of *homotopy quotients.* These are commutative spaces which provide, up to homotopy, geometric models for the corresponding noncommutative spaces. The noncommutative spaces themselves, as we are going to show in our case, appear as quotient spaces of foliations on the homotopy quotients with contractible leaves.

For the noncommutative space $\mathcal{S}_A/\mathbb{Z}$, with $\mathbb{Z}$ acting as powers of the invertible two-sided shift, the homotopy quotient is precisely the mapping torus (10.5),

$$\mathcal{S}_T = \mathcal{S} \times_{\mathbb{Z}} \mathbb{R}\,. \tag{10.8}$$

The noncommutative space $\mathcal{S}/\mathbb{Z}$ can be identified with the quotient space of the natural foliation on (10.8) whose generic leaf is contractible (a copy of $\mathbb{R}$).

Another noncommutative space associated to a subshift of finite type T (which, up to Morita equivalence, corresponds to another choice of the C^*-algebra of a Smale space, as mentioned above) is the Cuntz–Krieger algebra $\mathcal{O}_A$, where A is the admissibility matrix of the subshift of finite type (*cf.* [98], [99]).

A partial isometry is a linear operator S satisfying the relation $S = SS^*S$. The Cuntz–Krieger algebra $\mathcal{O}_A$ is defined as the universal C^*-algebra generated by partial isometries $S_1, \ldots, S_N$, satisfying the relations

$$\sum_j S_j S_j^* = I \,, \tag{10.9}$$

$$S_i^* S_i = \sum_j A_{ij}\, S_j S_j^* \,. \tag{10.10}$$

In the case of a Schottky group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ of genus g, the Cuntz–Krieger algebra $\mathcal{O}_A$ can be described in terms of the action of the free group Γ on its limit set $\Lambda_\Gamma \subset \mathbb{P}^1(\mathbb{C})$ (*cf.* [188], [201]), so that we can regard $\mathcal{O}_A$ as a noncommutative space replacing the classical quotient Λ_Γ/Γ,

$$\mathcal{O}_A \cong C(\Lambda_\Gamma) \rtimes \Gamma \,. \tag{10.11}$$

The quotient space

$$\Lambda_\Gamma \times_\Gamma \mathbb{H}^3 = \Lambda_\Gamma \times_\Gamma \underline{E}\Gamma, \tag{10.12}$$

is precisely the homotopy quotient of Λ_Γ with respect to the action of Γ, with $\underline{E}\Gamma = \mathbb{H}^3$ and the classifying space $\underline{B}\Gamma = \mathbb{H}^3/\Gamma$. Here $\mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold of infinite volume, which is topologically a handlebody of genus g. In this case also we find that the noncommutative space Λ_Γ/Γ is the quotient space of a foliation on the homotopy quotient (10.12) with contractible leaves $\mathbb{H}^3$.

11. Noncommutative Spaces from String Theory

Yang–Mills theory on noncommutative tori was first formulated (*cf.* [91]) using suitable notions of connections and curvature for noncommutative spaces.

In fact, the analogs of connection and curvature of vector bundles are straightforward to obtain ([59]): a connection is just given by the associated covariant differentiation ∇ on the space of smooth sections. Thus here it is given by a pair of linear operators on the Schwartz space of rapidly decaying functions,

$$\nabla_j : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \tag{11.1}$$

such that

$$\nabla_j(\xi b) = (\nabla_j \xi) b + \xi \delta_j(b) \quad \forall\, \xi \in \mathcal{S}\,, \quad b \in \mathcal{A}_\theta \,. \tag{11.2}$$

One checks that, as in the usual case, the trace of the curvature

$$\Omega = \nabla_1 \nabla_2 - \nabla_2 \nabla_1 \,,$$

is independent of the choice of the connection.

We can make the following choice for the connection:

$$(\nabla_1 \xi)(s) = -\frac{2\pi i s}{\theta}\, \xi(s)\,, \qquad (\nabla_2 \xi)(s) = \xi'(s)\,. \tag{11.3}$$

Notice that, up to the correct powers of $2\pi i$, the total curvature of $\mathcal{S}$ is an *integer*. In fact, the curvature Ω is constant, equal to $\frac{1}{\theta}$, so that the irrational number θ disappears in the total curvature, $\theta \times \frac{1}{\theta}$. This integrality phenomenon, where the pairing of dimension and curvature (both of which are non-integral) yields an integer:

$$\dim \times \Omega \sim \theta \times \frac{1}{\theta} = \text{integer}\,,$$

is the basis for the development of a theory of characteristic classes for noncommutative spaces. In the general case, this requires the development of more sophisticated tools, since analogs of the derivations δ_i used in the case of the noncommutative tori are not there in general. The general theory is obtained through cyclic homology, as developed in [60].

Consider then the modules $\mathcal{H}^\theta_{p,q}$ described in Section 6. It is possible to define an $\mathcal{A}_\theta$-valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ on $\mathcal{H}^\theta_{p,q}$, as in [186], which is used to show that $\mathcal{H}^\theta_{p,q}$ is a projective module. Connections are required to be compatible with the metric,

$$\delta_j \langle \xi, \eta \rangle_{\mathcal{A}} = \langle \nabla_j \xi, \eta \rangle_{\mathcal{A}} + \langle \xi, \nabla_j \eta \rangle_{\mathcal{A}}\,. \tag{11.4}$$

It is proved in [59] that such connections always exist. The curvature Ω has values in $\mathcal{E} = \mathrm{End}_{\mathcal{A}}(\mathcal{H})$. An $\mathcal{E}$-valued inner product on $\mathcal{H}$ is given by

$$\langle \xi, \eta \rangle_{\mathcal{E}} \zeta = \xi \langle \eta, \zeta \rangle_{\mathcal{A}}\,,$$

and a canonical faithful trace τ_E is defined as

$$\tau_E(\langle \xi, \eta \rangle_{\mathcal{E}}) = \tau(\langle \eta, \xi \rangle_{\mathcal{A}})\,,$$

where τ is the trace on the algebra $\mathcal{A}_\theta$, given by (8.8).

The Yang–Mills action is defined (*cf.* [91]) as

$$\tau(\langle \Omega, \Omega \rangle_{\mathcal{E}})\,. \tag{11.5}$$

One seeks minima of the Yang–Mills action among metric compatible connections (11.4). The main result is that this recovers the classical moduli spaces of Yang–Mills connections on the ordinary torus ([91]):

Theorem 11.1. *For a choice of a pair of integers (p, q) with $p + q\theta \geq 0$, the moduli space of Yang–Mills connections on the $\mathcal{A}_\theta$-module $\mathcal{H}^\theta_{pq}$ is a classical space given by the symmetric product*

$$s^N(T^2) = (T^2)^N / \Sigma_N\,,$$

where Σ_N is the group of permutations in N elements, for $N = \gcd(p, q)$.

The fact that noncommutativity of space coordinates is relevant for gravity goes back to the analysis of [103] which is independent of string theory and produces in a natural manner the Moyal deformations of space-time, of which the noncommutative tori give a compact Euclidean version. Since then, tremendous progress has been made in understanding quantum field theory on noncommutative spaces, thanks mainly to the breakthrough by H. Grosse and R. Wulkenhaar [114].

The main aspects of string and D-brane theory that involve noncommutative geometry are the bound states of configurations of parallel D-branes [211], the matrix models for M-theory [14] and the strong coupling limit of string theory (*cf. e.g.* [7], [8]). It also plays an important role in the M-theory compactifications [71]. We shall not discuss all these aspects in detail here. Since the focus of this review is on examples we only mention a couple of examples of noncommutative spaces arising from string and D-brane theory.

The noncommutative tori and the components of the Yang–Mills connections appear in the classification of the BPS states in M-theory [71].

In the matrix formulation of M-theory the basic equations to obtain periodicity of two of the basic coordinates X_i turn out to be

$$U_i X_j U_i^{-1} = X_j + a\delta_i^j, \quad i = 1, 2\,, \tag{11.6}$$

where the U_i are unitary gauge transformations. The multiplicative commutator $U_1 U_2 U_1^{-1} U_2^{-1}$ is then central and in the irreducible case its scalar value $\lambda = \exp 2\pi i\theta$ brings in the algebra of coordinates on the noncommutative torus. The X_j are then the components of the Yang–Mills connections. The same picture emerged from the other information one has about M-theory concerning its relation with 11-dimensional supergravity and that string theory dualities can then be interpreted using Morita equivalence, relating the values of θ on an orbit of $SL_2(\mathbb{Z})$.

Nekrasov and Schwarz [172] showed that Yang–Mills gauge theory on noncommutative $\mathbb{R}^4$ gives a conceptual understanding of the nonzero B-field desingularization of the moduli space of instantons obtained by perturbing the ADHM equations. In [193], Seiberg and Witten exhibited an unexpected relation between the standard gauge theory and the noncommutative one, and clarified the limit in which the entire string dynamics is described by a gauge theory on a noncommutative space. Techniques from noncommutative differential and Riemannian geometry, in the sense discussed in Section 2, were applied to string theory, for instance in [7]. The role of noncommutative geometry in the context of T-duality was considered in very interesting recent work of Mathai and collaborators, [28], [29], [164].

Recently, in the context of the holographic description of type IIB string theory on the plane-wave background, Shahin M.M. Sheikh-Jabbari obtained (*cf.* [194]) an interesting class of noncommutative spaces from the quantization of Nambu d-brackets. The classical Nambu brackets

$$\{f_1, \ldots, f_k\} = \epsilon^{i_1 \cdots i_k} \frac{\partial f_1}{\partial x^{i_1}} \cdots \frac{\partial f_k}{\partial x^{i_k}} \tag{11.7}$$

of k real-valued functions of variables $(x^1, \ldots, x^k)$ is quantized in the even case to the expression in $2k$ operators

$$\frac{1}{i^k}[F_1, \ldots, F_{2k}] = \frac{1}{i^k (2k)!} \epsilon^{i_1 \cdots i_{2k}} F_{i_1} \cdots F_{i_{2k}} \,. \tag{11.8}$$

This generalizes the Poisson bracket quantization $\{f_1, f_2\} \mapsto \frac{-i}{\hbar}[F_1, F_2]$. The odd case is more subtle and it involves an additional operator related to the chirality operator γ_5. One sets

$$\frac{1}{i^k}[F_1, \ldots, F_{2k-1}, \gamma] = \frac{1}{i^k (2k)!} \epsilon^{i_1 \cdots i_{2k}} F_{i_1} \cdots F_{i_{2k-1}} \gamma \,, \tag{11.9}$$

where γ is the chirality operator in $2k$ dimensions. For example, for $k = 2$ one gets

$$\begin{aligned}
[F_1, F_1, F_3, \gamma] = \frac{1}{24}&([F_1, F_2][F_3, \gamma] + [F_3, \gamma][F_1, F_2] \\
&- ([F_1, F_3][F_2, \gamma] + [F_2, \gamma][F_1, F_3]) \\
&+ [F_2, F_3][F_1, \gamma] + [F_1, \gamma][F_2, F_3]) \,.
\end{aligned}$$

If one describes the ordinary d-dimensional sphere of radius R by the equation

$$\sum_{i=1}^{d+1} (x^i)^2 = R^2 \,, \tag{11.10}$$

the coordinates satisfy

$$\{x^{i_1}, \ldots, x^{i_d}\} = R^{d-1} \epsilon^{i_1 \cdots i_{d+1}} x^{i_{d+1}} \,. \tag{11.11}$$

Equations (11.10) and (11.11) are then replaced by their quantized version, using the quantization of the Nambu bracket and the introduction of a quantization parameter ℓ. This defines algebras generated by unitaries X^i subject to the relations given by the quantization of (11.10) and (11.11). Matrix representations of these algebras correspond to certain fuzzy spheres. It would be interesting to study the general structure of these noncommutative spaces from the point of view of the various steps introduced in Section 2, *cf.* also the discussion in Section 19.

12. Groupoids and the Index Theorem

Since the construction of the C^*-algebra of foliations based on the holonomy groupoid (Section 5), groupoids have played a major role in noncommutative geometry. In fact the original construction of matrix mechanics by Heisenberg (Section 3) is exactly that of the convolution algebra of the groupoid of transitions imposed by experimental results. The convolution algebra of groupoids can be defined in the context of von Neumann algebras and of C^*-algebras (*cf.* [54] and [183]). It is particularly simple and canonical in the context of smooth groupoids (*cf.* [56] Section II.5). One virtue of the general construction is that it provides a geometric mental picture of complicated analytical constructions. The prototype example is given by the tangent groupoid of a manifold (*cf.* [56] Section II.5). It is obtained by blowing up the diagonal in the square $V \times V$ of the manifold and as a set is given by

$$G_V = V \times V \times]0,1] \cup TV\,,$$

where TV is the (total space of the) tangent bundle of V. A tangent vector $X \in T_x(V)$ appears as the limit of nearby triples (x_1, x_2, ε) provided that in any chart the ratios $(x_1 - x_2)/\varepsilon$ converge to X. When $\varepsilon \to 0$ the Heisenberg (matrix) law of composition:

$$(x_1, x_2, \varepsilon) \circ (x_2, x_3, \varepsilon) = (x_1, x_3, \varepsilon)$$

converges to the addition of tangent vectors, so that G_V becomes a smooth groupoid. The functoriality of the construction $G \to C^*(G)$ of the convolution algebra for smooth groupoids G is then enough to define the Atiyah-Singer analytic index of pseudo-differential operators. It is simply given by the map in K-theory for the exact sequence of C^*-algebras associated to the geometric sequence

$$V \times V \times]0,1] \to G_V \supset TV\,,$$

where TV is viewed as a closed subgroupoid of G_V. The corresponding exact sequence of C^*-algebras can be written as

$$0 \to C_0(]0,1]) \times \mathcal{K} \to C^*(G_V) \to C_0(T^*V) \to 0$$

and is a geometric form of the extension of pseudo-differential operators. By construction the algebra $C_0(]0,1])$ is *contractible* and the same holds for the tensor product $C_0(]0,1]) \times \mathcal{K}$ by the algebra $\mathcal{K}$ of compact operators. This shows that the restriction map $C^*(G_V) \to C_0(T^*V)$ is an isomorphism in K-theory:

$$K_0(C_0(T^*V)) \sim K_0(C^*(G_V))\,. \tag{12.1}$$

Since the K-theory of $\mathcal{K}$ is $\mathbb{Z}$ for K_0, one gets the analytic index by the evaluation map

$$C^*(G_V) \to \mathcal{K}\,, \quad K_0(C^*(G_V)) \to K_0(\mathcal{K}) = \mathbb{Z}$$

composed with the isomorphism (12.1). Using the Thom isomorphism yields a geometric proof (*cf.* [56]) of the Atiyah–Singer index theorem, where all the analyses have been taken care of once and for all by the functor $G \to C^*(G)$.

This paradigm for a geometric set-up of the index theorem has been successfully extended to manifolds with singularities (*cf.* [169] [170] and references therein) and to manifolds with boundary [1].

13. Riemannian Manifolds, Conical Singularities

A main property of the homotopy type of a compact oriented manifold is that it satisfies Poincaré duality not just in ordinary homology but also in K-homology. In fact, while Poincaré duality in ordinary homology is not sufficient to describe the homotopy type of manifolds (*cf.* [168]), Sullivan proved (*cf.* [203]) that for simply connected PL manifolds of dimension at least 5, ignoring 2-torsion, the same property in KO-homology suffices and the Chern character of the KO-homology fundamental class carries all the rational information on the Pontrjagin classes.

For an ordinary manifold the choice of the fundamental cycle in K-homology is a refinement of the choice of orientation of the manifold and, in its simplest form, it is a choice of Spin-structure. Of course the role of a Spin structure is to allow for the construction of the corresponding Dirac operator which gives a corresponding Fredholm representation of the algebra of smooth functions. The choice of a square root involved in the Dirac operator D corresponds to a choice of K-orientation.

K-homology admits a fairly simple definition in terms of Hilbert spaces and Fredholm representations of algebras. In fact, we have the following notion of Fredholm module (*cf.* [56]):

Definition 13.1. *Let $\mathcal{A}$ be an algebra. An odd Fredholm module over $\mathcal{A}$ is given by*:

(1) *a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$.*
(2) *an operator $F = F^*$, $F^2 = 1$, on $\mathcal{H}$ such that*

$$[F, a] \text{ is a compact operator for any } a \in \mathcal{A}\,.$$

An even Fredholm module is given by an odd Fredholm module $(\mathcal{H}, F)$ together with a $\mathbb{Z}/2$ grading γ, $\gamma = \gamma^*$, $\gamma^2 = 1$ of the Hilbert space $\mathcal{H}$ satisfying:

(1) $\gamma a = a\gamma$, *for all* $a \in \mathcal{A}$
(2) $\gamma F = -F\gamma$.

This definition is derived from Atiyah's definition [12] of abstract elliptic operators, and agrees with Kasparov's definition [130] for the cycles in K-homology, $KK(A, \mathbb{C})$, when A is a C^*-algebra.

The notion of Fredholm module can be illustrated by the following examples (*cf.* [56]).

Example 13.2. If X is a manifold, an elliptic operator on X can be twisted with vector bundles, so as to give rise to an index map $\mathrm{Ind} : K_0(C(X)) \to \mathbb{Z}$.

If P is an elliptic operator (the symbol is invertible) and a pseudodifferential operator of order zero, $P : L^2(X, E_+) \to L^2(X, E_-)$, then there exists a parametrix Q for P. This is also an operator of order zero, and a quasi-inverse for P, in the sense that it is an inverse at the symbol level, namely $PQ - I$ and $QP - I$ are compact operators. Consider then the operator

$$F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$$

on $\mathcal{H} = L^2(X, E_+) \oplus L^2(X, E_-)$. The algebra $C(X)$ acts on $\mathcal{H}$ and $[F, f]$ is a compact operator for all $f \in C(X)$. Since $F^2 - I$ is compact, it is possible to add to $\mathcal{H}$ a finite dimensional space to obtain $F^2 = I$. Notice that the functions of $C(X)$ act differently on this modified space. In particular the function $f \equiv 1$ no longer acts as the identity: one recovers the index of P in this way.

Example 13.3. Let $\Gamma = \mathbb{Z} * \mathbb{Z}$, a free group, and let $\mathcal{A} = \mathbb{C}\Gamma$. Let $\mathcal{T}$ be the tree of Γ with $\mathcal{T}^0$ the set of vertices and $\mathcal{T}^1$ the set of edges. Fix an origin x_0 in $\mathcal{T}^0$. For any vertex $v \in \mathcal{T}^0$ there exists a unique path connecting it to the origin x_0. This defines a bijection $\phi : \mathcal{T}^0 \backslash \{x_0\} \to \mathcal{T}^1$ that assigns $v \mapsto \phi(v)$ with $\phi(v)$ this unique edge. Let U_ϕ be the unitary operator implementing ϕ, and consider the operator

$$F = \begin{pmatrix} 0 & U_\phi \\ U_\phi^* & 0 \end{pmatrix}$$

acting on $\mathcal{H} = \ell^2(\mathcal{T}^0) \oplus \ell^2(\mathcal{T}^1) \oplus \mathbb{C}$. By construction, Γ acts naturally on $\mathcal{T}_j$ which gives a corresponding action of $\mathcal{A}$ in $\mathcal{H}$. The pair $(\mathcal{H}, F)$ is a Fredholm module over $\mathcal{A}$.

Example 13.4. On $S^1 \simeq \mathbb{P}^1(\mathbb{R})$, consider the algebra of functions $\mathrm{C}(\mathbb{P}^1(\mathbb{R}))$, acting on the Hilbert space $L^2(\mathbb{R})$, as multiplication operators $(f\,\xi)(s) = f(s)\xi(s)$. Let F be the Hilbert transform

$$(F\,\xi)(s) = \frac{1}{\pi i} \int \frac{f(t)}{s - t}\, dt\,.$$

This multiplies by $+1$ the positive Fourier modes and by -1 the negative Fourier modes. A function $f \in \mathrm{C}(\mathbb{P}^1(\mathbb{R}))$ has the property that $[F, f]$ is of finite rank if and only if f is a rational function $f(s) = P(s)/Q(s)$. This is Kronecker's characterization of rational functions.

Besides the K-homology class, specified by a Fredholm module, one also wants to generalize to the noncommutative setting the infinitesimal line element ds of a Riemannian manifold. In ordinary Riemannian geometry, one deals rather with the ds^2 given by the usual local expression $g_{\mu\nu}\, dx^\mu\, dx^\nu$.

However, in order to extend the notion of metric space to the noncommutative setting it is more natural to deal with ds, for which the ansatz is

$$ds = \times\!\!-\!\!\!-\!\!\times\,, \tag{13.1}$$

where the right-hand side has the meaning usually attributed to it in physics, namely the fermion propagator

$$\times\!\!-\!\!\!-\!\!\times = D^{-1}, \tag{13.2}$$

where D is the Dirac operator. In other words, the presence of a Spin (or Spin^c) structure makes it possible to extract the square root of ds^2, using the Dirac operator as a differential square root of a Laplacian.

This prescription recovers the usual geodesic distance on a Riemannian manifold, by the following result (*cf.* [61]).

Lemma 13.5. *On a Riemannian Spin manifold the geodesic distance $d(x, y)$ between two points is computed by the formula*

$$d(x, y) = \sup\{|f(x) - f(y)|\,;\ f \in \mathcal{A},\ \|[D, f]\| \leq 1\}\,, \tag{13.3}$$

where D is the Dirac operator, $D = ds^{-1}$, and $\mathcal{A}$ is the algebra of smooth functions.

This essentially follows from the fact that the quantity $\|[D, f]\|$ can be identified with the Lipschitz norm of the function f,

$$\|[D, f]\| = \operatorname{ess\,sup}_{x \in M} \|(\nabla f)_x\| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}\,.$$

Notice that, if ds has the dimension of a length L, then D has dimension L^{-1} and the above expression for $d(x, y)$ also has the dimension of a length. On a Riemannian Spin manifold X, the condition $\|[D, f]\| \leq 1$, for D the Dirac operator, is equivalent to the condition that f is a Lipschitz function with Lipschitz constant $c \leq 1$.

The advantage of the definition (13.1), (13.2) of the line element is that it is of a *spectral* and operator theoretic nature, hence it extends to the noncommutative setting.

The structure that combines the K-homology fundamental cycle with the spectral definition of the line element ds is the notion of *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ (*cf.* [65], [84]).

Definition 13.6. *A* (*compact*) *noncommutative geometry is a triple*

$$(\mathcal{A}, \mathcal{H}, D) \tag{13.4}$$

where $\mathcal{A}$ is a unital algebra represented concretely as an algebra of bounded operators on the Hilbert space $\mathcal{H}$. The unbounded operator D is the inverse of the line element

$$ds = 1/D\,. \tag{13.5}$$

Such a triple $(\mathcal{A}, \mathcal{H}, D)$ *is required to satisfy the properties*:

(1) $[D, a]$ *is bounded for any* $a \in A^\infty$, *a dense subalgebra of the* C^*-*algebra* $A = \bar{\mathcal{A}}$.

(2) $D = D^*$ *and* $(D + \lambda)^{-1}$ *is a compact operator, for all* $\lambda \notin \mathbb{R}$.

We say that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ *is even if the Hilbert space* $\mathcal{H}$ *has a* $\mathbb{Z}/2$-*grading by an operator* γ *satisfying*

$$\gamma = \gamma^*, \quad \gamma^2 = 1, \quad \gamma D = -D\gamma, \quad \gamma a = a\gamma \ \ \forall a \in A. \tag{13.6}$$

This definition is entirely spectral. The elements of the algebra (in general noncommutative) are operators and the line element is also an operator. The polar decomposition $D = |D|F$ recovers the Fredholm module F defining the fundamental class in K-homology. The formula for the geodesic distance extends to this context as follows.

Definition 13.7. *Let* $\varphi_i : A \to \mathbb{C}$, *for* $i = 1, 2$, *be states on* A, *i.e. normalized positive linear functionals on* A *with* $\varphi_i(1) = 1$ *and* $\varphi_i(a^* a) \geq 0$ *for all* $a \in A$. *Then the distance between them is given by the formula*

$$d(\varphi_1, \varphi_2) = \sup\{|\varphi_1(a) - \varphi_2(a)|; \ a \in A, \ \|[D, a]\| \leq 1\}. \tag{13.7}$$

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is of *metric dimension* p, or p-summable, if $|D|^{-1}$ is an infinitesimal of order $1/p$ (*i.e.* $|D|^{-p}$ is an infinitesimal of order one). Here $p < \infty$ is a positive real number. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is θ-summable if $\mathrm{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$. The latter case corresponds to an infinite dimensional geometry.

Spectral triples also provide a more refined notion of dimension besides the metric dimension (summability). It is given by the *dimension spectrum*, which is not a number but a subset of the complex plane.

More precisely, let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple satisfying the regularity hypothesis

$$a \text{ and } \ [D, a] \ \in \ \cap_k \mathrm{Dom}(\delta^k), \ \forall a \in A^\infty, \tag{13.8}$$

where δ is the derivation $\delta(T) = [|D|, T]$, for any operator T. Let $\mathcal{B}$ denote the algebra generated by $\delta^k(a)$ and $\delta^k([D, a])$. The dimension spectrum of the triple $(\mathcal{A}, \mathcal{H}, D)$ is the subset $\Sigma \subset \mathbb{C}$ consisting of all the singularities of the analytic functions $\zeta_b(z)$ obtained by continuation of

$$\zeta_b(z) = \mathrm{Tr}(b|D|^{-z}), \quad \Re(z) > p, \quad b \in \mathcal{B}. \tag{13.9}$$

Example 13.8. Let M be a smooth compact Riemannian Spin manifold, and $(\mathcal{A}, \mathcal{H}, D)$ be the corresponding spectral triple given by the algebra of smooth functions, the space of spinors, and the Dirac operator. Then the metric dimension agrees with the usual dimension n of M. The dimension spectrum of M is the set $\{0, 1, \ldots, n\}$, where $n = \dim M$, and it is simple. (Multiplicities appear for singular manifolds.)

It is interesting in the case of an ordinary Riemannian manifold M to see the meaning of the points in the dimension spectrum that are smaller than $n = \dim M$. These are dimensions in which the space "manifests itself nontrivially" with some interesting geometry.

For instance, at the point $n = \dim M$ of the dimension spectrum one can recover the volume form of the Riemannian metric by the equality (valid up to a normalization constant, *cf.* [56])

$$⨍ f\, |ds|^n = \int_{M_n} f \sqrt{g}\; d^n x \,, \tag{13.10}$$

where the integral $⨍ T$ is given (*cf.* [56]) by the Dixmier trace (*cf.* [102]) generalizing the Wodzicki residue of pseudodifferential operators (*cf.* [212]).

One can also consider integration $⨍ ds^k$ in any other dimension in the dimension spectrum, with $ds = D^{-1}$ the line element. In the case of a Riemannian manifold one finds other important curvature expressions. For instance, if M is a manifold of dimension $\dim M = 4$, when one considers integration in dimension 2 one finds the Einstein–Hilbert action. In fact, a direct computation yields the following result (*cf.* [131] [129]):

Proposition 13.9. *Let $dv = \sqrt{g}\; d^4x$ denote the volume form, $ds = D^{-1}$ the length element, i.e. the inverse of the Dirac operator, and r the scalar curvature. We obtain*:

$$⨍ ds^2 = \frac{-1}{48\pi^2} \int_{M_4} r \; dv \,. \tag{13.11}$$

In general, one obtains the scalar curvature of an n-dimensional manifold from the integral $⨍ ds^{n-2}$.

Many interesting examples of spectral triples just satisfy the conditions stated in Definition 13.6. However, there are significant cases where more refined properties of manifolds carry over to the noncommutative case, such as the presence of a real structure (which makes it possible to distinguish between K-homology and KO-homology) and the "order one condition" for the Dirac operator. These properties are described as follows (*cf.* [63] and [64]).

Definition 13.10. *A real structure on an n-dimensional spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$, with the property that*

$$J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad \textit{and} \quad J\gamma = \varepsilon'' \gamma J \ (\textit{even case})\,. \tag{13.12}$$

The numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ are functions of $n \bmod 8$ given by

n	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1	−1	1	−1				

Moreover, the action of $\mathcal{A}$ satisfies the commutation rule

$$[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}\,, \tag{13.13}$$

where

$$b^0 = Jb^* J^{-1} \qquad \forall\, b \in \mathcal{A}\,, \tag{13.14}$$

and the operator D satisfies

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}\,. \tag{13.15}$$

The anti-linear isometry J is given, in ordinary Riemannian geometry, by the charge conjugation operator acting on spinors. In the noncommutative case, this is replaced by the Tomita antilinear conjugation operator (*cf.* [205]).

In [63] and [109] Theorem 11.2, necessary and sufficient conditions are given that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ (with real structure J) should fulfill in order to come from an ordinary compact Riemannian Spin manifold:

(1) $ds = D^{-1}$ is an infinitesimal of order $1/n$.
(2) There is a real structure in the sense of Definition 13.10.
(3) The commutation relation (13.15) holds (this is $[[D, a], b] = 0$, for all $a, b \in \mathcal{A}$, when $\mathcal{A}$ is commutative).
(4) The regularity hypothesis of (13.8) holds: a and $[D, a]$ are in $\cap_k \mathrm{Dom}(\delta^k)$ for all $a \in \mathcal{A}_\infty$.
(5) There exists a Hochschild cycle $c \in Z_n(\mathcal{A}_\infty, \mathcal{A}_\infty)$ such that its representation $\pi(c)$ on $\mathcal{H}$ induced by

$$\pi(a^0 \otimes \cdots \otimes a^n) = a^0[D, a^1] \cdots [D, a^n]$$

satisfies $\pi(c) = \gamma$, for γ as in (13.6), in the even case, and $\pi(c) = 1$ in the odd case.
(6) The space $\mathcal{H}^\infty = \cap_k \mathrm{Dom}(D^k)$ is a finite projective $\mathcal{A}$-module, endowed with an $\mathcal{A}$-valued inner product $\langle \xi, \eta \rangle_{\mathcal{A}}$ defined by

$$\langle a\xi, \eta \rangle = \int\!\!\!\!\!\!- a\, \langle \xi, \eta \rangle_{\mathcal{A}}\, ds^n.$$

(7) The intersection form

$$K_*(\mathcal{A}) \times K_*(\mathcal{A}) \to \mathbb{Z} \tag{13.16}$$

obtained from the Fredholm index of D with coefficients in $K_*(\mathcal{A} \otimes \mathcal{A}^0)$ is invertible.

When $\mathcal{A} = \mathcal{C}^\infty(M)$ is given, the above conditions characterize the Dirac operators associated to a pair of a Riemannian structure and a Spin structure on M (we refer to [109] for the precise statement). The hope was formulated in [63] that, in the commutative case, one could drop the hypothesis $\mathcal{A} = \mathcal{C}^\infty(M)$ and use the orientation condition to construct an embedding of the spectrum of the algebra $\mathcal{A}$ as a submanifold of $\mathbb{R}^N$. Recent work by Rennie and Varilly [184] gives promising results in this direction. Moreover, the conditions can be stated without any commutativity assumption on $\mathcal{A}$. They are satisfied, for instance, by the isospectral deformations of [75], which we discuss in Section 18. Another very significant noncommutative example is the Standard Model of elementary particles (*cf.* [63]), which we discuss in Section 17.

Another example of spectral triple associated to a classical space, which is not classically a smooth manifold, is the case of manifolds with singularities. In particular, one can consider the case of an isolated conical singularity. This case was studied by Lescure [148].

Let X be a manifold with an isolated conical singularity. The cone point $c \in X$ has the property that there is a neighborhood U of c in X such that $U \smallsetminus \{c\}$ is of the form $(0,1] \times N$, with N a smooth compact manifold, and metric $g|_U = dr^2 + r^2 g_N$, where g_N is the metric on N.

A natural class of differential operators on manifolds with isolated conical singularities is given by the elliptic operators of Fuchs type, acting on sections of a bundle E. These are operators whose restriction over $(0,1] \times N$ takes the form

$$r^{-\nu} \sum_{k=0}^{d} a_k(r)(-r\partial_r)^k \,,$$

for $\nu \in \mathbb{R}$ and $a_k \in C^\infty([0,1], \mathrm{Diff}^{d-k}(N, E|_N))$, which are elliptic with symbol $\sigma_M(D) = \sum_{k=0}^d a_k(0) z^k$ that is an elliptic family parameterized by $\Im(z)$. In particular, operators of Dirac type are elliptic of Fuchs type. For such an operator D, which is of first order and symmetric, results of Chou [49], Brüning, Seeley [33] and Lesch [147] show that its self-adjoint extension has discrete spectrum, with $(n+1)$-summable resolvent, for $\dim X = n$.

The algebra that is used in the construction of the spectral triple is $\mathcal{A} = C_c^\infty(X) \oplus \mathbb{C}$, the algebra of functions that are smooth on $X \smallsetminus \{c\}$ and constant near the singularity. The Hilbert space on which D acts is chosen from a family of weighted Sobolev spaces. Roughly, one defines weighted Sobolev spaces that look like the standard Sobolev space on the smooth part and on the cone are defined locally by norms

$$\|f\|^2_{s,\gamma} = \int_{\mathbb{R}_+^* \times \mathbb{R}^{m-1}} \left(1 + (\log t)^2 + \xi^2\right)^s \left| (\widehat{r^{-\gamma+1/2} f})(t,\xi) \right|^2 \frac{dt}{t} d\xi \,,$$

where $\hat{f}$ denotes Fourier transform on the group $\mathbb{R}_+^* \times \mathbb{R}^{m-1}$.

Then one obtains the following result (*cf.* [148]).

Theorem 13.11. *The data* $(\mathcal{A}, \mathcal{H}, D)$ *given above define a spectral triple. In particular, the zeta functions* $\mathrm{Tr}(a|D|^{-z})$ *admit analytic continuation to* $\mathbb{C} \smallsetminus \Sigma$, *where the dimension spectrum is of the form*

$$\Sigma = \{\dim X - k, k \in \mathbb{N}\},$$

with multiplicities ≤ 2.

The analysis of the zeta functions uses the heat kernel

$$\mathrm{Tr}(|D|^{-z}) = \frac{1}{\Gamma(z/2)} \int_0^\infty t^{z/2-1} \mathrm{Tr}(e^{-tD^2}) dt,$$

for which one can rely on the results of [49] and [147]. The case of $\mathrm{Tr}(a|D|^{-z})$ of the form $\mathrm{Tr}(Q|D|^{-z})$, with $Q \in \Psi_c^\ell(E)$, is treated by splitting $Q|D|^{-z}$ as a sum of a contribution from the smooth part and one from the singularity.

The Chern character for this spectral triple gives a map

$$\mathrm{Ch} : K_*(X) \to H_*(X, \mathbb{C}),$$

where we have $K^*(X) \cong K_*(\mathcal{A})$ and $H_*(X, \mathbb{C}) \cong PHC^*(\mathcal{A})$, the periodic cyclic homology of the algebra $\mathcal{A}$.

The cocycles φ_n in the (b, B)-bicomplex for the algebra $\mathcal{A}$ have also been computed explicitly and are of the form

$$\varphi_n(a_0, \ldots, a_n) = \nu_n \int_X a_0 da_1 \wedge \cdots \wedge da_n \wedge \hat{A}(X) \wedge \mathrm{Ch}(E),$$

for $n \geq 1$, while for $n = 0$, $\lambda \in \mathbb{C}$, $\varphi_0(a + \lambda) = \int_X a\hat{A}(X) \wedge \mathrm{Ch}(E) + \lambda \mathrm{Ind}(D_+)$.

14. Cantor Sets and Fractals

An important class of C^*-algebras are those obtained as direct limits of a sequence of finite dimensional subalgebras and embeddings. These are called *approximately finite dimensional*, or simply AF-algebras.

An AF-algebra $\mathcal{A}$ is determined by a diagram of finite dimensional algebras and inclusions, its Bratteli diagram [31], and from the diagram itself it is possible to read off a lot of the structure of the algebra, for instance its ideal structure. Some simple examples of algebras that belong to this class are:

Example 14.1. An example of a commutative AF-algebra is the algebra of complex valued continuous functions on a Cantor set, where a Bratteli diagram is determined by a decreasing family of disjoint intervals covering the Cantor set.

A noncommutative example of AF-algebra is given by the algebra of the canonical anticommutation relations of quantum mechanics.

Example 14.2. Consider a real Hilbert space $\mathcal{E}$ and a linear map $\mathcal{E} \to \mathcal{B}(\mathcal{H})$, $f \mapsto T_f$, to bounded operators in a Hilbert space $\mathcal{H}$, satisfying

$$T_f T_g + T_g T_f = 0$$

$$T_f^* T_g + T_g T_f^* = \langle g, f \rangle I \,,$$

and the algebra $\mathcal{A}$ generated by all the operators T_f satisfying these relations.

A survey with many examples of AF-algebras and their properties is given for instance in [101].

Let $\mathcal{A}$ be a commutative AF C^*-algebra. A commutative AF-algebra $\mathcal{A}$ is spanned by its projections, since finite dimensional commutative algebras are generated by orthogonal projections. This condition is equivalent to the spectrum $\Lambda = \mathrm{Spec}(\mathcal{A})$ of the algebra being a totally disconnected compact Hausdorff space, typically a Cantor set. Realizing such Cantor set as the intersection of a decreasing family of disjoint intervals covering Λ also provides a Bratteli diagram for the AF-algebra $\mathcal{A} = C(\Lambda)$.

As described in [56], in order to construct the Hilbert space $\mathcal{H}$ for a Cantor set $\Lambda \subset \mathbb{R}$, let J_k be the collection of bounded open intervals in $\mathbb{R} \smallsetminus \Lambda$. We denote by $L = \{\ell_k\}_{k \geq 1}$ the countable collection of lengths of the intervals J_k. We can assume that the lengths are ordered

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k \cdots > 0 \,. \tag{14.1}$$

We also denote by $E = \{x_{k,\pm}\}$ the set of endpoints of the intervals J_k, with $x_{k,+} > x_{k,-}$. Consider the Hilbert space

$$\mathcal{H} := \ell^2(E) \tag{14.2}$$

Since the endpoints of the J_k are points of Λ, there is an action of $C(\Lambda)$ on $\mathcal{H}$ given by

$$f \cdot \xi(x) = f(x)\xi(x), \quad \forall\, f \in C(\Lambda),\ \forall\, \xi \in \mathcal{H},\ \forall\, x \in E \,. \tag{14.3}$$

A sign operator F can be obtained (*cf.* [56]) by choosing the closed subspace $\hat{\mathcal{H}} \subset \mathcal{H}$ given by

$$\hat{\mathcal{H}} = \{\xi \in \mathcal{H} : \xi(x_{k,-}) = \xi(x_{k,+}),\ \forall\, k\} \,. \tag{14.4}$$

Then F has eigenspaces $\hat{\mathcal{H}}$ with eigenvalue $+1$ and $\hat{\mathcal{H}}^\perp$ with eigenvalue -1, so that, when restricted to the subspace $\mathcal{H}_k$ of coordinates $\xi(x_{k,+})$ and $\xi(x_{k,-})$, the sign F is given by

$$F|_{\mathcal{H}_k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Finally, a Dirac operator $D = |D|F$ is obtained as

$$D|_{\mathcal{H}_k} \begin{pmatrix} \xi(x_{k,+}) \\ \xi(x_{k,-}) \end{pmatrix} = \ell_k^{-1} \cdot \begin{pmatrix} \xi(x_{k,-}) \\ \xi(x_{k,+}) \end{pmatrix}. \tag{14.5}$$

We then obtain the following result.

Proposition 14.3. *Let $\Lambda \subset \mathbb{R}$ be a Cantor set. Let $\mathcal{A}_\infty \subset C(\Lambda)$ be the dense subalgebra of locally constant functions on the Cantor set. Then the data $(\mathcal{A}, \mathcal{H}, D)$ form a spectral triple, with $\mathcal{H}$ as in (14.2), the action (14.3), and D as in (14.5). The zeta function satisfies*

$$\mathrm{Tr}(|D|^{-s}) = 2\zeta_L(s)\,,$$

where $\zeta_L(s)$ is the geometric zeta function of $L = \{\ell_k\}_{k\geq 1}$, defined as

$$\zeta_L(s) := \sum_k \ell_k^s\,. \tag{14.6}$$

These zeta functions are related to the theory of Dirichlet series and to other arithmetic zeta functions, and also to Ruelle's dynamical zeta functions (*cf.* [144]).

For example, for the classical middle-third Cantor set, we have set of lengths $\ell_k = 3^{-k}$ and multiplicities $m_k = 2^{k-1}$, for $k \geq 1$, so that we obtain

$$\mathrm{Tr}(|D|^{-s}) = 2\zeta_L(s) = \sum_{k\geq 1} 2^k 3^{-sk} = \frac{2 \cdot 3^{-s}}{1 - 2\cdot 3^{-s}}\,. \tag{14.7}$$

This shows that the dimension spectrum of the spectral triple of a Cantor set has points off the real line. In fact, the set of poles of (14.7) is

$$\left\{ \frac{\log 2}{\log 3} + \frac{2\pi i n}{\log 3} \right\}_{n\in\mathbb{Z}}\,. \tag{14.8}$$

In this case the dimension spectrum lies on a vertical line and it intersects the real axis in the point $D = \frac{\log 2}{\log 3}$ which is the Hausdorff dimension of the ternary Cantor set. The same is true for other Cantor sets, as long as the self-similarity is given by a unique contraction (in the ternary case the original interval is replaced by two intervals of lengths scaled by 1/3).

If one considers slightly more complicated fractals in $\mathbb{R}$, where the self-similarity requires more than one scaling map, the dimension spectrum may be correspondingly more complicated. This can be seen in the case of the Fibonacci Cantor set, for instance (*cf.* [144]).

The Fibonacci Cantor set Λ is obtained from the interval $I = [0, 4]$ by successively removing F_{n+1} open intervals $J_{n,j}$ of lengths $\ell_n = 1/2^n$ according to the rule of Figure 13. We can associate to this Cantor set the commutative AF-algebra $\mathcal{A} = C(\Lambda)$.

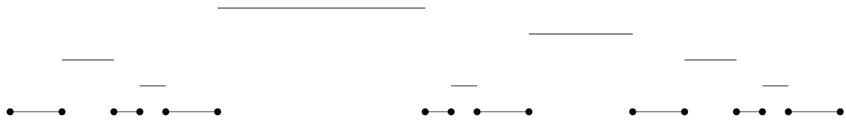


Figure 13. The Fibonacci Cantor set.

To obtain the Hilbert space we consider again the set E of endpoints $x_{n,j,\pm}$ of the intervals $J_{n,j}$ and we take $\mathcal{H} = \ell^2(E)$. We define the Dirac operator as in the previous case, and we again consider the dense involutive subalgebra $\mathcal{A}_\infty$ of locally constant functions.

The data $(\mathcal{A}, \mathcal{H}, D)$ give a spectral triple. The Dirac operator is related to the geometric zeta function of the Fibonacci Cantor set by

$$\operatorname{Tr}(|D|^{-s}) = 2\zeta_F(s) = \frac{2}{1 - 2^{-s} - 4^{-s}}\,,$$

where the geometric zeta function is $\zeta_F(s) = \sum_n F_{n+1} 2^{-ns}$, with F_n the Fibonacci numbers.

A simple argument shows that the dimension spectrum is given by the set

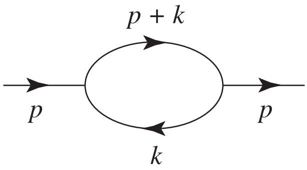
$$\Sigma = \left\{ \frac{\log\phi}{\log 2} + \frac{2\pi i n}{\log 2} \right\}_{n\in\mathbb{Z}} \cup \left\{ -\frac{\log\phi}{\log 2} + \frac{2\pi i(n+1/2)}{\log 2} \right\}_{n\in\mathbb{Z}}\,,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Recent results on the noncommutative geometry of fractals and Cantor sets and spectral triple constructions for AF-algebras can be found in [6], [115], [116]. The construction in [6] is in fact a spectral triple for the dual group of the Cantor set seen as the product of countably many copies of the group $\mathbb{Z}/2$. The recent work [50] shows that it is easy to describe a compact metric space exactly (*i.e.* recovering the metric) via a spectral triple, which is a sum of two-dimensional modules, but spectral triples carry much more information than just that regarding the metric. The construction of [51] provides a natural spectral triple for the Sierpiński gasket, which is of topological dimension 1.

15. Spaces of Dimension z and DimReg

In perturbative quantum field theory, one computes expectation values of observables via a formal series, where the terms are parameterized by Feynman graphs and reduce to ordinary finite dimensional integrals in momentum space of expressions assigned to the graphs by the Feynman rules. These expressions typically produce divergent integrals. For example, in the example of the scalar ϕ^3 theory in dimension $D = 4$ or $D = 4 + 2\mathbb{N}$, one encounters a divergence already in the simplest one-loop diagram, with corresponding integral (in Euclidean signature)



$$= \int \frac{1}{k^2 + m^2} \frac{1}{((p+k)^2 + m^2)} d^D k \,. \tag{15.1}$$

One therefore needs a regularization procedure for these divergent integrals. The regularization most commonly adopted in quantum field theory computation is "Dimensional Regularization (DimReg) and Minimal Subtraction (MS)". The method was introduced in the '70s in [23] and [125] and it has the advantage of preserving basic symmetries.

The regularization procedure of DimReg is essentially based on the use of the formula

$$\int e^{-\lambda k^2} \, d^d k \;=\; \pi^{d/2} \, \lambda^{-d/2} \,, \tag{15.2}$$

to *define* the meaning of the integral in $d = (D - z)$ dimensions, for $z \in \mathbb{C}$ in a neighborhood of zero. For instance, in the case of (15.1), the procedure of dimensional regularization yields the result

$$\pi^{(D-z)/2} \Gamma\left(\frac{4 - D + z}{2}\right) \int_0^1 \left((x - x^2) p^2 + m^2\right)^{\frac{D-z-4}{2}} dx \,.$$

In the recent survey [151], Yuri Manin refers to DimReg as "dimensions in search of a space"[a]. Indeed, in the usual approach in perturbative quantum field theory, the dimensional regularization procedure is just regarded as a formal rule of analytic continuation of formal (divergent) expressions in integral dimensions D to complex values of the variable D.

However, using noncommutative geometry, it is possible to construct actual spaces (in the sense of noncommutative Riemannian geometry) X_z whose dimension (in the sense of dimension spectrum) is a point $z \in \mathbb{C}$ (*cf.* [79]).

It is well known in the physics literature that there are problems related to using dimensionl regularization in chiral theory, which involves giving a consistent prescription on how to extend the γ_5 (the product of the matrices γ^i when $D = 4$) to noninteger dimension $D - z$. It turns out that a prescription known as Breitenlohner-Maison ([32], [52]) admits an interpretation in terms of the cup product of spectral triples, where one takes the product of the spectral triple associated to the ordinary geometry in the integer dimension D by a spectral triple X_z whose dimension spectrum is reduced to the complex number z (*cf.* [79]).

[a]Nicely reminiscent of Pirandello's play "Six characters in search of an author".

We illustrate here the construction for the case where $z \in \mathbb{R}_+^*$. The more general case of $z \in \mathbb{C}$ is more delicate.

One needs to work in a slightly modified setting for spectral triples, which is given by the type II spectral triples (*cf.* [21], [38], [39]). In this setting the usual type I trace of operators in $\mathcal{L}(\mathcal{H})$ is replaced by the trace on a type II_∞ von Neumann algebra.

One considers a self-adjoint operator Y, affiliated to a type II_∞ factor N, with spectral measure given by

$$\mathrm{Tr}_N(\chi_E(Y)) = \frac{1}{2}\int_E dy \tag{15.3}$$

for any interval $E \subset \mathbb{R}$, with characteristic function χ_E.

If $Y = F\,|Y|$ is the polar decomposition of Y, one sets

$$D_z = \rho(z)\,F\,|Y|^{1/z} \tag{15.4}$$

with the complex power $|Y|^{1/z}$ defined by the functional calculus. The normalization constant $\rho(z)$ is chosen to be

$$\rho(z) = \pi^{-\frac{1}{2}}\left(\Gamma\left(\frac{z}{2}+1\right)\right)^{\frac{1}{z}} \tag{15.5}$$

so that one obtains

$$\mathrm{Tr}\left(e^{-\lambda D^2}\right) = \pi^{z/2}\,\lambda^{-z/2} \quad \forall\,\lambda \in \mathbb{R}_+^*\,. \tag{15.6}$$

This gives a geometric meaning to the basic formula (15.2) of DimReg. The algebra $\mathcal{A}$ of the spectral triple X_z can be made to contain any operator a such that $[D_z, a]$ is bounded and both a and $[D_z, a]$ are smooth for the "geodesic flow"

$$T \mapsto e^{it|D_z|}\,T\,e^{-it|D_z|}\,. \tag{15.7}$$

The dimension spectrum of X_z is reduced to the single point z, since

$$\mathrm{Tr}'_N((D_z^2)^{-s/2}) = \rho^{-s}\int_1^\infty u^{-s/z}\,du = \rho^{-s}\,\frac{z}{s-z} \tag{15.8}$$

has a single (simple) pole at $s = z$ and is absolutely convergent in the half space $\mathrm{Re}(s/z) > 1$. Here Tr'_N denotes the trace with an infrared cutoff (*i.e.* integrating outside $|y| < 1$).

16. Local Algebras in Supersymmetric QFT

It is quite striking that the general framework of noncommutative geometry is suitable not only for handling finite dimensional spaces (commutative or not, of non-integer dimension etc.) but is also compatible with infinite dimensional spaces. We already saw in Section 7 that discrete groups of exponential growth naturally give rise to noncommutative spaces which are described by a θ-summable spectral triple, but not by a finitely summable spectral triple. This is characteristic of an infinite dimensional space and in that case, as we saw for discrete groups, cyclic cohomology needs to be extended to *entire* cyclic cohomology. A very similar kind of noncommutative space arises from Quantum Field Theory in the supersymmetric context [56] Section IV.9.β. We briefly recall this below and then explain open questions also in the context of supersymmetric theories.

The simplest example to understand the framework is that of the free Wess–Zumino model in two dimensions, a supersymmetric free field theory in a two-dimensional space-time where space is compact ([56]). Thus space is a circle S^1 and space-time is a cylinder $C = S^1 \times \mathbb{R}$ endowed with the Lorentzian metric. The fields are given by a complex scalar bosonic field ϕ of mass m and a spinor field ψ of the same mass. The Lagrangian of the theory is of the form $\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f$ where,

$$\mathcal{L}_b = \frac{1}{2}(|\partial_0\phi|^2 - |\partial_1\phi|^2 - m^2|\phi|^2)$$

and for the fermions,

$$\mathcal{L}_f = i\,\bar{\psi}\,\gamma^\mu\,\partial_\mu\,\psi - m\,\bar{\psi}\,\psi$$

where the spinor field is given by a column matrix, with $\bar{\psi} = \gamma^0\,\psi^*$ and the γ^μ are 2×2 Pauli matrices, anticommuting, self-adjoint and of square 1.

The Hilbert space of the quantum theory is the tensor product $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ of the bosonic one $\mathcal{H}_b$ by the fermionic one $\mathcal{H}_f$. The quantum field $\phi(x)$ and its conjugate momentum $\pi(x)$ are operator-valued distributions in $\mathcal{H}_b$ and the bosonic Hamiltonian is of the form

$$H_b = \int_{S^1} \,:\, |\pi(x)|^2 + |\partial_1\phi(x)|^2 + m^2|\phi(x)|^2 \,:\, dx$$

where the Wick ordering takes care of an irrelevant additive constant. The fermionic Hilbert space $\mathcal{H}_f$ is given by the Dirac sea representation which simply corresponds to a suitable spin representation of the infinite dimensional Clifford algebra containing the fermionic quantum fields $\psi_j(x)$. The fermionic Hamiltonian is then the positive operator in $\mathcal{H}_f$ given by

$$H_f = \int_{S^1} \,:\, \bar{\psi}\,\gamma^1\,i\,\partial\,\psi - m\,\bar{\psi}\,\psi\,.$$

The full Hamiltonian of the non-interacting theory acts on the Hilbert space $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ and is the positive operator

$$H = H_b \otimes 1 + 1 \otimes H_f\,.$$

This is where supersymmetry enters the scene in finding a self-adjoint square root of H in the same way as the Dirac operator is a square root of the Laplacian in the case of finite dimensional manifolds. This square root, called the *supercharge* operator, is given by

$$Q = \frac{1}{\sqrt{2}} \int_{S^1} (\psi_1(x)(\pi(x) - \partial\phi^*(x) - im\phi(x)) \\ + \psi_2(x)(\pi^*(x) - \partial\phi(x) - im\phi^*(x)) + \text{h.c.})dx\,,$$

where the symbol +h.c. means that one adds the Hermitian conjugate.

The basic relation with spectral triples is then given by the following result ([56] Section IV).

Theorem 16.1. *For any local region $\mathcal{O} \subset C$ let $\mathcal{A}(\mathcal{O})$ be the algebra of functions of quantum fields with support in $\mathcal{O}$ acting in the Hilbert space $\mathcal{H}$. Then the triple*

$$(\mathcal{A}(\mathcal{O}), \mathcal{H}, Q)$$

is an even θ-summable spectral triple, with $\mathbb{Z}_2$-grading given by the operator $\gamma = (-1)^{N_f}$ counting the parity of the fermion number operator N_f.

To be more specific the algebra $\mathcal{A}(\mathcal{O})$ is generated by the imaginary exponentials $e^{i(\phi(f)+\phi(f)^*)}$ and $e^{i(\pi(f)+\pi(f)^*)}$ for $f \in C_c^\infty(\mathcal{O})$. As shown in [56] Section IV.9.β, and exactly as in the case of discrete groups with exponential growth, one needs the entire cyclic cohomology rather than its finite dimensional version in order to obtain the Chern character of θ-summable spectral triples. Indeed, the index map is non-polynomial in the above example of the Wess–Zumino model in two dimensions and the K-theory of the above local algebras is highly non-trivial. In fact it is in that framework that the JLO-cocycle was discovered by Jaffe–Lesniewski and Osterwalder [127].

It is an open problem to extend the above result to interacting theories in higher dimension and to give a full computation of the K-theory of the local algebras as well as of the Chern character in entire cyclic cohomology. The results of Jaffe and his collaborators on constructive quantum field theory yield many interacting non-trivial examples of supersymmetric two-dimensional models. Moreover the recent breakthrough of Puschnigg in the case of lattices of semi-simple Lie groups of rank one opens the way to the computation of the Chern character in entire cyclic cohomology.

17. Spacetime and the Standard Model of Elementary Particles

The Standard Model of elementary particle physics provides a surprising example of a spectral triple in the noncommutative setting, which in addition to the conditions of Definition 13.6 also has a real structure satisfying all the additional conditions of Definition 13.10.

The noncommutative geometry of the Standard Model developed in [63] (*cf.* also [44], [45], [76], [132]) gives a concise conceptual way to describe, through a simple mathematical structure, the full complexity of the input from physics. As we recall here, the model also allows for predictions.

The physics of the Standard Model can be described by a Lagrangian. We consider here the Standard Model minimally coupled to gravity, so that the Lagrangian we shall be concerned with is the sum

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{SM} \tag{17.1}$$

of the Einstein–Hilbert Lagrangian $\mathcal{L}_{EH}$ and the standard model Lagrangian $\mathcal{L}_{SM}$.

The Standard Model Lagrangian $\mathcal{L}_{SM}$ has a very complicated expression, which, if written in full, might take a full page (*cf. e.g.* [206]). It comprises five types of terms,

$$\mathcal{L}_{SM} = \mathcal{L}_G + \mathcal{L}_{GH} + \mathcal{L}_H + \mathcal{L}_{Gf} + \mathcal{L}_{Hf}\,, \tag{17.2}$$

where the various terms involve:

- spin 1 bosons G: the eight gluons, γ, $W^{\pm}$, Z;
- spin 0 bosons H such as the Higgs fields;
- spin 1/2 fermions f: quarks and leptons.

The term $\mathcal{L}_G$ is the pure gauge boson part, $\mathcal{L}_{GH}$ for the minimal coupling with the Higgs fields, and $\mathcal{L}_H$ gives the quartic Higgs self interaction. In addition to the coupling constants for the gauge fields, the fermion kinetic term $\mathcal{L}_{Gf}$ contains the hypercharges Y_L, Y_R. These numbers, which are constant over generations, are assigned phenomenologically, so as to obtain the correct values of the electromagnetic charges. The term $\mathcal{L}_{Hf}$ contains the Yukawa coupling of the Higgs fields with fermions. A more detailed and explicit description of the various terms of (17.2) is given in [56] §VI.5.β. See also [206].

The symmetry group of the Einstein–Hilbert Lagrangian $\mathcal{L}_{EH}$ by itself would be, by the equivalence principle, the diffeomorphism group $\mathrm{Diff}(X)$ of the space-time manifold. In the Standard Model Lagrangian $\mathcal{L}_{SM}$, on the other hand, the gauge theory has another huge symmetry group which is the group of local gauge transformations. According to our current understanding of elementary particle physics, this is given by

$$G_{SM}(X) = C^{\infty}(X, U(1) \times SU(2) \times SU(3)) \tag{17.3}$$

(at least in the case of a trivial principal bundle, *e.g.* when the spacetime manifold X is contractible).

Thus, when one considers the Lagrangian $\mathcal{L}$ of (17.1), the full symmetry group G will be a semidirect product

$$G(X) = G_{SM}(X) \rtimes \mathrm{Diff}(X)\,. \tag{17.4}$$

In fact, a diffeomorphism of the manifold relabels the gauge parameters.

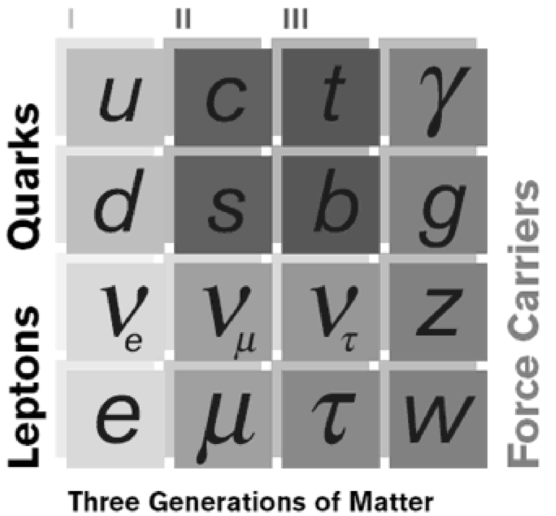


Figure 14. Elementary particles.

To achieve a *geometrization* of the Standard Model, one would like to be able to exhibit a space X for which

$$G(X) = \mathrm{Diff}(X)\,. \tag{17.5}$$

If such a space existed, then we would be able to say that the whole theory is pure gravity on X. However, it is impossible to find such a space X among ordinary manifolds. In fact, a result of W. Thurston, D. Epstein and J. Mather (*cf.* [165]) shows that the connected component of the identity in the diffeomorphism group of a (connected) manifold is a simple group (see [165] for the precise statement). A simple group cannot have a nontrivial normal subgroup, so it cannot have the structure of semi-direct product like $G(X)$ in (17.4).

However, it is possible to obtain a space with the desired properties among *noncommutative spaces.* What plays the role of the connected component of the identity in the diffeomorphism group $\mathrm{Diff}(X)$ in the noncommutative setting is the group $\mathrm{Aut}^+(\mathcal{A})$ of automorphisms of the (noncommutative) algebra that preserve the fundamental class in K-homology, *i.e.* that can be implemented by a unitary compatible with the grading and real structure.

When the algebra $\mathcal{A}$ is noncommutative, among its automorphisms there are, in particular, inner ones. They associate to an element x of the algebra the element uxu^{-1}, for some $u \in \mathcal{A}$. Of course uxu^{-1} is not, in general, equal to x because the algebra is noncommutative. The inner automorphisms form a normal subgroup of the group of automorphisms. Thus, we see that the group $\mathrm{Aut}^+(\mathcal{A})$ has in general the same type of structure as our desired group of symmetries $G(X)$, namely, it has a normal subgroup of inner automorphisms and it has a quotient. It is amusing how the physical and the mathematical vocabularies agree here: in physics one talks about internal symmetries and in mathematics one talks about inner automorphisms (one might as well call them internal automorphisms).

There is a very simple noncommutative algebra $\mathcal{A}$ whose group of inner automorphisms corresponds to the group of gauge transformations $G_{SM}(X)$, and such that the quotient $\mathrm{Aut}^+(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$ corresponds exactly to diffeomorphisms (*cf.* [192]). The noncommutative space is a product $X \times F$ of an ordinary spacetime manifold X by a "finite noncommutative space" F. The noncommutative algebra $\mathcal{A}_F$ is a direct sum of the algebras $\mathbb{C}$, $\mathbb{H}$ (here denoting the quaternions), and $M_3(\mathbb{C})$ (the algebra of 3×3 complex matrices).

The algebra $\mathcal{A}_F$ corresponds to a *finite* space where the Standard Model fermions and the Yukawa parameters (masses of fermions and mixing matrix of Cabibbo–Kobayashi–Maskawa) determine the spectral geometry in the following manner. The Hilbert space $\mathcal{H}_F$ is finite dimensional and admits the set of elementary fermions as a basis. This comprises the generations of quarks (down–up, strange–charmed, bottom–top),

$$\begin{array}{cccccccc} u_L & u_R & d_L & d_R & \bar{u}_L & \bar{u}_R & \bar{d}_L & \bar{d}_R \\ c_L & c_R & s_L & s_R & \bar{c}_L & \bar{c}_R & \bar{s}_L & \bar{s}_R \\ t_L & t_R & b_L & b_R & \bar{t}_L & \bar{t}_R & \bar{b}_L & \bar{b}_R\,, \end{array} \tag{17.6}$$

with the additional color index (y, r, b), and the generations of leptons (electron, muon, tau, and corresponding neutrinos)

$$\begin{array}{cccccc} e_L & e_R & \nu_L^e & \bar{e}_L & \bar{e}_R & \bar{\nu}_L^e \\ \mu_L & \mu_R & \nu_L^\mu & \bar{\mu}_L & \bar{\mu}_R & \bar{\nu}_L^\mu \\ \tau_L & \tau_R & \nu_L^\tau & \bar{\tau}_L & \bar{\tau}_R & \bar{\nu}_L^\tau \end{array} \tag{17.7}$$

(We discuss here only the minimal Standard Model with no right handed neutrinos.)

The $\mathbb{Z}/2$ grading γ_F on the Hilbert space $\mathcal{H}_F$ has sign $+1$ on left-handed particles (*e.g.* the u_L, d_L, etc.) and sign -1 on the right-handed particles. The involution J_F giving the real structure is the charge conjugation, namely, if we write $\mathcal{H}_F = \mathcal{E} \oplus \bar{\mathcal{E}}$, then J_F acts on the fermion basis as $J_F(f, \bar{h}) = (h, \bar{f})$. This satisfies $J_F^2 = 1$ and $J_F\gamma_F = \gamma_F J_F$, as should be the case for dimension $n = 0$.

The algebra $\mathcal{A}_F$ admits a natural representation in $\mathcal{H}_F$ (see [64]). An element $(z, q, m) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ acts as

$$(z,q,m)\cdot\begin{pmatrix} u_R \\ d_R \end{pmatrix} = \begin{pmatrix} z\, u_R \\ \bar{z}\, d_R \end{pmatrix} \qquad (z,q,m)\cdot e_R = \bar{z}\, e_R$$

$$(z,q,m)\cdot\begin{pmatrix} u_L \\ d_L \end{pmatrix} = q\begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad (z,q,m)\cdot\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix} = q\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix},$$

$$(z,q,m)\cdot\begin{pmatrix} \bar{e}_L \\ \bar{e}_R \end{pmatrix} = \begin{pmatrix} z\, \bar{e}_L \\ z\, \bar{e}_R \end{pmatrix}$$

$$(z,q,m)\cdot \bar{u}_R = m\, \bar{u}_R \qquad (z,q,m)\cdot \bar{d}_R = m\, \bar{d}_R$$

and similarly for the other generations. Here $q \in \mathbb{H}$ acts as multiplication by the matrix

$$q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$

where $q = \alpha + \beta\, j$, with $\alpha, \beta \in \mathbb{C}$. The matrix $m \in M_3(\mathbb{C})$ acts on the color indices (y, r, b).

The data $(\mathcal{A}_F, \mathcal{H}_F)$ can be completed to a spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ where the Dirac operator (in this finite dimensional case a matrix) is given by

$$D_F = \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix} \tag{17.8}$$

on $\mathcal{H}_F = \mathcal{E} \oplus \bar{\mathcal{E}}$, where Y is the Yukawa coupling matrix, which combines the masses of the elementary fermions together with the Cabibbo–Kobayashi–Maskawa (CKM) quark mixing matrix.

The fermionic fields acquire mass through the spontaneous symmetry breaking produced by the Higgs fields. The Yukawa coupling matrix takes the form $Y = Y_q \otimes 1 \oplus Y_f$, where the matrix Y_f is of the form

$$\begin{pmatrix} 0 & 0 & M_e \\ 0 & 0 & 0 \\ M_e^* & 0 & 0 \end{pmatrix},$$

in the basis (e_R, ν_L, e_L) and successive generations, while Y_q is of the form

$$\begin{pmatrix} 0 & 0 & M_u & 0 \\ 0 & 0 & 0 & M_d \\ M_u^* & 0 & 0 & 0 \\ 0 & M_d^* & 0 & 0 \end{pmatrix},$$

in the basis given by (u_R, d_R, u_L, d_L) and successive generations. In the case of the lepton masses, up to rotating the fields to mass eigenstates, one obtains a mass term for each fermion, and the off-diagonal terms in M_e can be reabsorbed in the definition of the fields. In the quark case, the situation is more complicated and the Yukawa coupling matrix can be reduced to the mass eigenvalues and the CKM quark mixing. By rotating the fields, it is possible to eliminate the off-diagonal terms in M_u. Then M_d satisfies $VM_dV^* = M_u$, where V is the CKM quark mixing, given by a 3×3 unitary matrix

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

acting on the charge $-e/3$ quarks (down, strange, bottom). The entries of this matrix can be expressed in terms of three angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a phase, and can be determined experimentally from weak decays and deep inelastic neutrino scatterings.

The detailed structure of the Yukawa coupling matrix Y (in particular the fact that color is not broken) allows one to check that the finite geometry $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ satisfies all the axioms of Definition 13.10 for a noncommutative spectral manifold. The key point is that elements $a \in \mathcal{A}_F$ and $[D_f, a]$ commute with $J_F \mathcal{A}_F J_F$. These operators preserve the subspace $\mathcal{E} \subset \mathcal{H}_F$. On this subspace, for $b = (z, q, m)$, the action of $J_F b^* J_F$ is by multiplication by z or by the transpose m^t. It is then not hard to check explicitly the commutation with a or $[D, a]$ (*cf.* [56] §VI.5.δ). By exchanging the roles of a and b, one sees analogously that a commutes with $J_F b J_F$ and $[D, J_F b J_F]$ on $\bar{\mathcal{E}}$, hence the desired commutation relations hold on all of $\mathcal{H}_F$.

We can then consider the product $X \times F$, where X is an ordinary 4-dimensional Riemannian spin manifold and F is the finite geometry described above. This product geometry is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ obtained as the cup product of a triple $(\mathcal{C}^\infty(X), L^2(X, S), D_1)$, where D_1 is the Dirac operator on X acting on square integrable spinors in $L^2(X, S)$, with the spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ described above. Namely, the resulting (smooth) algebra and Hilbert space are of the form

$$\mathcal{A}_\infty = \mathcal{C}^\infty(X, \mathcal{A}_F), \qquad \mathcal{H} = L^2(X, S) \otimes \mathcal{H}_F, \tag{17.9}$$

and the Dirac operator is given by

$$D = D_1 \otimes 1 + \gamma \otimes D_F, \tag{17.10}$$

where γ is the usual $\mathbb{Z}/2$ grading on the spinor bundle S. The induced $\mathbb{Z}/2$ grading on $\mathcal{H}$ is the tensor product $\gamma \otimes \gamma_F$, and the real structure is given by $J = C \otimes J_F$, where C is the charge conjugation operator on spinors.

Notice that, so far, we have only used the information on the fermions of the Standard Model. We'll see now that the bosons, with the correct quantum numbers, are *deduced* as inner fluctuations of the metric of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

It is a general fact that, for noncommutative geometries $(\mathcal{A}, \mathcal{H}, D)$, one can consider inner fluctuations of the metric of the form

$$D \mapsto D + A + JAJ^{-1}$$

where A is of the form

$$A = \sum a_i \, [D, a_i'] \qquad a_i, a_i' \in \mathcal{A} \, . \tag{17.11}$$

In the case of the Standard Model, a direct computation of the inner fluctuations gives the Standard Model gauge bosons $\gamma, W^{\pm}, Z$, the eight gluons and the Higgs fields φ with accurate quantum numbers (*cf.* [63]). In fact, a field A of the form (17.11) can be separated into a "discrete part" $A^{(0,1)} = \sum a_i \, [\gamma \otimes D_F, a_i']$ and a continuous part $A^{(1,0)} = \sum a_i \, [D_1 \otimes 1, a_i']$, with $a_i = (z_i, q_i, m_i)$ and $a_i' = (z_i', q_i', m_i')$, $q_i = \alpha_i + \beta_i j$ and $q_i' = \alpha_i' + \beta_i' j$. The discrete part gives a quaternion-valued function

$$q(x) = \sum z_i \left((\alpha_i' - z_i') + z_i \beta_i' \, j \right) = \varphi_1 + \varphi_2 \, j$$

which provides the Higgs doublet. The continuous part gives three types of fields:

- A $U(1)$ gauge field $U = \sum z_i \, dz_i'$
- An $SU(2)$ gauge field $Q = \sum q_i \, dq_i'$
- A $U(3)$ gauge field $M = \sum m_i \, dm_i'$, which can be reduced to an $SU(3)$ gauge field M' by subtracting the scalar part of the overall gauge field which eliminates inessential fluctuations that do not change the metric.

The resulting internal fluctuation of the metric $A + JAJ^{-1}$ is then of the form (*cf.* [63])

$$\begin{pmatrix} -2U & 0 & 0 \\ 0 & Q_{11} - U & Q_{12} \\ 0 & Q_{21} & Q_{22} - U \end{pmatrix}$$

on the basis of leptons (e_R, ν_L, e_L) and successive generations, and

$$\begin{pmatrix} \frac{4}{3}U + M' & 0 & 0 & 0 \\ 0 & \frac{-2}{3}U + M' & 0 & 0 \\ 0 & 0 & Q_{11} + \frac{1}{3}U + M' & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22} + \frac{1}{3}U + M' \end{pmatrix},$$

on the basis of quarks given by (u_R, d_R, u_L, d_L) and successive generations. A striking feature that these internal fluctuations exhibit is the fact that the expressions above recover all the exact values of the hypercharges Y_L, Y_R that appear in the fermion kinetic term of the Standard Model Lagrangian.

Finally, one can also recover the bosonic part of the Standard Model Lagragian from a very general principle, the *spectral action principle* of Chamseddine–Connes (*cf.* [44–46]). The result is that the Hilbert–Einstein action functional for the Riemannian metric, the Yang–Mills action for the vector potentials, and the self-interaction and the minimal coupling for the Higgs fields all appear with the correct signs in the asymptotic expansion for large Λ of the number $N(\Lambda)$ of eigenvalues of D which are $\leq \Lambda$ (*cf.* [44]),

$$N(\Lambda) = \#\text{ eigenvalues of } D \text{ in } [-\Lambda, \Lambda]\,. \tag{17.12}$$

The spectral action principle, applied to a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, can be stated as saying that the physical action depends only on $\mathrm{Spec}(D) \subset \mathbb{R}$. This spectral datum corresponds to the data $(\mathcal{H}, D)$ of the spectral triple, independent of the action of $\mathcal{A}$. Different $\mathcal{A}$ that correspond to the same spectral data can be thought of as the noncommutative analog of isospectral Riemannian manifolds (*cf.* the discussion of isospectral deformations in Section 18). A natural expression for an action that depends only on $\mathrm{Spec}(D)$ and is *additive* for direct sums of spaces is of the form

$$\mathrm{Tr}\,\chi\left(\frac{D}{\Lambda}\right) + \langle \psi, D\,\psi\rangle\,, \tag{17.13}$$

where χ is a positive even function and Λ is a scale.

In the case of the Standard Model, this formula (17.13) is applied to the full "metric" including the internal fluctuations and gives the full Standard Model action minimally coupled with gravity. The fermionic part of the action (17.13) gives (*cf.* [44], [45])

$$\langle \psi, D\,\psi\rangle = \int_X (\mathcal{L}_{Gf} + \mathcal{L}_{Hf})\,\sqrt{|g|}d^4x\,. \tag{17.14}$$

The bosonic part of the action (17.13) evaluated via heat kernel invariants gives the Standard Model Lagrangian minimally coupled with gravity. Namely, one writes the function $\chi(\frac{D}{\Lambda})$ as the superposition of exponentials. One then computes the trace by a semiclassical approximation from local expressions involving the familiar heat equation expansion. This delivers all the correct terms in the action (*cf.* [45] for an explicit calculation of all the terms involved).

Notice that here one treats the spacetime manifold X in Euclidean signature. The formalism of spectral triple can be extended in various ways to Lorentzian signature (*cf. e.g.* [122]). Perhaps the most convenient choice is to drop the self-adjointness condition for D while still requiring D^2 to be self-adjoint.

While it is certainly remarkable to obtain the Standard Model action from simple geometric principles the above work has several shortcomings:

(1) The finite geometry F is put in "by hand" with no conceptual understanding of the representation of $\mathcal{A}_F$ in $\mathcal{H}_F$.
(2) There is a fermion doubling problem (*cf.* [150]) in the fermionic part of the action.
(3) It does not incorporate the neutrino mixing and see-saw mechanism for neutrino masses.

These three problems have recently been solved, in [68] and [47], simply by keeping the distinction between the following two notions of dimension of a noncommutative space:

- The metric dimension
- The KO-dimension

The metric dimension manifests itself by the growth of the spectrum of the Dirac operator. As far as space-time goes it appears that the situation of interest will be the 4-dimensional one. In particular the metric dimension of the finite geometry F will be zero.

The KO-dimension of Definition 13.10 is only well defined modulo 8 and it takes into account both the $\mathbb{Z}/2$-grading γ of $\mathcal{H}$ as well as the real structure J. The real surprise is that in order for things to work, the only needed change (besides the easy addition of a right-handed neutrino) is to change the $\mathbb{Z}/2$ grading of the finite geometry F to its opposite in the "antiparticle" sector. It is only due to this that the fermion doubling problem pointed out in [150] can be successfully handled. Moreover it will automatically generate the *full Standard Model*, *i.e.* the model with neutrino mixing and the see-saw mechanism. The fermionic part of the action now involves all the structure of the real spectral triple and takes the form

$$\frac{1}{2}\langle J\xi, D\xi\rangle, \quad \xi \in \mathcal{H}^+ \tag{17.15}$$

where the vectors in $\mathcal{H}^+ = \{\xi \in \mathcal{H} | \gamma\xi = \xi\}$ are treated as Grassmann variables.

18. Isospectral Deformations

A very rich class of examples of noncommutative manifolds is obtained by considering isospectral deformations of a classical Riemannian manifold. These examples satisfy all the axioms of ordinary Riemannian geometry (*cf.* [63]) except commutativity. They are obtained by the following result (Connes–Landi [75]):

Theorem 18.1. *Let M be a compact Riemannian Spin manifold. Then if the isometry group of M has rank $r \geq 2$, M admits a non-trivial one-parameter isospectral deformation to noncommutative geometries M_θ.*

The main idea of the construction is to deform the standard spectral triple describing the Riemannian geometry along a two-torus embedded in the isometry group, to a family of spectral triples describing noncommutative geometries.

More precisely, under the assumption on the rank of the group of isometries of the compact Spin manifold X, there exists a two-torus

$$T^2 \subset \mathrm{Isom}(X)\,,$$

where we identify $T^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$. Let $U(s)$ be the unitary operators in this subgroup of isometries, for $s = (s_1, s_2) \in T^2$, acting on the Hilbert space $\mathcal{H} = L^2(X, S)$ of the spectral triple

$$(C^\infty(X), L^2(X, S), D, J)\,.$$

Equivalently, we write $U(s) = \exp(i(s_1P_1 + s_2P_2))$, where P_i are the corresponding Lie algebra generators, with $\mathrm{Spec}(P_i) \subset \mathbb{Z}$, satisfying $[D, P_i] = 0$ and $P_iJ = -JP_i$, so that $[U(s), D] = [U(s), J] = 0$.

The action $\alpha_s(T) = U(s)TU(s)^{-1}$ has the following property. Any operator T such that the map $s \mapsto \alpha_s(T)$ is smooth can be uniquely written as a norm convergent series

$$T = \sum_{n_1,n_2\in\mathbb{Z}} \hat{T}_{n_1,n_2} \tag{18.1}$$

where each term $\hat{T}_{n_1,n_2}$ is an operator of bi-degree (n_1, n_2), that is,

$$\alpha_s(\hat{T}_{n_1,n_2}) = \exp(i(s_1n_1 + s_2n_2))\hat{T}_{n_1,n_2}\,,$$

for each $s = (s_1, s_2) \in T^2$, and the sequence of norms $\|\hat{T}_{n_1,n_2}\|$ is of rapid decay.

This property makes it possible to define left and right twists for such operators T, defined as

$$\ell(T) := \sum_{n_1,n_2} \hat{T}_{n_1,n_2} \exp\left(2\pi i\theta n_2P_1\right) \tag{18.2}$$

and

$$r(T) := \sum_{n_1,n_2} \hat{T}_{n_1,n_2} \exp\left(2\pi i\theta n_1P_2\right). \tag{18.3}$$

Both series still converge in norm, since the P_i are self-adjoint operators.

It is then possible to introduce a (left) deformed product

$$x * y = \exp(2\pi i\theta n_1'n_2)xy\,, \tag{18.4}$$

for x a homogeneous operator of bi-degree (n_1, n_2) and y a homogeneous operator of bi-degree (n_1', n_2'). A (right) deformed product is similarly defined by setting $x *_r y = \exp(2\pi i\theta n_1n_2')xy$. These deformed products satisfy $\ell(x)\ell(y) = x * y$ and $r(x)r(y) = x *_r y$.

The deformed spectral triples are then obtained by maintaining the same Hilbert space $\mathcal{H} = L^2(X, S)$ and Dirac operator D, while modifying the algebra $C^\infty(X)$ to the noncommutative algebra $\mathcal{A}_\theta := \ell(C^\infty(X))$ and the involution J that defines the real structure to $J_\theta := \exp(2\pi i\theta P_1 P_2)J$.

19. Algebraic Deformations

There is a very general context in which one constructs noncommutative spaces via deformations of commutative algebras. Unlike the isospectral deformations discussed in Section 18, here one proceeds mostly at a formal algebraic level, without involving the operator algebra structure and without invoking the presence of a Riemannian structure.

The idea of deformation quantization originates from the idea that classical mechanics have as setting a smooth manifold (phase space) with a symplectic structure, which defines a Poisson bracket $\{,\}$. The system is quantized by deforming the pointwise product in the algebra $\mathcal{A} = \mathcal{C}^\infty(M)$ (or in a suitable subalgebra) to a family $*_\hbar$ of products satisfying $f *_\hbar g \to fg$ as $\hbar \to 0$, which are associative but no longer necessarily commutative. These are also required to satisfy

$$\frac{f *_\hbar g - g *_\hbar f}{i\hbar} \to \{f, g\}\,,$$

as $\hbar \to 0$, namely, the ordinary product is deformed in the direction of the Poisson bracket. On the algebra $\mathcal{C}^\infty(M)$ a Poisson bracket is specified by assigning a section Λ of $\Lambda^2(TM)$ with the property that

$$\{f, g\} = \langle \Lambda, df \wedge dg \rangle$$

satisfies the Jacobi identity. Typically, this produces a *formal deformation*: a formal power series in $\hbar$. Namely, the deformed product can be written in terms of a sequence of bi–differential operators B_k satisfying

$$f * g = fg + \hbar B_1(f, g) + \hbar^2 B_2(f, g) + \cdots\,. \tag{19.1}$$

Under this perspective, there is a good understanding of formal deformations. For instance, Kontsevich [139] proved that formal deformations always exist, by providing an explicit combinatorial formula that generates all the $\{B_2, B_3, \ldots\}$ in the expansion from the B_1, hence in terms of the Poisson structure Λ. The formal solution (19.1) can then be written as

$$\sum_{n=0}^{\infty} \hbar^n \sum_{\Gamma \in G[n]} \omega_\Gamma B_{\Gamma,\Lambda}(f, g)\,,$$

where $G[n]$ is a set of $(n(n+1))^n$ labeled graphs with $n+2$ vertices and n edges, ω_Γ is a coefficient obtained by integrating a differential form (depending on the graph Γ) on the configuration space of n distinct points in the upper half-plane, and $B_{\Gamma,\Lambda}$ is a bi–differential operator whose coefficients are derivatives of Λ of orders specified by the combinatorial information of the graph Γ.

A setting of deformation quantization which is compatible with C^*–algebras was developed by Rieffel in [187]. We recall briefly Rieffel's setting. For simplicity, we restrict to the simpler case of a compact manifold.

Definition 19.1. *A strict (Rieffel) deformation quantization of $\mathcal{A} = \mathcal{C}^\infty(M)$ is obtained by assigning an associative product $*_\hbar$, an involution (depending on $\hbar$) and a C^*-norm $\|\cdot\|_\hbar$ on $\mathcal{A}$, for $\hbar \in I$ (some interval containing zero), such that:*

(i) *For $\hbar = 0$ these give the C^*–algebra $C(M)$,*
(ii) *For all $f, g \in \mathcal{A}$, as $\hbar \to 0$,*

$$\left\| \frac{f *_\hbar g - g *_\hbar f}{i\hbar} - \{f, g\} \right\|_\hbar \to 0 \,.$$

One denotes by $\mathcal{A}_\hbar$ the C^*–algebra obtained by completing $\mathcal{A}$ in the norm $\|\cdot\|_\hbar$.

The functions of $\hbar$ are all assumed to be analytic, so that formal power series expansions make sense.

Remark 19.2. *The notion of a strict deformation quantization should be regarded as a notion of integrability for formal solutions.*

Rieffel also provides a setting for compatible actions by a Lie group of symmetries, and proves that noncommutative tori (also of higher rank) are strict deformation quantizations of ordinary tori, that are compatible with the action of the ordinary torus as group of symmetry. Typically, for a given Poisson structure, strict deformation quantizations are not unique. This happens already in the case of tori.

In the same paper [187], Rieffel uses a basic result of Wassermann [209] to produce an example where formal solutions are not integrable. The example is provided by the two-sphere S^2. There is on S^2 a symplectic structure, and a corresponding Poisson structure Λ which is invariant under $SO(3)$. Rieffel proves the following striking result (Theorem 7.1 of [187]):

Theorem 19.3. *There are no $SO(3)$-invariant strict deformations of the ordinary product on $\mathcal{C}^\infty(S^2)$ in the direction of the $SO(3)$–invariant Poisson structure.*

In fact, the proof of this result shows more, namely that no $SO(3)$-invariant deformation of the ordinary product in $C(S^2)$ can produce a noncommutative C^*-algebra. This rigidity result reflects a strong rigidity result for $SU(2)$ proved by Wassermann [209], namely that the only ergodic actions of $SU(2)$ are on von Neumann algebras of type I. The interest of this result lies in the fact that there are formal deformations of the Poisson structure that are $SO(3)$-invariant (see *e.g.* [118], [17]), but these only exist as a formal power series in the sense of (19.1) and, by the results of Wassermann and Rieffel, are not integrable.

Summarizing, we have the following type of phenomenon: on the one hand we have formal solutions, formal deformation quantizations about which a lot is known, but for which, in general, there may not be an integrability result. More precisely, when we try to pass from formal to actual solutions, there are cases where existence fails (the sphere), and others (tori) where uniqueness fails. The picture that emerges is remarkably similar to the case of formal and actual solutions of ordinary differential equations.

It is very instructive to build an analogy between the *problem of ambiguity* for formal solutions of ODE's and the present situation of formal noncommutative spaces and actual noncommutative spaces. The main conclusion to be drawn from this analogy is that there ought to be a *theory of ambiguity* which formulates precisely the relation between the formal noncommutative geometry and its integrated (C^*-algebraic) version.

To illustrate this concept, we take a closer look at the analogous story in the theory of ODE's. A good reference for a modern viewpoint is [182]. A formal solution of a differential equation is a power series expansion: for instance $\sum_{n=0}^{\infty}(-1)^n n! x^{n+1}$ is a formal solution of the Euler equation $x^2y' + y = x$. Convergent series give rise to actual solutions, and more involved summation processes such as Borel summation can be used to transform a given formal solution of an analytic ODE into an actual solution on a sufficiently narrow sector in $\mathbb{C}$ of sufficiently small radius, but such solution is in general *not unique.* It is known from several classical methods that some divergent series can be "summed" modulo a function with exponential decrease of a certain order. This property (Gevrey summability) is also satisfied by formal solutions of analytic ODE's, and, stated in a more geometric fashion, it is essentially a cohomological condition. It also shows that, whereas on *small sectors* one has existence of actual solutions but not uniqueness, on *large sectors* one gains uniqueness, at the cost of possibly losing existence. A complete answer to summability of formal solutions can then be given in terms of a more refined multi–summability (combining Gevrey series and functions of different order) and the Newton polygon of the equation.

The general flavor of this theory is surprisingly similar to the problem of formal solutions in noncommutative geometry. It is to be expected that an ambiguity theorem exists, which accounts for the cases of lack of uniqueness, or lack of existence, of actual solutions illustrated by the results of Rieffel.

Already in dealing with our first truly non-trivial example of noncommutative spaces, the noncommutative tori, we encountered subtleties related to the difference between the quotient and the deformation approach to the construction of noncommutative spaces.

In fact, the noncommutative tori we described in Section 6 admit a description as algebras obtained as deformations of the ordinary product of functions, by setting

$$(f * g)(x, y) := \left(e^{2\pi i \theta \frac{\partial}{\partial x} \frac{\partial}{\partial y'}} f(x, y) g(x', y') \right)_{x=x', y=y'} = \sum \frac{(2\pi i \theta)^n}{n!} D_1^n f D_2^n g. \tag{19.2}$$

Notice however that while $U \frac{\partial}{\partial U}$ and $V \frac{\partial}{\partial V}$ are derivations for the algebra of the noncommutative torus, this is *not* the case for $\frac{\partial}{\partial U}$ and $\frac{\partial}{\partial V}$. The same holds for the quantum plane (*cf.* [155]) whose algebra of coordinates admits two generators u, v with relation

$$uv = qvu\,.$$

These generators can be rotated ($u \mapsto \lambda u$, $v \mapsto \mu v$) without affecting the presentation but translations of the generators are not automorphisms of the algebra. In other words, one can view the noncommutative torus as a deformation of an ordinary torus, which in turn is a quotient of the classical plane $\mathbb{R}^2$ by a lattice of translations, but the action of translations does not extend to the quantum plane. This is an instance of the fact that the general operations of quotient and deformation, in constructing noncommutative spaces, do not satisfy any simple compatibility rules and need to be manipulated with care.

Moreover, phenomena like the Morita equivalence between, for instance θ and $1/\theta$, are not detectable in a purely deformation theoretic perturbative expansion like the one given by the Moyal product (19.2). They are non-perturbative and cannot be seen at the perturbative level of the star product.

In this respect, a very interesting recent result is that of Gayral, Gracia-Bondía, Iochum, Schücker, and Várilly, [112], where they consider a version of the structure of spectral triple for non-compact spaces. In that case, for instance, one no longer can expect the Dirac operator to have compact resolvent and one can only expect a local version to hold, *e.g.* $a(D-i)^{-1}$ is compact for $a \in \mathcal{A}$. Other properties of Definitions 13.6 and 13.10 are easily adapted to a "local version" but become more difficult to check than in the compact case. They show that the Moyal product deformation of $\mathbb{R}^{2n}$ fits in the framework of spectral triples and provides an example of such non-unital spectral triples. Thus, it appears that the structure of noncommutative Riemannian geometry provided by spectral triples should adapt nicely to some classes of algebraic deformations.

It appears at first that spectral triples may not be the right type of structure to deal with noncommutative spaces associated to algebraic deformations, because it corresponds to a form of Riemannian geometry, while many such spaces originate from Kähler geometry. However, the Kähler structure can often also be encoded in the setting of spectral triple, for example by considering also a second Dirac operator, as in [26] or through the presence of a Lefschetz operator as in [94].

Noncommutative spaces obtained as deformations of commutative algebras fit in the context of a well developed algebraic theory of noncommutative spaces (*cf. e.g.* [140], [141], [152], [153], [155], [189], [190], [200]). This theory touches on a variety of subjects like quantum groups and the deformation approach to noncommutative spaces and is interestingly connected to the theory of mirror symmetry. However, it is often not clear how to integrate this approach with the functional analytic theory of noncommutative geometry briefly summarized in Section 2. Only recently have several results confirmed the existence of a rich interplay between the algebraic and functional analytic aspects of noncommutative geometry, especially through the work of Connes and Dubois-Violette (*cf.* [72]–[74]) and of Polishchuk (*cf.* [178]). Also, the work of Chakraborty and Pal [42] and Connes [67] and more recently of Dabrowski, Landi, Sitarz, van Suijlekom and Várilly [202], [100] showed that quantum groups fit very nicely within the framework of noncommutative geometry described by spectral triples, contrary to what was previously believed. Ultimately, successfully importing tools from the theory of operator algebras into the realm of algebraic geometry might well land within the framework of what Manin refers to as a "second quantization of algebraic geometry".

20. Quantum Groups

For a long time, it was widely believed that quantum groups could not fit into the setting of noncommutative manifolds defined in terms of spectral geometry. On the contrary, recent work of Chakraborty and Pal showed in [42] that the quantum group $SU_q(2)$, for $0 \leq q < 1$, admits a spectral triple with Dirac operator that is equivariant with respect to its own (co)action.

The algebra $\mathcal{A}$ of functions on the quantum group $SU_q(2)$ is generated by two elements α and β with the relations

$$\begin{gathered} \alpha^*\alpha + \beta^*\beta = 1, \quad \alpha\alpha^* + q^2\beta\beta^* = 1, \\ \alpha\beta = q\beta\alpha, \quad \alpha\beta^* = q\beta^*\alpha, \quad \beta^*\beta = \beta\beta^* \,. \end{gathered} \tag{20.1}$$

By the representation theory of the quantum group $SU_q(2)$ (*cf.* [137]) there exists a Hilbert space $\mathcal{H}$ with orthonormal basis $e_{ij}^{(n)}$, $n \in \frac{1}{2}\mathbb{N}$,

$i, j \in \{-n, \dots, n\}$, and a unitary representation

$$\begin{aligned} \alpha\, e_{ij}^{(n)} &= a_+(n,i,j)\, e_{i-1/2,j-1/2}^{(n+1/2)} + a_-(n,i,j)\, e_{i-1/2,j-1/2}^{(n-1/2)}\,, \\ \beta\, e_{ij}^{(n)} &= b_+(n,i,j)\, e_{i+1/2,j-1/2}^{(n+1/2)} + b_-(n,i,j)\, e_{i+1/2,j-1/2}^{(n-1/2)}\,, \end{aligned} \tag{20.2}$$

with coefficients

$$\begin{aligned} a_+(n,i,j) &= q^{2n+i+j+1}\, Q(2n-2j+2, 2n-2i+2, 4n+2, 4n+4)\,, \\ a_-(n,i,j) &= Q(2n+2j, 2n+2i, 4n, 4n+2)\,, \\ b_+(n,i,j) &= -q^{n+j}\, Q(2n-2j+2, 2n+2i+2, 4n+2, 4n+4)\,, \\ b_-(n,i,j) &= q^{n+i}\, Q(2n+2j, 2n-2i, 4n, 4n+2)\,, \end{aligned}$$

where we use the notation

$$Q(n,m,k,r) = \frac{(1-q^n)^{1/2}(1-q^m)^{1/2}}{(1-q^k)^{1/2}(1-q^r)^{1/2}}\,.$$

Consider then, as in [67], the operator

$$D\, e_{ij}^{(n)} = \begin{cases} -2n & n \neq i, \\ 2n & n = i. \end{cases} \tag{20.3}$$

More generally, one can consider operators of the form $D\, e_{ij}^{(n)} = d(n,i)\, e_{ij}^{(n)}$, as in [42], with $d(n,i)$ satisfying the conditions $d(n+1/2, i+1/2) - d(n,i) = O(1)$ and $d(n+1/2, i-1/2) - d(n,i) = O(n+i+1)$. Then one has the following result (Chakraborty–Pal [42]):

Theorem 20.1. *The data $(\mathcal{A}, \mathcal{H}, D)$ as above define an $SU_q(2)$ equivariant odd 3-summable spectral triple.*

The equivariance condition means that there is an action on $\mathcal{H}$ of the enveloping algebra $\mathcal{U} = U_q(SL(2))$, which commutes with the Dirac operator D. This is generated by operators

$$\begin{aligned} k\, e_{ij}^{(n)} &= q^j\, e_{ij}^{(n)} \\ e\, e_{ij}^{(n)} &= q^{-n+1/2}(1-q^{2(n+j+1)})^{1/2}(1-q^{2(n-j)})^{1/2}(1-q^2)^{-1}\, e_{ij+1}^{(n)}\,, \end{aligned}$$

satisfying the relations

$$ke = qek, \quad kf = q^{-1}fk, \quad [e,f] = \frac{k^2 - k^{-2}}{q - q^{-1}}\,,$$

with $f = e^*$, and with coproduct

$$\Delta(k) = k \otimes k, \quad \Delta(e) = k^{-1} \otimes e + e \otimes k, \quad \Delta(f) = k^{-1} \otimes f + f \otimes k\,.$$

It is interesting that, while the classical $SU(2)$ is of (topological and metric) dimension three, the topological dimension of the algebra $\mathcal{A}$ of $SU_q(2)$ drops to one (*cf.* [42]), but the metric dimension of the spectral triple remains equal to three as in the classical case.

Chakraborty and Pal showed in [42] that the Chern character of the spectral triple is nontrivial. Moreover, Connes in [67] gave an explicit formula for its local index cocycle, where a delicate calculation provides the cochain whose coboundary is the difference between the Chern character and the local version in terms of remainders in the rational approximation to the logarithmic derivative of the Dedekind eta function.

The local index formula is obtained by constructing a symbol map

$$\rho : \mathcal{B} \to C^\infty(S_q^*) ,$$

where the algebra $C^\infty(S_q^*)$ gives a noncommutative version of the cosphere bundle, with a restriction map $r : C^\infty(S_q^*) \to C^\infty(D_{q+}^2 \times D_{q-}^2)$ to the algebra of two noncommutative disks. Here $\mathcal{B}$ is the algebra generated by the elements $\delta^k(a)$, $a \in \mathcal{A}$, with $\delta(a) = [|D|, a]$. On the cosphere bundle there is a geodesic flow, induced by the group of automorphisms $a \mapsto e^{it|D|}\, a\, e^{-it|D|}$. Then $\rho(b)^0$ denotes the component of degree zero with respect to the grading induced by this flow.

The algebra $C^\infty(D_q^2)$ is an extension

$$0 \to \mathcal{S} \to C^\infty(D_q^2) \xrightarrow{\sigma} C^\infty(S^1) \to 0 ,$$

where the ideal $\mathcal{S}$ is the algebra of rapidly decaying matrices. There are linear functionals τ_0 and τ_1 on $C^\infty(D_q^2)$,

$$\tau_1(a) = \frac{1}{2\pi} \int_0^{2\pi} \sigma(a) d\theta ,$$

$$\tau_0(a) = \lim_{N\to\infty} \sum_{k=0}^{N} \langle a\, \epsilon_k, \epsilon_k \rangle - \tau_1(a) N ,$$

where τ_0 is defined in terms of the representation of $C^\infty(D_q^2)$ on the Hilbert space $\ell^2(\mathbb{N})$ with orthonormal basis $\{\epsilon_k\}$.

Recall that (*cf.* [56]) a cycle $(\Omega, d, \int)$ is a triple with where (Ω, d) is a graded differential algebra, and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace on Ω. A cycle over an algebra $\mathcal{A}$ is given by a cycle $(\Omega, d, \int)$ together with a homomorphism $\rho : \mathcal{A} \to \Omega^0$.

In the case of the algebra $\mathcal{A}$ of $SU_q(2)$, a cycle $(\Omega, d, \int)$ is obtained in [67] by considering $\Omega^1 = \mathcal{A} \oplus \Omega^{(2)}(S^1)$, with $\Omega^{(2)}(S^1)$ the space of weight two differential forms $f(\theta)d\theta^2$, with the $\mathcal{A}$–bimodule structure

$$a\,(\xi, f) = (a\xi, \sigma(a) f) , \qquad (\xi, f)\, a = (\xi a, -i\sigma(\xi)\sigma(a)' + f\sigma(a)) ,$$

with differential

$$da = \partial a + \frac{1}{2}\sigma(a)'' d\theta^2 ,$$

with ∂ the derivation $\partial = \partial_\beta - \partial_\alpha$, and

$$\int (\xi, f) = \tau(\xi) + \frac{1}{2\pi i} \int f \, d\theta \,,$$

where $\tau(a) = \tau_0(r_-(a^{(0)}))$, with $a^{(0)}$ the component of degree zero for ∂ and r_- the restriction to $C^\infty(D^2_{q-})$. This definition of the cycle corrects for the fact that τ itself (as well as τ_0) fails to be a trace.

The following result then holds (Connes [67]):

Theorem 20.2. (1) *The spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *of Theorem* 20.1 *has dimension spectrum* $\Sigma = \{1, 2, 3\}$.
(2) *The residue formula for pseudodifferential operators* $a \in \mathcal{B}$ *in terms of their symbol is given by*

$$\begin{aligned} ⨍ a\,|D|^{-3} &= (\tau_1 \otimes \tau_1)(r\rho(a)^0) \\ ⨍ a\,|D|^{-2} &= (\tau_1 \otimes \tau_0 + \tau_0 \otimes \tau_1)(r\rho(a)^0) \\ ⨍ a\,|D|^{-1} &= (\tau_0 \otimes \tau_0)(r\rho(a)^0) \end{aligned}$$

(3) *The character* $\chi(a_0, a_1) = \int a_0 \, da_1$ *of the cycle* $(\Omega, d, \int)$ *is equal to the cocycle*

$$\psi_1(a_0, a_1) = 2 ⨍ a_0 \delta(a_1) P |D|^{-1} - ⨍ a_0 \delta^2(a_1) P |D|^{-1} \,,$$

with $P = (1 + F)/2$. *The local index formula is given by*

$$\varphi_{odd} = \psi_1 + (b + B)\varphi_{even} \,,$$

where φ *is the local index cocycle.*
(4) *The character* $\mathrm{Tr}(a_0[F, a_1])$ *differs from the local form* ψ_1 *by the coboundary* $b\psi_0$, *with* $\psi_0(a) = 2\mathrm{Tr}(aP|D|^{-s})_{s=0}$. *This cochain is determined by the values* $\psi_0((\beta^*\beta)^n)$, *which are of the form*

$$\psi_0((\beta^*\beta)^n) = q^{-2n}(q^2 R_n(q^2) - G(q^2)) \,,$$

where G *is the logarithmic derivative of the Dedekind eta function*

$$\eta(q^2) = q^{1/12} \prod_{k=1}^{\infty} (1 - q^{2k}) \,, \tag{20.4}$$

and the R_n *are rational functions with poles only at roots of unity.*

More recently, another important breakthrough in the relation between quantum groups and the formalism of spectral triples was obtained by Ludwik Dabrowski, Giovanni Landi, Andrzej Sitarz, Walter van Suijlekom, and Joseph C. Varilly, in [202] and [100].

They construct a 3^+-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is, as before, the algebra of coordinates of the quantum group $SU_q(2)$. The geometry in this case is an isospectral deformation of the classical case, in the sense that the Dirac operator is the same as the usual Dirac operator for the round metric on the ordinary 3-sphere S^3. Moreover, the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is especially nice, in as much as it is equivariant with respect to both left and right action of the Hopf algebra $\mathcal{U}_q(su_q(2))$.

The classical Dirac operator for the round metric on S^3 has spectrum $\Sigma = \Sigma_+ \cup \Sigma_-$ with $\Sigma_+ = \{(2j + 3/2) : j = 0, 1/2, 1, 3/2, \ldots\}$ with multiplicities $(2j + 1)(2j + 2)$ and $\Sigma_- = \{-(2j + 1/2) : j = 1/2, 1, 3/2, \ldots\}$ with multiplicities $2j(2j + 1)$. The Hilbert space is obtained by taking $V \otimes \mathbb{C}^2$, where V is the left regular representation of $\mathcal{A}$. It is very important here to take $V \otimes \mathbb{C}^2$ instead of $\mathbb{C}^2 \otimes V$. Not only does the latter violate the equivariance condition, but it was shown by Goswami that it produces unbounded commutators $[D, a]$, hence one does not obtain a spectral triple in that way.

The spectral triple contructed in [202] and [100] has a real structure J and the Dirac operator satisfies a weak form of the "order one condition" (*cf.* Section 13 above). The local index formula of [67] (*cf.* Theorem 20.2 above) extends to the spectral triple of [202], as proved in [100] and the structures of the cotangent space and the geodesic flow are essentially the same.

Even more recently, the construction of finitely summable spectral triples of [202], [100] was generalized by Sergey Neshveyev and Lars Tuset [173] to a functorial construction that works for any quantum group G_q obtained as q-deformation of a simply connected simple compact Lie group G.

21. Spherical Manifolds

The noncommutative spheres $S^3_\varphi \subset \mathbb{R}^4_\varphi$ are obtained as solutions of a very simple problem, namely the vanishing of the first component of the Chern character of a unitary $U \in M_2(\mathcal{A})$ where $\mathcal{A}$ is the algebra of functions on the sphere and the Chern character is taken in the cyclic homology (b, B)-bicomplex. The origin of this problem is to quantize the volume form of a three-manifold (*cf.* [72]). The solutions are parameterized by three angles φ_k, $k \in \{1, 2, 3\}$ and the corresponding algebras are obtained by imposing the "unit sphere relation"

$$\sum x_\mu^2 = 1 \tag{21.1}$$

on the four generators x_0, x_1, x_2, x_3 of the quadratic algebra $C_{\rm alg}(\mathbb{R}^4_\varphi)$ with the six relations

$$\sin(\varphi_k)\,[x_0, x_k]_+ = i \cos(\varphi_\ell - \varphi_m)\,[x_\ell, x_m]\,, \tag{21.2}$$

$$\cos(\varphi_k)\,[x_0, x_k] = i \sin(\varphi_\ell - \varphi_m)\,[x_\ell, x_m]_+\,, \tag{21.3}$$

where $[a, b]_+ = ab + ba$ is the anticommutator and by convention the indices $k, l, m \in \{1, 2, 3\}$ always appear in cyclic order.

The analysis of these algebras is a special case of the general theory of central quadratic forms for quadratic algebras developed in [73], [74] and which we briefly recall below.

Let $\mathcal{A} = A(V, R) = T(V)/(R)$ be a quadratic algebra where V is the linear span of the generators and $(R) \subset T(V)$ the ideal generated by the relations. The geometric data $\{E\,,\,\sigma\,,\,\mathcal{L}\}$ is given by an algebraic variety E, a correspondence σ on E and a line bundle $\mathcal{L}$ over E. These data are defined so as to yield a homomorphism h from $\mathcal{A}$ to a crossed product algebra constructed from sections of powers of the line bundle $\mathcal{L}$ on the graphs of the iterations of the correspondence σ. This crossed product only involves the positive powers of the correspondence σ, hence it remains "triangular" and far removed from the "semi-simple" set-up of C^*-algebras.

This morphism h can be considerably refined using the notion of positive central quadratic form.

Definition 21.1. *Let $Q \in S^2(V)$ be a symmetric bilinear form on V^* and C a component of $E \times E$. We say that Q is central on C if for all (Z, Z') in C and $\omega \in R$ one has*

$$\omega(Z, Z')\, Q(\sigma(Z'), \sigma^{-1}(Z)) + Q(Z, Z')\, \omega(\sigma(Z'), \sigma^{-1}(Z)) = 0 \tag{21.4}$$

This makes it possible to construct purely algebraically a crossed product algebra and a homomorphism from $\mathcal{A} = A(V, R)$ to this crossed product [73], [74]. The relation with C^*-algebras arises from positive central quadratic forms which make sense on involutive quadratic algebras.

Let $\mathcal{A} = A(V, R)$ be an *involutive* quadratic algebra, *i.e.* an algebra over $\mathbb{C}$ which is a $*$-algebra with involution $x \mapsto x^*$ preserving the subspace V of the generators. The real structure of V is given by the antilinear involution $v \mapsto j(v)$, the restriction of $x \mapsto x^*$. As $(xy)^* = y^* x^*$ for $x, y \in \mathcal{A}$, the space R of relations fulfills

$$(j \otimes j)(R) = t(R) \tag{21.5}$$

in $V \otimes V$, where $t : V \otimes V \to V \otimes V$ is the transposition $v \otimes w \mapsto t(v \otimes w) = w \otimes v$. This implies that the characteristic variety is stable under the involution j and one has

$$\sigma(j(Z)) = j(\sigma^{-1}(Z))\,.$$

Now let C be an invariant component of $E \times E$. We say that C is *j-real* when it is globally invariant under the involution

$$\tilde{j}(Z,\, Z') := (j(Z'),\, j(Z))\,. \tag{21.6}$$

Let Q be a central quadratic form on C. We say that Q is positive on C if

$$Q(Z, j(Z)) > 0\,, \quad \forall\, Z \in K\,.$$

One can then endow the line bundle $\mathcal{L}$ dual to the tautological bundle on $\mathbb{P}(V^*)$ with the Hermitian metric given by

$$\langle f\, L,\, g\, L' \rangle_Q(Z) = \, f(Z)\, \overline{g(Z)}\, \frac{L(Z)\, \overline{L'(Z)}}{Q(Z,\, j(Z))} \qquad L, L' \in V, \quad Z \in K\,, \tag{21.7}$$

$(\forall\, f, g \in C(K))$.

One then defines a generalized crossed product C^*-algebra $C(K) \times_{\sigma,\, \mathcal{L}} \mathbb{Z}$ following M. Pimsner [176]. Given a compact space K, a homeomorphism σ of K and a Hermitian line bundle $\mathcal{L}$ on K we define the C^*-algebra $C(K) \times_{\sigma,\, \mathcal{L}} \mathbb{Z}$ as the twisted crossed product of $C(K)$ by the Hilbert C^*-bimodule associated to $\mathcal{L}$ and σ ([3], [176]).

We let for each $n \geq 0$, $\mathcal{L}^{\sigma^n}$ be the hermitian line bundle pullback of $\mathcal{L}$ by σ^n and (cf. [10], [199])

$$\mathcal{L}_n := \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}\,. \tag{21.8}$$

We first define a $*$-algebra as the linear span of the monomials

$$\xi\, W^n\,, \quad W^{*n}\, \eta^*\,, \quad \xi\,, \eta \in C(K, \mathcal{L}_n) \tag{21.9}$$

with product given as in ([10], [199]) for $(\xi_1\, W^{n_1})\,(\xi_2\, W^{n_2})$ so that

$$(\xi_1\, W^{n_1})\,(\xi_2\, W^{n_2}) := (\xi_1 \otimes (\xi_2 \circ \sigma^{n_1}))\, W^{n_1+n_2}\,. \tag{21.10}$$

We use the Hermitian structure of $\mathcal{L}_n$ to give meaning to the products $\eta^* \xi$ and $\xi \eta^*$ for $\xi, \eta \in C(K, \mathcal{L}_n)$. The product then extends uniquely to an associative product of $*$-algebras fulfilling the additional rules

$$(W^{*k}\, \eta^*)\,(\xi\, W^k) := \, (\eta^*\, \xi) \circ \sigma^{-k}\,, \qquad (\xi\, W^k)\,(W^{*k}\, \eta^*) \, := \, \xi\, \eta^*\,. \tag{21.11}$$

The C^*-norm of $C(K) \times_{\sigma,\, \mathcal{L}} \mathbb{Z}$ is defined as for ordinary crossed products and due to the amenability of the group $\mathbb{Z}$ there is no distinction between the reduced and maximal norms. The latter is obtained as the supremum of the norms in involutive representations in Hilbert space. The natural positive conditional expectation on the subalgebra $C(K)$ shows that the C^*-norm restricts to the usual sup norm on $C(K)$.

Theorem 21.2. *Let $K \subset E$ be a compact σ-invariant subset and Q be central and strictly positive on $\{(Z,\, \bar{Z});\, Z \in K\}$. Let $\mathcal{L}$ be the restriction to K of the dual of the tautological line bundle on $\mathbb{P}(V^*)$ endowed with the Hermitian metric $\langle\ ,\ \rangle_Q$.*

(i) *The equality $\sqrt{2}\, \theta(Y) := Y\, W + W^*\, \bar{Y}^*$ yields a $*$-homomorphism*

$$\theta : \mathcal{A} = A(V, R) \to C(K) \times_{\sigma,\, \mathcal{L}} \mathbb{Z}\,.$$

(ii) *For any* $Y \in V$ *the* C^**-norm of* $\theta(Y)$ *fulfills*

$$\sup_K \|Y\| \leq \sqrt{2}\|\,\theta(Y)\| \leq 2\sup_K \|Y\|$$

(iii) *If* $\sigma^4 \neq \mathbb{1}$, *then* $\theta(Q) = 1$ *where* Q *is viewed as an element of* $T(V)/(R)$.

In the above case of the sphere S^3_φ one lets Q be the quadratic form

$$Q(x,\, x') := \sum x_\mu\, x'_\mu \tag{21.12}$$

In the generic case one has:

Proposition 21.3. (1) *The characteristic variety is the union of 4 points with an elliptic curve* F_φ.
(2) *The quadratic form* Q *is central and positive on* $F_\varphi \times F_\varphi$.

In suitable coordinates the equations defining the elliptic curve F_φ are

$$\frac{Z_0^2 - Z_1^2}{s_1} = \frac{Z_0^2 - Z_2^2}{s_2} = \frac{Z_0^2 - Z_3^2}{s_3}\,, \tag{21.13}$$

where $s_k := 1 + t_\ell\, t_m$, $t_k := \tan\varphi_k$.

The positivity of Q is automatic since in the coordinates x the involution j_φ of the $*$-algebra $C_{\rm alg}(\mathbb{R}^4_\varphi)$ is just $j_\varphi(Z) = \bar{Z}$, so that $Q(X,\, j_\varphi(X)) > 0$ for $X \neq 0$.

Corollary 21.4. *Let* $K \subset F_\varphi$ *be a compact* σ*-invariant subset. The homomorphism* θ *of Theorem* 21.2 *is a unital* $*$*-homomorphism from* $C_{\rm alg}(S^3_\varphi)$ *to the crossed product* $C^\infty(K) \times_{\sigma,\,\mathcal{L}} \mathbb{Z}$.

It follows that one obtains a non-trivial C^*-algebra $C^*(S^3_\varphi)$ as the completion of $C_{\rm alg}(S^3_\varphi)$ for the semi-norm

$$\|P\| := \sup \|\,\pi(P)\|\,, \tag{21.14}$$

where π varies over all unitary representations of $C_{\rm alg}(S^3_\varphi)$. It was clear from the start that (21.14) defines a finite C^*-semi-norm on $C_{\rm alg}(S^3_\varphi)$ since the equation of the sphere $\sum x_\mu^2 = 1$ together with the self-adjointness $x_\mu = x_\mu^*$ show that in any unitary representation one has

$$\|\,\pi(x_\mu)\| \leq 1\,, \quad \forall\, \mu\,.$$

What the above corollary gives is a lower bound for the C^*-norm such as that given by statement (ii) of Theorem 21.2 on the linear subspace V of generators.

The correspondence σ on F_φ is, for generic φ, a translation of modulus η of the elliptic curve F_φ and one distinguishes two cases: the *even* case when it preserves the two real components of the curve $F_\varphi \cap P_3(\mathbb{R})$ and the odd case when it permutes them.

Proposition 21.5. *Let φ be generic and even.*
(i) *The crossed product $C(F_\varphi) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ is isomorphic to the mapping torus of the automorphism β of the noncommutative torus $\mathbb{T}^2_\eta = C_\varphi \times_\sigma \mathbb{Z}$ acting on the generators by the matrix $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$.*
(ii) *The crossed product $F_\varphi \times_{\sigma, \mathcal{L}} \mathbb{Z}$ is a noncommutative 3-manifold with an elliptic action of the three-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ and an invariant trace τ.*

It follows that one is exactly in the framework developed in [59]. We refer to [187] and [2] where these noncommutative manifolds were analyzed in terms of crossed products by Hilbert C^*-bimodules.

Integration on the translation invariant volume form dv of F_φ gives the $\mathfrak{h}_3$-invariant trace τ,

$$\tau(f) = \int f dv\,, \quad \forall\, f \in C^\infty(F_\varphi)\,,$$

$$\tau(\xi\, W^k) = \tau(W^{*k}\, \eta^*) \;=\; 0\,, \quad \forall\, k \neq 0\,. \tag{21.15}$$

It follows in particular that the results of [59] apply to obtain the calculus. In particular the following gives the "fundamental class" as a 3-cyclic cocycle,

$$\tau_3(a_0,\, a_1,\, a_2,\, a_3) = \sum \epsilon_{ijk}\, \tau(a_0\, \delta_i(a_1)\, \delta_j(a_2)\, \delta_k(a_3))\,, \tag{21.16}$$

where the δ_j are the generators of the action of $\mathfrak{h}_3$.

The relation between the noncommutative spheres S^3_φ and the noncommutative nil manifolds $F_\varphi \times_{\sigma, \mathcal{L}} \mathbb{Z}$ is analyzed in [73], [74] thanks to the computation of the Jacobian of the homomorphism θ.

22. $\mathbb{Q}$-lattices

A class of examples of noncommutative spaces of relevance to number theory is given by the moduli spaces of $\mathbb{Q}$-lattices up to commensurability. These fall within the general framework of noncommutative spaces obtained as quotients of equivalence relations discussed in Section 4.

A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ consists of a pair (Λ, ϕ) of a lattice $\Lambda \subset \mathbb{R}^n$ (a cocompact free abelian subgroup of $\mathbb{R}^n$ of rank n) together with a system of labels of its torsion points given by a homomorphism of abelian groups

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda\,. \tag{22.1}$$

Two $\mathbb{Q}$-lattices are commensurable,

$$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)\,,$$

iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and

$$\phi_1 \equiv \phi_2 \mod \Lambda_1 + \Lambda_2\,.$$

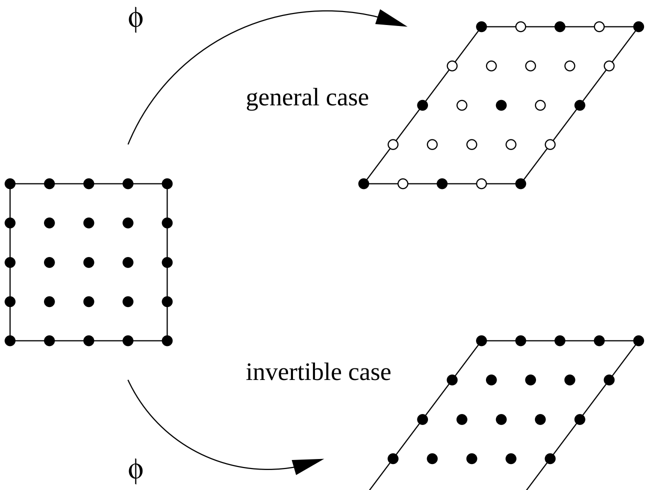


Figure 15. $\mathbb{Q}$-lattices: generic and invertible case.

In general, the map ϕ of (22.1) is just a group homomorphism. A $\mathbb{Q}$-lattice is said to be *invertible* if ϕ is an isomorphism. Two invertible $\mathbb{Q}$-lattices are commensurable if and only if they are equal.

In this section we denote by $\mathcal{L}_n$ the space of commensurabilty classes of $\mathbb{Q}$-lattices in $\mathbb{R}^n$ (not to be confused with the unrelated notation $\mathcal{L}_n$ used above in (21.8)). The space $\mathcal{L}_n$ has the typical property of noncommutative spaces: it has the cardinality of the continuum but one cannot construct a countable collection of measurable functions that separate points of $\mathcal{L}_n$. Thus, one can use noncommutative geometry to describe the quotient space $\mathcal{L}_n$ through a noncommutative C^*-algebra $C^*(\mathcal{L}_n)$.

We consider especially the case of $n = 1$ and $n = 2$. One is also interested in the C^*-algebras describing $\mathbb{Q}$-lattices up to scaling, $\mathcal{A}_1 = C^*(\mathcal{L}_1/\mathbb{R}_+^*)$ and $\mathcal{A}_2 = C^*(\mathcal{L}_2/\mathbb{C}^*)$.

In the 1-dimensional case, a $\mathbb{Q}$-lattice can always be written in the form

$$(\Lambda, \phi) = (\lambda\,\mathbb{Z}, \lambda\,\rho) \tag{22.2}$$

for some $\lambda > 0$ and some

$$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}\,. \tag{22.3}$$

By considering lattices up to scaling, we eliminate the factor $\lambda > 0$ so that 1-dimensional $\mathbb{Q}$-lattices up to scale are completely specified by the choice of the element $\rho \in \hat{\mathbb{Z}}$. Thus, the algebra of coordinates of the space of

1-dimensional $\mathbb{Q}$-lattices up to scale is the commutative C^*-algebra

$$C(\hat{\mathbb{Z}}) \simeq C^*(\mathbb{Q}/\mathbb{Z}), \tag{22.4}$$

where we use Pontrjagin duality to get the identification in (22.4).

The equivalence relation of commensurability is implemented by the action of the semigroup $\mathbb{N}^\times$ on $\mathbb{Q}$-lattices. The corresponding action on the algebra (22.4) is by

$$\alpha_n(f)(\rho) = \begin{cases} f(n^{-1}\rho) & \rho \in n\hat{\mathbb{Z}}, \\ 0 & \text{otherwise.} \end{cases} \tag{22.5}$$

Thus, the quotient of the space of 1-dimensional $\mathbb{Q}$-lattices up to scale by the commensurability relation and its algebra of coordinates is given by the semigroup crossed product

$$C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times . \tag{22.6}$$

This is the Bost–Connes C^*-algebra introduced in [27].

It has a natural time evolution given by the covolume of a pair of commensurable $\mathbb{Q}$-lattices. It has symmetries (compatible with the time evolution) given by the group $\hat{\mathbb{Z}}^* = \mathrm{GL}_1(\mathbb{A}_f)/\mathbb{Q}^*$ and the KMS (Kubo–Martin–Schwinger) equilibrium states of the system have interesting arithmetic properties. Namely, the partition function of the system is the Riemann zeta function. There is a unique KMS state for sufficiently high temperature, while at low temperature the system undergoes a phase transition with spontaneous symmetry breaking. The pure phases (estremal KMS states) at low temperature are parameterized by elements in $\hat{\mathbb{Z}}^*$. They have an explicit expression in terms of polylogarithms at roots of unity. At zero temperature the extremal KMS states, evaluated on the elements of a rational subalgebra, assume values that are algebraic numbers. The action on these values of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ factors through its abelianization and is obtained (via the class field theory isomorphism $\hat{\mathbb{Z}}^* \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$) as the action of symmetries on the algebra (*cf.* [27], [77], [78] for details).

In the 2-dimensional case, a $\mathbb{Q}$-lattice can be written in the form

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}\tau), \lambda\rho),$$

for some $\lambda \in \mathbb{C}^*$, some $\tau \in \mathbb{H}$, and some $\rho \in M_2(\hat{\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2)$. Thus, the space of 2-dimensional $\mathbb{Q}$-lattices, up to the scale factor $\lambda \in \mathbb{C}^*$ and up to isomorphisms, is given by

$$M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \bmod \Gamma = \mathrm{SL}_2(\mathbb{Z}). \tag{22.7}$$

The commensurability relation giving the space $\mathcal{L}_2/\mathbb{C}^*$ is implemented by the partially defined action of $\mathrm{GL}_2^+(\mathbb{Q})$.

One considers in this case the quotient of the space

$$\tilde{\mathcal{U}} := \{(g, \rho, \alpha) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathrm{GL}_2^+(\mathbb{R}) \;:\; g\rho \in M_2(\hat{\mathbb{Z}})\} \tag{22.8}$$

by the action of $\Gamma \times \Gamma$ given by

$$(\gamma_1, \gamma_2)\,(g, \rho, \alpha) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 \alpha)\,. \qquad (22.9)$$

The groupoid $\mathcal{R}_2$ of the equivalence relation of commensurability on 2-dimensional $\mathbb{Q}$-lattices (not considered up to scaling for the moment) is a locally compact groupoid, which can be parameterized by the quotient of (22.8) by $\Gamma \times \Gamma$ via the map $r : \tilde{\mathcal{U}} \to \mathcal{R}_2$,

$$r(g, \rho, \alpha) = \big((\alpha^{-1} g^{-1} \Lambda_0, \alpha^{-1}\rho), (\alpha^{-1}\Lambda_0, \alpha^{-1}\rho)\big)\,. \qquad (22.10)$$

We then consider the quotient by scaling. The quotient $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{C}^*$ can be identified with the hyperbolic plane $\mathbb{H}$ in the usual way. If (Λ_k, ϕ_k), for $k = 1, 2$, are a pair of commensurable 2-dimensional $\mathbb{Q}$-lattices, then for any $\lambda \in \mathbb{C}^*$, the $\mathbb{Q}$-lattices $(\lambda\Lambda_k, \lambda\phi_k)$ are also commensurable, with

$$r(g, \rho, \alpha\lambda^{-1}) = \lambda r(g, \rho, \alpha)\,.$$

However, the action of $\mathbb{C}^*$ on $\mathbb{Q}$-lattices is not free due to the presence of lattices (such as Λ_0 above) with nontrivial automorphisms. Thus, the quotient $Z = \mathcal{R}_2/\mathbb{C}^*$ is no longer a groupoid. Still, one can define a convolution algebra for Z by restricting the convolution product of $\mathcal{R}_2$ to homogeneous functions of weight zero, where a function f has weight k if it satisfies

$$f(g, \rho, \alpha\lambda) = \lambda^k f(g, \rho, \alpha), \quad \forall\, \lambda \in \mathbb{C}^*\,.$$

The space Z is the quotient of the space

$$\mathcal{U} := \{(g, \rho, z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H} | g\rho \in M_2(\hat{\mathbb{Z}})\} \qquad (22.11)$$

by the action of $\Gamma \times \Gamma$. Here the space $M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ has a partially defined action of $\mathrm{GL}_2^+(\mathbb{Q})$ given by

$$g(\rho, z) = (g\rho, g(z))\,,$$

where $g(z)$ denotes the action as a fractional linear transformation.

Thus, the algebra of coordinates $\mathcal{A}_2$ for the noncommutative space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices up to scaling is given by the following convolution algebra.

Consider the space $C_c(Z)$ of continuous compactly supported functions on Z. These can be seen, equivalently, as functions on $\mathcal{U}$ as in (22.11) invariant under the $\Gamma \times \Gamma$ action $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 z)$. One endows $C_c(Z)$ with the convolution product

$$(f_1 * f_2)(g, \rho, z) = \sum_{s \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q}) : s\rho \in M_2(\hat{\mathbb{Z}})} f_1(gs^{-1}, s\rho, s(z)) f_2(s, \rho, z) \quad (22.12)$$

and the involution $f^*(g, \rho, z) = \overline{f(g^{-1}, g\rho, g(z))}$.

Again there is a time evolution on this algebra, which is given by the covolume,

$$\sigma_t(f)(g,\rho,z) = \det(g)^{it}\, f(g,\rho,z)\,. \tag{22.13}$$

The partition function for this GL_2 system is given by

$$Z(\beta) = \sum_{m\in\Gamma\backslash M_2^+(\mathbb{Z})} \det(m)^{-\beta} = \sum_{k=1}^{\infty} \sigma(k)\, k^{-\beta} = \zeta(\beta)\zeta(\beta-1)\,, \tag{22.14}$$

where $\sigma(k) = \sum_{d|k} d$. The form of the partition function suggests the possibility that two distinct phase transitions might happen at $\beta = 1$ and $\beta = 2$.

The structure of KMS states for this system is analysed in [77]. The main result is the following.

Theorem 22.1. *The KMS$_\beta$ states of the* GL_2*-system have the following properties*:
(1) *In the range $\beta \le 1$ there are no KMS states.*
(2) *In the range $\beta > 2$ the set of extremal KMS states is given by the classical Shimura variety*

$$\mathcal{E}_\beta \cong \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*. \tag{22.15}$$

The symmetries are more complicated than in the Bost–Connes case. In fact, in addition to symmetries given by automorphisms that commute with the time evolution, there are also symmetries by *endomorphisms* that play an important role. The resulting symmetry group is the quotient $\mathrm{GL}_2(\mathbb{A}_f)/\mathbb{Q}^*$. An important result of Shimura [195] shows that this group is in fact the Galois group of the field F of modular functions. The group $\mathrm{GL}_2(\mathbb{A}_f)$ decomposes as a product

$$\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q})\mathrm{GL}_2(\hat{\mathbb{Z}})\,, \tag{22.16}$$

where $\mathrm{GL}_2(\hat{\mathbb{Z}})$ acts by automorphisms related to the deck transformations of the tower of the modular curves, while $\mathrm{GL}_2^+(\mathbb{Q})$ acts by endomorphisms that move across levels in the modular tower.

The modular field F is the field of modular functions over $\mathbb{Q}^{ab}$, namely the union of the fields F_N of modular functions of level N rational over the cyclotomic field $\mathbb{Q}(\zeta_n)$, that is, such that the q-expansion in powers of $q^{1/N} = \exp(2\pi i\tau/N)$ has all coefficients in $\mathbb{Q}(e^{2\pi i/N})$.

The action of the Galois group $\hat{\mathbb{Z}}^* \simeq \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ on the coefficients determines a homomorphism

$$\mathrm{cycl} : \hat{\mathbb{Z}}^* \to \mathrm{Aut}(F)\,. \tag{22.17}$$

If $\tau \in \mathbb{H}$ is a generic point, then the evaluation map $f \mapsto f(\tau)$ determines an embedding $F \hookrightarrow \mathbb{C}$. We denote by F_τ the image in $\mathbb{C}$. This yields an identification

$$\theta_\tau : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_f)\,. \tag{22.18}$$

There is an arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$ (defined over $\mathbb{Q}$) of unbounded multipliers of the C^*-algebra $\mathcal{A}_2$, obtained by considering continuous functions on Z (*cf.* (22.11)), with finite support in the variable $g \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})$ and with the following properties. Let $p_N : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ be the canonical projection. With the notation $f_{(g,\rho)}(z) = f(g,\rho,z)$, we say that $f_{(g,\rho)} \in C(\mathbb{H})$ is of level N if

$$f_{(g,\rho)} = f_{(g,p_N(\rho))} \qquad \forall\,(g,\rho)\,.$$

We require that elements of $\mathcal{A}_{2,\mathbb{Q}}$ have the $f_{(g,\rho)}$ of finite level with $f_{(g,m)} \in F$ for all (g,m). We also require that the action (22.17) on the coefficients of the q-expansion of the $f_{(g,m)}$ satisfies

$$f_{(g,\alpha(u)m)} = \mathrm{cycl}(u)\, f_{(g,m)}\,,$$

for all $g \in \mathrm{GL}_2^+(\mathbb{Q})$ diagonal and all $u \in \hat{\mathbb{Z}}^*$, with

$$\alpha(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

to avoid some "trivial" elements that would spoil the Galois action on values of states (*cf.* [77], [78]). The action of symmetries extends to $\mathcal{A}_{2,\mathbb{Q}}$. We have then the following result ([77]):

Theorem 22.2. *Consider a state $\varphi = \varphi_{\infty,L} \in \mathcal{E}_\infty$, for a generic invertible $\mathbb{Q}$-lattice $L = (\rho,\tau)$. Then the values of the state on elements of the arithmetic subalgebra generate the image in $\mathbb{C}$ of the modular field,*

$$\varphi(\mathcal{A}_{2,\mathbb{Q}}) \subset F_\tau\,, \tag{22.19}$$

and the isomorphism

$$\theta_\varphi : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_f)\,, \tag{22.20}$$

given by

$$\theta_\varphi(\gamma) = \rho^{-1}\,\theta_\tau(\gamma)\,\rho\,, \tag{22.21}$$

for θ_τ as in (22.18), *intertwines the Galois action on the values of the state with the action of symmetries,*

$$\gamma\,\varphi(f) = \varphi(\theta_\varphi(\gamma)f)\,, \qquad \forall\, f \in \mathcal{A}_{2,\mathbb{Q}}, \quad \forall\,\gamma \in \mathrm{Gal}(F_\tau/\mathbb{Q})\,. \tag{22.22}$$

A notion analogous to that of $\mathbb{Q}$-lattices can be given for other number fields $\mathbb{K}$. This notion was used in [81] to construct a quantum statistical mechanical system for $\mathbb{K}$ an imaginary quadratic field. This system shares properties with both the Bost–Connes system of [27] and the GL_2 system (2-dimensional $\mathbb{Q}$-lattices) of [77].

We assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$, d a positive integer. Let $\tau \in \mathbb{H}$ be such that $\mathbb{K} = \mathbb{Q}(\tau)$ and $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ is the ring of integers of $\mathbb{K}$.

A 1-dimensional $\mathbb{K}$-lattice (Λ, ϕ) is a finitely generated $\mathcal{O}$-submodule $\Lambda \subset \mathbb{C}$, such that

$$\Lambda \otimes_{\mathcal{O}} \mathbb{K} \cong \mathbb{K}\,,$$

together with a morphism of $\mathcal{O}$-modules

$$\phi : \mathbb{K}/\mathcal{O} \to \mathbb{K}\Lambda/\Lambda\,. \tag{22.23}$$

A 1-dimensional $\mathbb{K}$-lattice is *invertible* if ϕ is an isomorphism of $\mathcal{O}$-modules. A 1-dimensional $\mathbb{K}$-lattice is, in particular, a 2-dimensional $\mathbb{Q}$-lattice.

We consider the notion of commensurability as in the case of $\mathbb{Q}$-lattices. Two 1-dimensional $\mathbb{K}$-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable if $\mathbb{K}\Lambda_1 = \mathbb{K}\Lambda_2$ and $\phi_1 \equiv \phi_2 \bmod \Lambda_1 + \Lambda_2$. In particular, two 1-dimensional $\mathbb{K}$-lattices are commensurable iff the underlying $\mathbb{Q}$-lattices are commensurable.

The algebra of the corresponding noncommutative space is a restriction of the algebra of the GL_2-system to the subgroupoid of the equivalence of commensurability restricted to $\mathbb{K}$-lattices. The time evolution is also a restriction from the GL_2-system.

The resulting system has partition function the Dedekind zeta function $\zeta_{\mathbb{K}}(\beta)$ of the number field $\mathbb{K}$. Above the critical temperature $T = 1$ there is a unique KMS state, while at lower temperatures the extremal KMS states are parameterized by elements of $\mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^*$, where $\mathbb{A}_K = \mathbb{A}_{K,f} \times \mathbb{C}$ are the adèles of $\mathbb{K}$, with $\mathbb{A}_{K,f} = \mathbb{A}_f \otimes \mathbb{K}$. The KMS states at zero temperature, evaluated on the restriction to $\mathbb{K}$-lattices of the arithmetic algebra of the GL_2-system, have an action of the Galois group $\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$ realized (via the class field theory isomorphism) through the action of symmetries (automorphisms and endomorphisms) of the system (*cf.* [81]).

23. Modular Hecke Algebras

Connes and Moscovici [87] defined modular Hecke algebras $\mathcal{A}(\Gamma)$ of level Γ, a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. These extend both the ring of classical Hecke operators and the algebra of modular forms.

Modular Hecke algebras encode two *a priori* unrelated structures on modular forms, namely the algebra structure given by the pointwise product on one hand, and the action of the Hecke operators on the other. To any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ corresponds a crossed product algebra $\mathcal{A}(\Gamma)$, the *modular Hecke algebra* of level Γ, which is a direct extension of both the ring of classical Hecke operators and of the algebra $\mathcal{M}(\Gamma)$ of Γ-modular forms.

These algebras can be obtained by considering the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on the algebra of modular forms on the full (adelic) modular tower, which yields the "holomorphic part" of the "ring of functions" of the noncommutative space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices, introduced in Section 22.

With $\mathcal{M}$ denoting the algebra of modular forms of arbitrary level, the elements of $\mathcal{A}(\Gamma)$ are maps with finite support

$$F : \Gamma\backslash \mathrm{GL}^+(2,\mathbb{Q}) \to \mathcal{M}\,, \qquad \Gamma\alpha \mapsto F_\alpha \in \mathcal{M}\,,$$

satisfying the covariance condition

$$F_{\alpha\gamma} = F_\alpha|\gamma\,, \quad \forall\, \alpha \in \mathrm{GL}^+(2,\mathbb{Q}), \gamma \in \Gamma$$

and their product is given by convolution.

More in detail, let $G = \mathrm{PGL}_2^+(\mathbb{Q})$ and let $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ be a finite index subgroup. The quotient map $\Gamma\backslash G \to \Gamma\backslash G/\Gamma$ is finite to one, and Γ acts on $\mathbb{C}[\Gamma\backslash G]$. Let $\mathcal{H}_k$ be the space of holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ with polynomial growth, and with the action $|_k$, for $k \in 2\mathbb{Z}$, of $\mathrm{PGL}_2^+(\mathbb{R})$ of the form

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)(z) = \frac{(ad-bc)^{k/2}}{(cz+d)^k}\, f\left(\frac{az+b}{cz+d}\right).$$

This determines induced actions of G and Γ on $\mathcal{H}_k$. The space of modular forms is obtained as $\mathcal{M}_k(\Gamma) = \mathcal{H}_k^\Gamma$, the invariants of this action.

One can then define

$$\mathcal{A}_k(\Gamma) := \left(\mathbb{C}[\Gamma\backslash G] \otimes_{\mathbb{C}} \mathcal{H}_k\right)^\Gamma\,, \tag{23.1}$$

with respect to the right action of Γ,

$$\gamma : \sum_i (\Gamma g_i) \otimes f_i \mapsto \sum_i \Gamma g_i \gamma \otimes (f_i|_k\gamma)\,.$$

One considers the graded vector space $\mathcal{A}_*(\Gamma) = \oplus_k \mathcal{A}_k(\Gamma)$. The elements of $\mathcal{A}_k(\Gamma)$ can be thought of as finitely supported Γ-equivariant maps

$$\phi : \Gamma\backslash G \to \mathcal{H}_k\,, \qquad \sum_i (\Gamma g_i) \otimes f_i \mapsto f_i\,.$$

We can embed

$$\mathcal{A}_*(\Gamma) \subset \hat{\mathcal{A}}_*(\Gamma) := \mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma\backslash G], \mathcal{H}_*)\,,$$

where we think of $\mathcal{A}_*(\Gamma) = \mathcal{H}_*[\Gamma\backslash G]$ as polynomials in $\Gamma\backslash G$ with $\mathcal{H}_*$ coefficients, and of $\hat{\mathcal{A}}_*(\Gamma) = \mathcal{H}_*[[\Gamma\backslash G]]$ as formal power series, that is, Γ-equivariant maps $\phi : \Gamma\backslash G \to \mathcal{H}_k$. There is on $\mathcal{A}_*(\Gamma)$ an associative multiplication (*cf.* [87]), which makes $\mathcal{A}_*(\Gamma)$ into a noncommutative ring. This is given by a convolution product. For any $\phi \in \mathcal{A}_k(\Gamma)$, we have $\phi_g = \phi_{\gamma g}$, with $\phi_g = 0$ off a finite subset of $\Gamma\backslash G$, and $\phi_g|\gamma = \phi_{g\gamma}$, so these terms are left Γ-invariant and right Γ-equivariant. For $\phi \in \mathcal{A}_k(\Gamma)$ and $\psi \in \mathcal{A}_\ell(\Gamma)$ we then define the convolution product as

$$(\phi * \psi)_g := \sum_{(g_1,g_2)\in G\times_\Gamma G, g_1g_2=g} (\phi_{g_1}|g_2)\phi_{g_2}\,. \tag{23.2}$$

The algebra $\mathcal{A}_*(\Gamma)$ constructed in this way has two remarkable subalgebras.

- $\mathcal{A}_0(\Gamma) = \mathbb{C}[\Gamma\backslash G/\Gamma]$ is the algebra $\mathbb{T}$ of Hecke operators.
- $\mathcal{M}_k(\Gamma) \subset \mathcal{A}_k(\Gamma)$ also gives a subalgebra $\mathcal{M}_*(\Gamma) \subset \mathcal{A}_*(\Gamma)$.

In particular observe that all the coefficients ϕ_g are modular forms. In fact, they satisfy $\phi_g|\gamma = \phi_{g\gamma}$. Thus, for $\gamma \in \Gamma$, this gives $\phi_g|\gamma = \phi_g$.

Notice, however, that the convolution product on $\mathcal{A}_*(\Gamma)$ does not agree with the Hecke action, namely the diagram

$$\begin{tikzcd} \mathcal{M}_*(\Gamma) \otimes \mathbb{T} \arrow[r, "H"] \arrow[d, "\iota"] & \mathcal{M}_*(\Gamma) \arrow[d, "\iota"] \\ \mathcal{A}_*(\Gamma) \otimes \mathcal{A}_*(\Gamma) \arrow[r, "*"] & \mathcal{A}_*(\Gamma)\,, \end{tikzcd}$$

with ι the inclusion of subalgebras and H the Hecke action, is *not* commutative, nor is the symmetric one

$$\begin{tikzcd} \mathbb{T} \otimes \mathcal{M}_*(\Gamma) \arrow[r, "H"] \arrow[d, "\iota"] & \mathcal{M}_*(\Gamma) \arrow[d, "\iota"] \\ \mathcal{A}_*(\Gamma) \otimes \mathcal{A}_*(\Gamma) \arrow[r, "*"] & \mathcal{A}_*(\Gamma)\,. \end{tikzcd}$$

To get the correct Hecke action on modular forms from the algebra $\mathcal{A}_*(\Gamma)$, one needs to introduce the augmentation map

$$\epsilon : \mathbb{C}[\Gamma\backslash G] \to \mathbb{C}$$

extended to a map

$$\epsilon \otimes 1 : \mathbb{C}[\Gamma\backslash G] \otimes \mathcal{H}_k \to \mathcal{H}_k, \qquad \sum [g] \otimes \phi_g \mapsto \sum \phi_g\,.$$

One then obtains a commutative diagram

$$\begin{tikzcd} \mathbb{T} \otimes \mathcal{M}_*(\Gamma) \arrow[r, "H"] \arrow[d, "\iota"] & \mathcal{M}_*(\Gamma) \\ \mathcal{A}_*(\Gamma) \otimes \mathcal{A}_*(\Gamma) \arrow[r, "*"] & \mathcal{A}_*(\Gamma)\,. \arrow[u, "\epsilon\otimes 1"'] \end{tikzcd}$$

In [85] Connes and Moscovici introduced a Hopf algebra $\mathcal{H}_1$ associated to the transverse geometry of codimension one foliations. This is the universal enveloping algebra of a Lie algebra with basis $\{\mathcal{X}, \mathcal{Y}, \delta_n\, n \geq 1\}$ satisfying, for $n, k, \ell \geq 1$,

$$[\mathcal{Y}, \mathcal{X}] = \mathcal{X}, \;\; [\mathcal{Y}, \delta_n] = n\,\delta_n, \;\; [\mathcal{X}, \delta_n] = \delta_{n+1}, \;\; [\delta_k, \delta_\ell] = 0\,, \tag{23.3}$$

with coproduct an algebra homomorphism $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ satisfying

$$\begin{aligned} \Delta\mathcal{Y} &= \mathcal{Y} \otimes 1 + 1 \otimes \mathcal{Y}\,, \\ \Delta\delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1\,, \\ \Delta\mathcal{X} &= \mathcal{X} \otimes 1 + 1 \otimes \mathcal{X} + \delta_1 \otimes \mathcal{Y}\,, \end{aligned} \tag{23.4}$$

antipode the anti-isomorphism satisfying

$$S(\mathcal{Y}) = -\mathcal{Y}\,, \quad S(\mathcal{X}) = -\mathcal{X} + \delta_1 \mathcal{Y}\,, \quad S(\delta_1) = -\delta_1\,, \tag{23.5}$$

and co-unit $\epsilon(h)$ the constant term of $h \in \mathcal{H}_1$.

The Hopf algebra $\mathcal{H}_1$ acts as symmetries of the modular Hecke algebras. This is a manifestation of the general fact that, while symmetries of ordinary commutative spaces are encoded by group actions, symmetries of noncommutative spaces are given by Hopf algebras.

By comparing the actions of the Hopf algebra $\mathcal{H}_1$, it is possible to derive an analogy (*cf.* [87]) between the modular Hecke algebras and the crossed product algebra of the action of a discrete subgroup of $\mathrm{Diff}(S^1)$ on polynomial functions on the frame bundle of S^1.

In fact, for Γ a discrete subgroup of $\mathrm{Diff}(S^1)$, and X a smooth compact 1-dimensional manifold, consider as in [87] the algebra

$$\mathcal{A}_\Gamma = C_c^\infty(J_+^1(X)) \rtimes \Gamma\,, \tag{23.6}$$

where $J_+^1(X)$ is the oriented 1-jet bundle. This has an action of the Hopf algebra $\mathcal{H}_1$ by

$$\begin{aligned} \mathcal{Y}(fU_\phi^*) &= y_1 \frac{\partial f}{\partial y_1} U_\phi^*\,, \\ \mathcal{X}(fU_\phi^*) &= y_1 \frac{\partial f}{\partial y} U_\phi^*\,, \\ \delta_n(fU_\phi^*) &= y_1^n \frac{d^n}{dy^n}\left(\log \frac{d\phi}{dy}\right) fU_\phi^*\,, \end{aligned} \tag{23.7}$$

with coordinates (y, y_1) on $J_+^1(X) \simeq X \times \mathbb{R}^+$. The trace τ defined by the volume form

$$\tau(fU_\phi^*) = \begin{cases} \displaystyle\int_{J_+^1(X)} f(y, y_1)\, \frac{dy \wedge dy_1}{y_1^2} & \phi = 1 \\ 0 & \phi \neq 1 \end{cases} \tag{23.8}$$

satisfies

$$\tau(h(a)) = \nu(h)\tau(a) \quad \forall\, h \in \mathcal{H}_1\,, \tag{23.9}$$

with $\nu \in \mathcal{H}_1^*$ satisfying

$$\nu(\mathcal{Y}) = 1\,, \quad \nu(\mathcal{X}) = 0\,, \quad \nu(\delta_n) = 0\,. \tag{23.10}$$

The twisted antipode $\tilde{S} = \nu * S$ satisfies $\tilde{S}^2 = 1$ and

$$\tilde{S}(\mathcal{Y}) = -\mathcal{Y} + 1\,, \quad \tilde{S}(\mathcal{X}) = -\mathcal{X} + \delta_1 \mathcal{Y}\,, \quad \tilde{S}(\delta_1) = -\delta_1\,. \tag{23.11}$$

The Hopf cyclic cohomology of a Hopf algebra is another fundamental tool in noncommutative geometry, which was developed by Connes and Moscovici in [85]. They applied it to the computation of the local index formula for tranversely hypoelliptic operators on foliations. An action of a Hopf algebra on an algebra induces a characteristic map from the Hopf cyclic cohomology of the Hopf algebra to the cyclic cohomology of the algebra, hence the index computation can be done in terms of Hopf cyclic cohomology. The periodic Hopf cyclic cohomology of the Hopf algebra of transverse geometry is related to the Gelfand-Fuchs cohomology of the Lie algebra of formal vector fields [86].

In the case of the Hopf algebra $\mathcal{H}_1$, there are three basic cyclic cocycles, which in the original context of transverse geometry correspond, respectively, to the Schwarzian derivative, the Godbillon–Vey class, and the transverse fundamental class.

In particular, the Hopf cyclic cocycle associated to the Schwarzian derivative is of the form

$$\delta_2' := \delta_2 - \frac{1}{2}\delta_1^2 \tag{23.12}$$

with

$$\delta_2'(fU_\phi^*) = y_1^2\,\{\phi(y);y\}\;fU_\phi^*\,, \tag{23.13}$$

where

$$\{F;x\} := \frac{d^2}{dx^2}\left(\log\frac{dF}{dx}\right) - \frac{1}{2}\left(\frac{d}{dx}\left(\log\frac{dF}{dx}\right)\right)^2. \tag{23.14}$$

The action of the Hopf algebra $\mathcal{H}_1$ on the modular Hecke algebra described in [87] involves the natural derivation on the algebra of modular forms initially introduced by Ramanujan, which corrects the ordinary differentiation by a logarithmic derivative of the Dedekind η-function,

$$\mathcal{X} := \frac{1}{2\pi i}\frac{d}{dz} - \frac{1}{2\pi i}\frac{d}{dz}(\log\eta^4)\;\mathcal{Y}, \quad \mathcal{Y}(f) = \frac{k}{2}\,f,\ \ \forall\, f\in\mathcal{M}_k\,. \tag{23.15}$$

The element $\mathcal{Y}$ is the grading operator that multiplies by $k/2$ forms of weight k, viewed as sections of the $(k/2)$th power of the line bundle of 1-forms. The element δ_1 acts as multiplication by a form-valued cocycle on $\mathrm{GL}_2^+(\mathbb{Q})$, which measures the lack of invariance of the section $\eta^4 dz$. More precisely, one has the following action of $\mathcal{H}_1$ (Connes–Moscovici [87]):

Theorem 23.1. *There is an action of the Hopf algebra $\mathcal{H}_1$ on the modular Hecke algebra $\mathcal{A}(\Gamma)$ of level Γ, induced by an action on $\mathcal{A}_{G^+(\mathbb{Q})} := \mathcal{M}\rtimes G^+(\mathbb{Q})$, for $\mathcal{M} = \varinjlim_{N\to\infty}\mathcal{M}(\Gamma(N))$, of the form*

$$\begin{aligned}
\mathcal{Y}(fU_\gamma^*) &= \mathcal{Y}(f)\,U_\gamma^*\,,\\
\mathcal{X}(fU_\gamma^*) &= \mathcal{X}(f)\,U_\gamma^*\,,\\
\delta_n(fU_\gamma^*) &= \frac{d^n}{dZ^n}\left(\log\frac{d(Z|_0\gamma)}{dZ}\right)(dZ)^n\,fU_\gamma^*\,,
\end{aligned} \tag{23.16}$$

with $\mathcal{X}(f)$ *and* $\mathcal{Y}(f)$ *as in* (23.15), *and*

$$Z(z) = \int_{i\infty}^{z} \eta^4 dz\,. \tag{23.17}$$

The cocycle (23.12) associated to the Schwarzian derivative is represented by an inner derivation of $\mathcal{A}_{G^+(\mathbb{Q})}$,

$$\delta_2'(a) = [a, \omega_4]\,, \tag{23.18}$$

where ω_4 is the weight four modular form

$$\omega_4 = -\frac{E_4}{72}\,, \quad \text{with} \quad E_4(q) = 1 + 240 \sum_{n=1}^{\infty} n^3 \frac{q^n}{1-q^n}\,, \quad q = e^{2\pi i z}\,, \tag{23.19}$$

which is expressed as a Schwarzian derivative

$$\omega_4 = (2\pi i)^{-2} \{Z; z\}\,. \tag{23.20}$$

This result is used in [87] to investigate perturbations of the Hopf algebra action. The freedom one has in modifying the action by a 1-cocycle corresponds exactly to the data introduced by Zagier in [213], defining "canonical" Rankin-Cohen algebras, with the derivation ∂ and the element Φ in Zagier's notation corresponding, respectively, to the action of the generator $\mathcal{X}$ on modular forms and to $\omega_4 = 2\Phi$.

The cocycle associated to the Godbillon-Vey class is described in terms of a 1-cocycle on $\mathrm{GL}_2^+(\mathbb{Q})$ with values in Eisenstein series of weight two, which measures the lack of $\mathrm{GL}_2^+(\mathbb{Q})$-invariance of the connection associated to the generator $\mathcal{X}$. The authors derive from this an arithmetic presentation of the rational Euler class in $H^2(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Q})$ in terms of generalized Dedekind sums.

The cocycle associated to the transverse fundamental class, on the other hand, gives rise to a natural extension of the first Rankin-Cohen bracket [213] from modular forms to the modular Hecke algebras.

Rankin–Cohen algebras can be treated in different perspectives: Zagier introduced them and studied them with a direct algebraic approach (*cf.* [213]). There appears to be an interesting and deep connection to vertex operator algebras, which manifests itself in a form of duality between these two types of algebras.

The Rankin–Cohen brackets are a family of brackets $[f,g]_n^{(k,\ell)}$ for $n \geq 0$, defined for $f, g \in R$, where R is a graded ring with a derivation D. For $R = \oplus_{k\geq 0} R_k$, $D : R_k \to R_{k+2}$, $f \in R_k$ $g \in R_\ell$, the brackets $[\,,\,]_n : R_k \otimes R_\ell \to R_{k+\ell+2}$ are given by

$$[f,g]_n^{(k,\ell)} = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} D^r f\, D^s g\,. \tag{23.21}$$

These Rankin–Cohen brackets induced by $(R_*, D) \Rightarrow (R_*, [\,,\,]_*)$ give rise to a *standard Rankin–Cohen algebra*, in Zagier's terminology (*cf.* [213]). There is an isomorphism of categories between graded rings with a derivation and standard Rankin–Cohen algebras.

In the case of Lie algebras, one can first define a *standard Lie algebra* as the Lie algebra associated to an associative algebra $(A, *) \Rightarrow (A, [\,,\,])$ by setting $[X, Y] = X * Y - Y * X$ and then define an abstract Lie algebra as a structure $(A, [\,,\,])$ that satisfies *all* the algebraic identities satisfied by a standard Lie algebras, though it is not necessarily induced by an associative algebra. It is then a theorem that the antisymmetry of the bracket and the Jacobi identity are sufficient to determine all the other algebraic identities, hence one can take these as a definition of an abstract Lie algebra.

Just as in the case of Lie algebras, we can define a Rankin–Cohen algebra $(R_*, [\,,\,]_*)$ as a graded ring R_* with a family of degree $2n$ brackets $[\,,\,]_n$ satisfying all the algebraic identities of the standard Rankin–Cohen algebra. However, in this case there is no simple set of axioms that implies all the algebraic identities.

The motivation for this structure lies in the fact that there is a very important example of a Rankin–Cohen algebra which is in fact non-standard. The example is provided by modular forms (*cf.* [213]).

If $f \in \mathcal{M}_k$ is a modular form satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)\,,$$

then its derivative is no longer a modular form, due to the presence of the second term in

$$f'\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2} f(z) + kc(cz+d)^{k+1} f(z)\,.$$

On the other hand, if we have $f \in \mathcal{M}_k$ and $g \in \mathcal{M}_\ell$, the bracket

$$[f, g](z) := \ell f'(z) g(z) - k f(z) g'(z)$$

is a modular form in $\mathcal{M}_{k+\ell+2}$. Similarly, we can define an n-th bracket

$$[\,,\,]_n : \mathcal{M}_k \otimes \mathcal{M}_\ell \to \mathcal{M}_{k+\ell+2}\,.$$

Here are the first few brackets:

$$\begin{aligned}
[f, g]_0 &= fg\,, \\
[f, g]_1 &= kfg' - \ell f'g\,, \\
[f, g]_2 &= \binom{k+1}{2} fg'' - (k+1)(\ell+1) f'g' + \binom{\ell+1}{2} f''g\,.
\end{aligned}$$

Notice that for the graded ring of modular forms we have $\mathcal{M}_*(\Gamma) \subset \mathcal{H}$, where $\mathcal{H}$ is the vector space $\mathcal{H} = \mathrm{Hol}(\mathbb{H})_{polyn}$ of holomorphic functions on the upper half-plane $\mathbb{H}$ with polynomial growth. This is closed under differentiation and $(\mathcal{H}, D)$ induces a standard Rankin–Cohen algebra $(\mathcal{H}, [\, ,\,]_*)$. The inclusion $(\mathcal{M}_*, [\, ,\,]_*) \subset (\mathcal{H}, [\, ,\,]_*)$ is not closed under differentiation but it is closed under the brackets.

A way of constructing non-standard Rankin–Cohen algebras is provided by Zagier's *canonical construction* (*cf.* [213]). One considers here the data (R_*, D, Φ), where R_* is a graded ring with a derivation D and with a choice of an element $\Phi \in R_4$, the *curvature*. One then defines the brackets by the formula

$$[f, g]_n^{(k,\ell)} = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f_r\, g_s\,, \tag{23.22}$$

where $f_0 = f$ and

$$f_{r+1} = D f_r + r(r+1)\Phi f_{r-1}\,. \tag{23.23}$$

The structure $(R_*, [\, ,\,]_*)$ obtained this way is a Rankin–Cohen algebra (see [213]).

There is a gauge action on the curvature Φ, namely, for any $\varphi \in R_2$ the transformation $D \mapsto D'$ and $\Phi \mapsto \Phi'$ with

$$D'(f) = D(f) + k\varphi f$$

for $f \in \mathcal{M}_k$ and

$$\Phi' = \Phi + \varphi^2 - D(\varphi) \tag{23.24}$$

give rise to *the same* Rankin–Cohen algebra. Thus, all the cases where the curvature Φ can be gauged away to zero correspond to the standard case.

The modular form ω_4 of (23.20) provides the curvature element $\omega_4 = 2\Phi$, and the gauge equivalence condition (23.24) can be rephrased in terms of Hopf algebras as the freedom to change the $\mathcal{H}_1$ action by a cocycle. In particular (*cf.* [87]), for the specified action, the resulting Rankin–Cohen structure is canonical but not standard, in Zagier's terminology.

The 1-form $dZ = \eta^4 dz$ is, up to scalars, the only holomorphic differential on the elliptic curve $E = X_{\Gamma(6)} \cong X_{\Gamma_0(36)}$ of equation $y^2 = x^3 + 1$, so that $dZ = \frac{dx}{y}$ in Weierstrass coordinates.

The Rankin–Cohen brackets on modular forms can be extended to brackets RC_n on the modular Hecke algebra, defined in terms of the action of the Hopf algebra $\mathcal{H}_1$ of transverse geometry. In fact, more generally, it is shown in [88] that it is possible to define such Rankin–Cohen brackets on any associative algebra $\mathcal{A}$ endowed with an action of the Hopf algebra $\mathcal{H}_1$ for which there exists an element $\Omega \in \mathcal{A}$ such that

$$\delta_2'(a) = [\Omega, a], \quad \forall\, a \in \mathcal{A}\,, \tag{23.25}$$

and with δ_2' as in (23.12), and

$$\delta_n(\Omega) = 0, \;\; \forall\, n \geq 1\,. \tag{23.26}$$

Under these hypotheses, the following result holds (Connes–Moscovici [88]):

Theorem 23.2. *Suppose given an associative algebra $\mathcal{A}$ with an action of the Hopf algebra $\mathcal{H}_1$ satisfying the conditions* (23.25) *and* (23.26)*.*

(1) *There exist Rankin–Cohen brackets RC_n of the form*

$$RC_n(a,b) = \sum_{k=0}^{n} \frac{A_k}{k!}(2\mathcal{Y}+k)_{n-k}(a)\,\frac{B_{n-k}}{(n-k)!}(2\mathcal{Y}+n-k)_k(b)\,, \quad (23.27)$$

with $(\alpha)_r = \alpha(\alpha+1)\cdots(\alpha+r-1)$ and the coefficients $A_{-1}=0$, $A_0=1$, $B_0=1$, $B_1=\mathcal{X}$,

$$A_{n+1} = S(\mathcal{X})A_n - n\Omega^0\left(\mathcal{Y} - \frac{n-1}{2}\right)A_{n-1}\,,$$

$$B_{n+1} = \mathcal{X}B_n - n\Omega\left(\mathcal{Y} - \frac{n-1}{2}\right)B_{n-1}\,,$$

and Ω^0 the right multiplication by Ω.

(2) *When applied to the modular Hecke algebra $\mathcal{A}(\Gamma)$, with $\Omega = \omega_4 = 2\Phi$, the above construction yields brackets* (23.27) *that are completely determined by their restriction to modular forms, where they agree with the Rankin–Cohen brackets* (23.22)*.*

(3) *The brackets* (23.27) *determine associative deformations*

$$a * b = \sum_n \hbar^n\, RC_n(a,b)\,. \quad (23.28)$$

The first of the steps described in Section 2, namely resolving the diagonal in $\mathcal{A}(\Gamma)$, is not yet done for the modular Hecke algebras and should shed light on the important number theoretic problem of the interrelation of the Hecke operators with the algebra structure given by the pointwise product.

The algebra $\mathcal{A}(\Gamma)$ is deeply related to the algebra of the space of two-dimensional $\mathbb{Q}$-lattices of Section 22.

24. Noncommutative Moduli Spaces, Shimura Varieties

It appears from the study of some significant cases that an important source of interesting noncommutative spaces is provided by the "boundary" of classical (algebro-geometric) moduli spaces, when one takes into account the possible presence of degenerations of classical algebraic varieties that give rise to objects no longer defined within the context of algebraic varieties, but which still make sense as noncommutative spaces.

An example of algebro-geometric moduli spaces which is sufficiently simple to describe but which at the same time exhibits a very rich structure is that of the modular curves. The geometry of modular curves has already appeared behind our discussion of the 2-dimensional $\mathbb{Q}$-lattices and of the modular Hecke algebras, through an associated class of functions: the modular functions that appeared in our discussion of the arithmetic algebra for the quantum statistical mechanical system of 2-dimensional $\mathbb{Q}$-lattices and the modular forms in the modular Hecke algebras.

The modular curves, quotients of the hyperbolic plane $\mathbb{H}$ by the action of a subgroup Γ of finite index of $\mathrm{SL}_2(\mathbb{Z})$, are complex algebraic curves, which admit an arithmetic structure, as they are defined over cyclotomic number fields $\mathbb{Q}(\zeta_N)$. They are also naturally moduli spaces. The object they parameterize are elliptic curves (with some level structure). The modular curves have an algebro-geometric compactification obtained by adding finitely many cusp points, given by the points in $\mathbb{P}^1(\mathbb{Q})/\Gamma$. These correspond to the algebro-geometric degeneration of the elliptic curve to $\mathbb{C}^*$. However, in addition to these degenerations, one can consider degenerations to noncommutative tori, obtained by a limit $q \to \exp(2\pi i\theta)$ in the modulus $q = \exp(2\pi i\tau)$ of the elliptic curve, where now θ is allowed to be also irrational. The resulting boundary $\mathbb{P}^1(\mathbb{R})/\Gamma$ is a noncommutative space (in the sense of Section 4). It appeared in the string theory compactifications considered in [71]. The arithmetic properties of the noncommutative spaces $\mathbb{P}^1(\mathbb{R})/\Gamma$ were studied in [156], [159] [160].

The modular curves, for varying finite index $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, form a tower of branched coverings. The projective limit of this tower sits as a connected component in the more refined *adelic* version of the modular tower, given by the quotient

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*\,, \tag{24.1}$$

where, as usual, $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ denotes the adèles of $\mathbb{Q}$, with $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ the finite adèles.

The space (24.1) is also a moduli space. In fact, it belongs to an important class of algebro-geometric moduli spaces of great arithmetic significance, the Shimura varieties $Sh(G, X)$, where the data (G, X) are given by a reductive algebraic group G and a Hermitian symmetric domain X. The pro-variety (24.1) is the Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})$, where $\mathbb{H}^{\pm} = \mathrm{GL}_2(\mathbb{R})/\mathbb{C}^*$ is the union of the upper and lower half-planes in $\mathbb{P}^1(\mathbb{C})$.

We have mentioned above that the spaces $\mathbb{P}^1(\mathbb{R})/\Gamma$ describe degenerations of elliptic curves to noncommutative tori. This type of degeneration corresponds to degenerating a lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ to a pseudolattice $L = \mathbb{Z} + \mathbb{Z}\theta$ (see [152] for a detailed discussion of this viewpoint and its implications in noncommutative geometry and in arithmetic). In terms of the space (24.1), it corresponds to degenerating the archimedean component, namely replacing $\mathrm{GL}_2(\mathbb{R})$ by $M_2(\mathbb{R})^{\cdot} = M_2(\mathbb{R}) \smallsetminus \{0\}$, nonzero 2×2 matrices. However, when one is working with the adelic description as in

(24.1), one can equally consider the possibility of degenerating a lattice at the non-archimedean components. This brings back directly the notion of $\mathbb{Q}$-lattices of Section 22.

In fact, it was shown in [82] that the notions of 2-dimensional $\mathbb{Q}$-lattices and commensurability can be reformulated in terms of Tate modules of elliptic curves and isogeny. In these terms, the space of $\mathbb{Q}$-lattices corresponds to non-archimedean degenerations of the Tate module, which correspond to the "bad quotient"

$$\mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})/\mathbb{C}^* . \tag{24.2}$$

The combination of these two types of degenerations yields a "noncommutative compactification" of the Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})$ which is the algebra of the "bad quotient"

$$\mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A})^{\cdot}/\mathbb{C}^* , \tag{24.3}$$

where $M_2(\mathbb{A})^{\cdot}$ are the elements of $M_2(\mathbb{A})$ with nonzero archimedean component. One recovers the Shimura variety $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})$ as the set of classical points (extremal KMS states at zero temperature) of the quantum statistical mechanical system associated to the noncommutative space (24.2) (*cf.* [77], [82]).

More generally, Shimura varieties are moduli spaces for certain types of motives or arise as moduli spaces of Hodge structures (*cf. e.g.* [167]). A Hodge structure (W, h) is a pair of a finite dimensional $\mathbb{Q}$-vector space W and a homomorphism $h : \mathbb{S} \to \mathrm{GL}(W_{\mathbb{R}})$, of the real algebraic group $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, with $W_{\mathbb{R}} = W \otimes \mathbb{R}$. This determines a decomposition $W_{\mathbb{R}} \otimes \mathbb{C} = \oplus_{p,q} W^{p,q}$ with $\overline{W^{p,q}} = W^{q,p}$ and $h(z)$ acting on $W^{p,q}$ by $z^{-p}\bar{z}^{-q}$. This gives a Hodge filtration and a weight filtration $W_{\mathbb{R}} = \oplus_k W_k$ where $W_k = \oplus_{p+q=m} W^{p,q}$. The Hodge structure (W, h) has weight m if $W_{\mathbb{R}} = W_m$. It is rational if the weight filtration is defined over $\mathbb{Q}$. A Hodge structure of weight m is polarized if there is a morphism of Hodge structures $\psi : W \otimes W \to \mathbb{Q}(-m)$, such that $(2\pi i)^m \psi(\cdot, h(i)\cdot)$ is symmetric and positive definite. Here $\mathbb{Q}(m)$ is the rational Hodge structure of weight $-2m$, with $W = (2\pi i)^m \mathbb{Q}$ with the action $h(z) = (z\bar{z})^m$. For a rational (W, h), the subspace of $W \otimes \mathbb{Q}(m)$ fixed by the $h(z)$, for all $z \in \mathbb{C}^*$, is the space of "Hodge cycles".

One can then view Shimura varieties $Sh(G, X)$ as moduli spaces of Hodge structures in the following way. Let (G, X) be a Shimura datum and $\rho : G \to \mathrm{GL}(V)$ a faithful representation. Since G is reductive, there is a finite family of tensors τ_i such that

$$G = \{g \in \mathrm{GL}(V) : g\tau_i = \tau_i\} . \tag{24.4}$$

A point $x \in X$ is by construction a $G(\mathbb{R})$ conjugacy class of morphisms $h_x : \mathbb{S} \to G$, with suitable properties.

Consider data of the form $((W,h),\{s_i\},\phi)$, where (W,h) is a rational Hodge structure, $\{s_i\}$ a finite family of Hodge cycles, and ϕ a K-level structure, for some $K \subset G(\mathbb{A}_f)$, namely a K-orbit of $\mathbb{A}_f$-module *isomorphisms* $\phi : V(\mathbb{A}_f) \to W(\mathbb{A}_f)$, which maps τ_i to s_i. Isomorphisms of such data are isomorphisms $f : W \to W'$ of rational Hodge structures, sending $s_i \mapsto s'_i$, and such that $f \circ \phi = \phi' k$, for some $k \in K$.

We assume that there exists an isomorphism of $\mathbb{Q}$-vector spaces $\beta : W \to V$ mapping $s_i \mapsto \tau_i$ and h to h_x, for some $x \in X$.

One denotes by $\mathrm{Hodge}(G,X,K)$ the set data $((W,h),\{s_i\},\phi)$. The Shimura variety

$$Sh_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

is the moduli space of isomorphism classes of data $((W,h),\{s_i\},\phi)$. Namely, there is a map of $\mathrm{Hodge}(G,X,K)$ to $Sh_K(G,X)$ (seen over $\mathbb{C}$), that descends to a bijection on isomorphism classes $\mathrm{Hodge}(G,X,K)/\sim$.

In such cases also one can consider degenerations of these data, both at the archimedean and at the non-archimedean components. One then considers data $((W,h),\{s_i\},\phi,\tilde{\beta})$, with a non-trivial *homomorphism* $\tilde{\beta} : W \to V$, which is a morphism of Hodge structures, and such that $\tilde{\beta}(\ell_{s_i}) \subset \ell_{\tau_i}$. This yields noncommutative spaces inside which the classical Shimura variety sits as the set of classical points.

Quantum statistical machanical systems associated to Shimura varieties have been recently studied by Ha and Paugam in [119]. Given a faithful representation $\rho : G \to \mathrm{GL}(V)$ as above, there is an "enveloping semigroup" M, that is, a normal irreducible semigroup $M \subset \mathrm{End}(V)$ such that $M^\times = G$. Such a semigroup can be used to encode the degenerations of the Hodge data described above. The data (G,X,V,M) then determine a noncommutative space which describes the "bad quotient" $Sh_K^{nc}(G,X) = G(\mathbb{Q})\backslash X \times M(\mathbb{A}_f)$ and is a moduli space for the possibly non-invertible data $((W,h),\{s_i\},\phi)$. Its "set of classical points" is the Shimura variety $Sh(G,X)$. The actual construction of the algebras involves some delicate steps, especially to handle the presence of stacky singularities (*cf.* [119]).

25. The Adèle Class Space and the Spectral Realization

In this section we describe a noncommutative space, the adèle class space $X_{\mathbb{K}}$, associated to any global field $\mathbb{K}$, which leads to a spectral realization of the zeros of the Riemann zeta function for $\mathbb{K} = \mathbb{Q}$ and more generally of L-functions associated to Hecke characters. It also gives a geometric interpretation of the Riemann-Weil explicit formulas of number theory as a trace formula. This space is closely related for $\mathbb{K} = \mathbb{Q}$ with the space of commensurability classes of $\mathbb{Q}$-lattices described above. Rather than starting directly with its description we first put the problem of finding the geometry of the set of prime numbers in the proper perspective.

The set of primes

One of the main problems of arithmetic is to understand the distribution of the set of prime numbers

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, \ldots\}$$

as a subset of the integers. To that effect one introduces the counting function

$$\pi(n) = \text{ number of primes } p \leq n\,.$$

The problem is to understand the behavior of $\pi(x)$ when $x \to \infty$. One often hears that the problem comes from the lack of a “simple” formula for $\pi(x)$. This is not really true and for instance in 1898 H. Laurent [146] gave the following formula whose validity is an easy exercise in arithmetic,

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{2\pi i \Gamma(k)/k} - 1}{e^{-2\pi i/k} - 1}\,,$$

where $\Gamma(k) = (k-1)!$ is the Euler Gamma function.

The problem with this formula is that it has no bearing on the asymptotic expansion of $\pi(x)$ when $x \to \infty$. Such an expansion was guessed by Gauss in the following form,

$$\pi(x) = \int_0^x \frac{du}{\log(u)} + R(x)\,,$$

where the logarithmic integral admits the asymptotic expansion

$$\mathrm{Li}(x) = \int_0^x \frac{du}{\log(u)} \sim \sum (k-1)!\, \frac{x}{\log(x)^k}\,.$$

The key issue then is the size of the remainder $R(x)$.

The Riemann Hypothesis

It asserts that this size is governed by

$$R(x) = O(\sqrt{x}\, \log(x))\,. \tag{25.1}$$

The Riemann Hypothesis is in fact a conjecture on the zeros of the zeta function

$$\zeta(s) = \sum_1^\infty n^{-s}\,, \tag{25.2}$$

whose definition goes back to Euler, who showed the fundamental factorization

$$\zeta(s) = \prod_{\mathcal{P}} (1 - p^{-s})^{-1}\,. \tag{25.3}$$

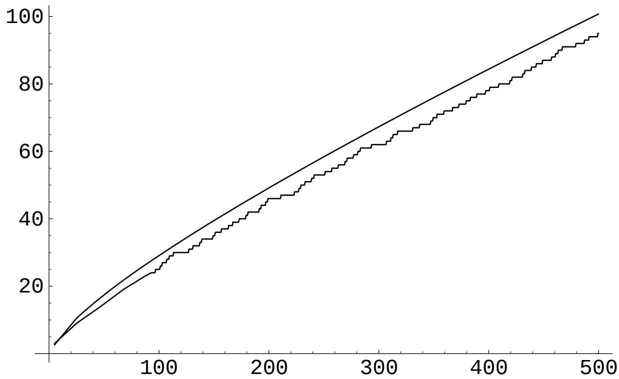


Figure 16. Graphs of $\pi(x)$ and $\mathrm{Li}(x)$.

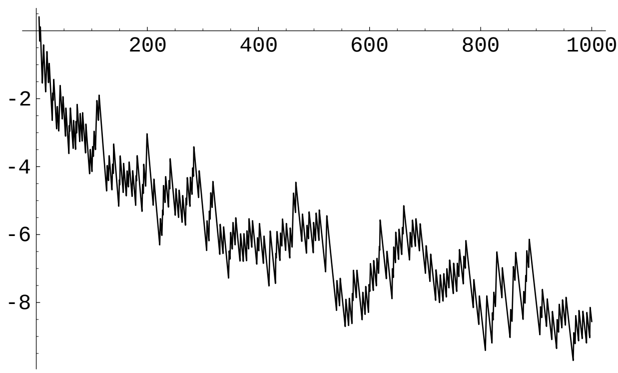


Figure 17. Graph of $\pi(x)-\mathrm{Li}(x)$.

It extends to a meromorphic function in the whole complex plane $\mathbb{C}$ and fulfills the functional equation

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \pi^{-(1-s)/2}\,\Gamma((1-s)/2)\,\zeta(1-s)\,, \tag{25.4}$$

so that the function

$$\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) \tag{25.5}$$

admits the symmetry $s \mapsto 1-s$.

The Riemann conjecture asserts that all zeros of $\zeta_{\mathbb{Q}}$ are on the critical line $\frac{1}{2} + i\,\mathbb{R}$. The reason why the location of the zeros of $\zeta_{\mathbb{Q}}$ controls the size of the remainder in (25.1) is the explicit formulas that relate primes with these zeros. Riemann proved the following first instance of an "explicit

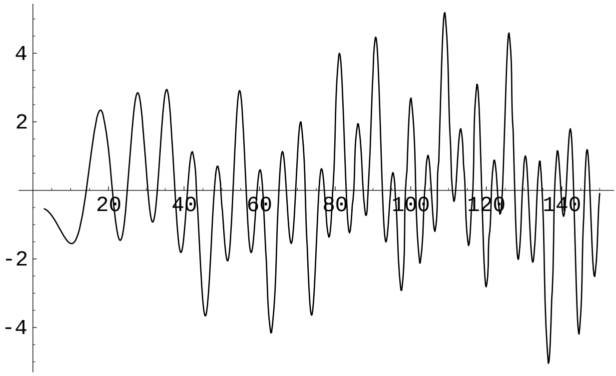


Figure 18. Zeros of zeta.

formula"

$$\pi'(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) + \int_x^{\infty} \frac{1}{u^2-1}\,\frac{du}{u\log u} + \log\xi(0)\,, \qquad (25.6)$$

where $\xi(0) = -\frac{1}{8}\Gamma(1/4)\pi^{-1/4}\zeta(1/2)$, the sum is over the non-trivial (*i.e.* complex) zeros of the zeta function, and where the function $\pi'(x)$ is given by

$$\pi'(x) = \pi(x) + \frac{1}{2}\,\pi(x^{\frac{1}{2}}) + \frac{1}{3}\,\pi(x^{\frac{1}{3}}) + \cdots$$

which gives by the Möbius inversion formula

$$\pi(x) = \sum \mu(m)\,\frac{1}{m}\,\pi'(x^{\frac{1}{m}})\,.$$

The generalized Riemann hypothesis

The explicit formulas of Riemann were put in more modern form by A. Weil, as

$$\widehat{h}(0) + \widehat{h}(1) - \sum_{\rho} \widehat{h}(\rho) = \sum_{v} \int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u\,, \qquad (25.7)$$

where $\mathbb{K}$ is now an arbitrary global field, $v \in \Sigma_{\mathbb{K}}$ varies among the places of $\mathbb{K}$ and the integral is over the locally compact field $\mathbb{K}_v$ obtained by completion of $\mathbb{K}$ at the place v. Also $\int'$ is the pairing with the distribution on $\mathbb{K}_v$ which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform (relative to a selfdual choice of additive characters α_v) vanishes at 1. By definition a *global field* is a (countable) discrete cocompact subfield in a locally compact ring. This ring depends functorially on $\mathbb{K}$ and is called the ring $\mathbb{A}_{\mathbb{K}}$ of adèles of $\mathbb{K}$. The quotient

$$C_{\mathbb{K}} = \mathrm{GL}_1(\mathbb{A}_{\mathbb{K}})/\mathrm{GL}_1(\mathbb{K}) \tag{25.8}$$

is the locally compact group of idèle classes of $\mathbb{K}$ which play a central role in class field theory. In Weil's explicit formula the test function h is in the Bruhat–Schwartz space $\mathcal{S}(C_{\mathbb{K}})$. The multiplicative groups $\mathrm{GL}_1(\mathbb{K}_v) = \mathbb{K}_v^*$ are embedded canonically as cocompact subgroups of $C_{\mathbb{K}}$. The sum on the left hand side is over the zeros of L-functions associated to Hecke characters. The function $\widehat{h}$ is the Fourier transform of h. The generalized Riemann conjecture asserts that all the zeros of these L-functions are on the critical line $\frac{1}{2} + i\,\mathbb{R}$. This was proved by Weil when the global field $\mathbb{K}$ has non-zero characteristic, but remains open in the case when $\mathbb{K}$ is of characteristic zero, in which case it is a number field, *i.e.* a finite algebraic extension of the field $\mathbb{Q}$ of rational numbers.

Quantum chaos → Riemann flow?

For $E > 0$ let $N(E)$ be the number of zeros of the Riemann zeta function $\zeta_{\mathbb{Q}}$ whose imaginary parts are in the open interval $]0, E[$. Riemann proved that the step function $N(E)$ can be written as the sum

$$N(E) = \langle N(E)\rangle + N_{\mathrm{osc}}(E)$$

of a smooth approximation $\langle N(E)\rangle$ and a purely oscillatory function $N_{\mathrm{osc}}(E)$ and gave the explicit form

$$\langle N(E)\rangle = \frac{E}{2\,\pi}\left(\log\frac{E}{2\,\pi} - 1\right) + \frac{7}{8} + o(1) \tag{25.9}$$

for the smooth approximation.

There is a striking analogy between the behavior of the step function $N(E)$ and that of the function counting the number of eigenvalues of the Hamiltonian H of the quantum system obtained after quantization of a chaotic dynamical system, which is at center stage in the theory of *quantum chaos*. A comparison of the asymptotic expansions of the oscillatory terms in the two cases, namely

$$N_{\mathrm{osc}}(E) \sim \frac{1}{\pi}\sum_{\gamma_p}\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{1}{2\mathrm{sh}\left(\frac{m\lambda_p}{2}\right)}\,\sin(m\,E\,T_\gamma^{\#})$$

for the quantization of a chaotic dynamical system, and

$$N_{\mathrm{osc}}(E) \sim \frac{-1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{1}{p^{m/2}}\,\sin\left(m\,E\,\log p\right)$$

for the Riemann zeta function, gives important indications on the hypothetical *Riemann flow* that would make it possible to identify the zeros of

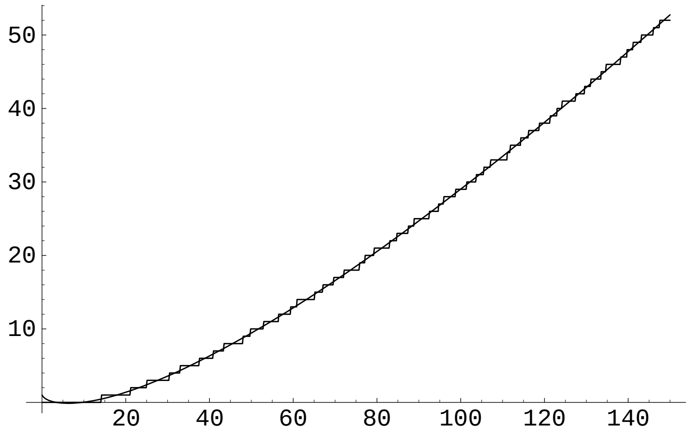


Figure 19. Counting zeros of zeta.

zeta as the spectrum of a Hamiltonian. For instance the periodic orbits of the flow should be labeled by the prime numbers and the corresponding periods T_p should be given by $\log p$. However, a closer look reveals an overall minus sign that forbids any direct comparison.

Spectral realization as an absorption spectrum

The above major sign obstruction was bypassed in [66] using the following basic distinction between observed spectra in physics. When the light coming from a hot chemical element is decomposed through a prism, it gives rise to bright emission lines on a dark background, and the corresponding frequencies are a signature of its chemical composition. When the light coming from a distant star is decomposed through a prism, it gives rise to dark lines, called absorption lines, on a white background. The spectrum of the light emitted by the sun was the first observed example of an absorption spectrum. In this case the absorption lines were discovered by Fraunhofer. The chemicals in the outer atmosphere of the star absorb the corresponding frequencies in the white light coming from the core of the star.

The simple idea then is that, because of the minus sign above, one should look for the spectral realization of the zeros of zeta not as a usual emission spectrum but as an absorption spectrum. Of course by itself this idea does not suffice to get anywhere since one needs the basic dynamical system anyway. The adèle class space, namely the quotient

$$X_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$$

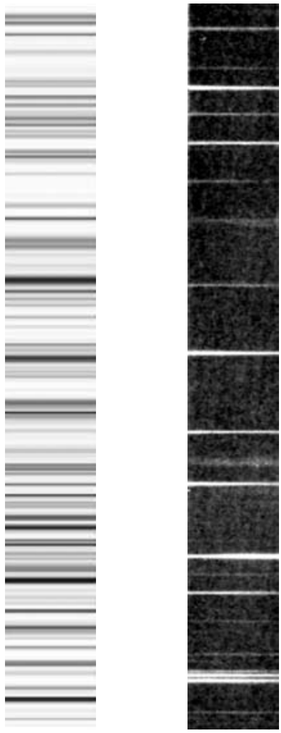


Figure 20. The two kinds of spectra.

introduced in [66], does the job as shown there. The action of the idèle class group $C_{\mathbb{K}}$ on the adèle class space is simply given by multiplication. In particular the idèle class group $C_{\mathbb{K}}$ acts on the suitably defined Hilbert space $L^2(X_{\mathbb{K}})$ and the zeros of L-functions give the absorption spectrum, with non-critical zeros appearing as resonances.

Exactly as adèles, the adèle class space $X_{\mathbb{K}}$ involves all the places of $\mathbb{K}$. If, in order to simplify, one restricts to a finite set of places one still finds a noncommutative space but one can analyze the action of the analogue of $C_{\mathbb{K}}$ and compute its trace after performing a suitable cutoff (necessary in all cases to see the missing lines of an absorption spectrum). One gets a trace formula

$$\mathrm{Tr}\,(R_\Lambda\, U(h)) = 2h(1)\log' \Lambda + \sum_{v\in S} \int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)\,,$$

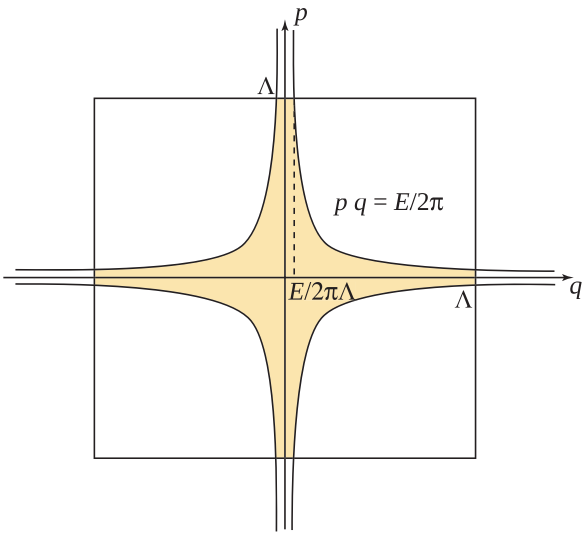


Figure 21. Counting quantum states.

where the terms on the right-hand side are exactly the same as in Weil's explicit formula (25.7). This is very encouraging since at least it gives geometric meaning to the complicated terms of (25.7) as the contributions of the periodic orbits to the computation of the trace.

In particular it gives a perfect interpretation of the smooth function $\langle N(E)\rangle$ approximating the counting $N(E)$ of the zeros of zeta, from counting the number of states of the one dimensional quantum system with Hamiltonian

$$h(q,p) = 2\pi q\, p$$

which is just the generator of the scaling group. Indeed, the function $\frac{E}{2\,\pi}(\log\frac{E}{2\,\pi} - 1)$ of the Riemann formula (25.9) appears as the number of missing degrees of freedom in the number of quantum states for the above system, as one obtains from the simple computation of the area of the region

$$B_+ = \{(p,q) \in [0,\Lambda]^2\,;\, h(p,q) \leq E\}$$

$$\mathrm{Area}(B_+) = \frac{E}{2\pi} \times 2\log\Lambda - \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right),$$

while the term $\frac{E}{2\pi} \times 2\log\Lambda$ corresponds to the number of degrees of freedom of white light. A careful computation gives not only the correction term of $\frac{7}{8}$ in (25.9) but all the remaining $o(1)$ terms.

Finally, it was shown in [66] that the generalized Riemann Hypothesis is equivalent to the validity of a *global* trace formula, but this is only one of many equivalent reformulations of the Riemann hypothesis.

26. Thermodynamics of Endomotives and the Tehran Program

In many ways the great virtue of a problem like RH comes from the developments that it generates. At first sight it does not appear as having any relation with geometry, and its geometric nature gradually emerged in the twentieth century mainly because of the solution of Weil in the case of global fields of positive characteristic.

We outline a program, current joint work of Katia Consani and the two authors, to adapt Weil's proof for the case of global fields of positive characteristic to the case of number fields (*cf.* [70], [96]).[b]

Function fields

Given a global field $\mathbb{K}$ of positive characteristic, there exists a finite field $\mathbb{F}_q$ and a smooth projective curve C defined over $\mathbb{F}_q$ such that $\mathbb{K}$ is the field of $\mathbb{F}_q$-valued rational functions on C. The analogue (Artin, Hasse, Schmidt) of the zeta function is

$$\zeta_{\mathbb{K}}(s) = \prod_{v \in \Sigma_{\mathbb{K}}} (1 - q^{-f(v)s})^{-1}$$

where $\Sigma_{\mathbb{K}}$ is the set of places of $\mathbb{K}$ and $f(v)$ is the degree (see below) of the place $v \in \Sigma_{\mathbb{K}}$.

The functional equation takes the form

$$q^{(g-1)(1-s)}\,\zeta_{\mathbb{K}}(1-s) = q^{(g-1)s}\,\zeta_{\mathbb{K}}(s)\,,$$

where g is the genus of C.

The analogue of the Riemann conjecture for such global fields was proved by Weil (1942) who developed algebraic geometry in that context. Weil's proof rests on two steps.

- (A) Explicit Formula
- (B) Positivity

Both are based on the geometry of the action of the Frobenius on the set $C(\bar{\mathbb{F}}_q)$ of points of C over an algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$. This set $C(\bar{\mathbb{F}}_q)$ maps canonically to the set $\Sigma_{\mathbb{K}}$ of places of $\mathbb{K}$ and the degree of a place $v \in \Sigma_{\mathbb{K}}$ is the number of points in the orbit of the Frobenius acting on the fiber of the projection

$$C(\bar{\mathbb{F}}_q) \to \Sigma_{\mathbb{K}}\,.$$

[b]This program was first announced in a lecture at IPM Tehran in September 2005, hence we refer to it as "the Tehran program".

The analogue

$$\#\{C(\mathbb{F}_{q^j})\} = \sum (-1)^k \operatorname{Tr}(\mathrm{Fr}^{*j} | H^k_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell))$$

of the Lefschetz fixed point formula makes it possible to compute the number $\#\{C(\mathbb{F}_{q^j})\}$ of points with coordinates in the finite extension $\mathbb{F}_{q^j}$ from the action of Fr^* in the étale cohomology group $H^1_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell)$, which does not depend upon the choice of the ℓ-adic coefficients $\mathbb{Q}_\ell$.

This shows that the zeta function is a rational fraction

$$\zeta_{\mathbb{K}}(s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where the polynomial P is the characteristic polynomial of the action of Fr^* in H^1.

The analogue of the Riemann conjecture for global fields of characteristic p means that its eigenvalues, *i.e.* the complex numbers λ_j of the factorization

$$P(T) = \prod (1 - \lambda_j T)$$

are of modulus $|\lambda_j| = q^{1/2}$.

The main ingredient in the proof of Weil is the notion of correspondence, given by divisors in $C \times C$. They can be viewed as multivalued maps,

$$Z : C \to C, \quad P \mapsto Z(P).$$

Two correspondences are equivalent if they differ by a principal divisor,

$$U \sim V \Leftrightarrow U - V = (f).$$

The composition of correspondences is

$$Z = Z_1 \star Z_2, \quad Z_1 \star Z_2(P) = Z_1(Z_2(P))$$

and the adjoint is given using the transposition $\sigma(x,y) = (y,x)$ by

$$Z' = \sigma(Z).$$

The degree $d(Z)$ of a correspondence is defined, independently of a generic point $P \in C$ by,

$$d(Z) = Z \bullet (P \times C),$$

where $\bullet$ is the intersection number. One has a similar definition of the codegree

$$d'(Z) = Z \bullet (C \times P)$$

Weil defines the *Trace* of a correspondence as follows

$$\operatorname{Tr}(Z) = d(Z) + d'(Z) - Z \bullet \Delta$$

where Δ is the identity correspondence. The main step in Weil's proof is

Theorem 26.1. (*Weil*) *The following positivity holds*: $\operatorname{Tr}(Z \star Z') > 0$ *unless* Z *is a trivial class.*

Clearly if one wants to have any chance at imitating the steps of Weil's proof of RH for the case of number fields one needs to have an analogue of the points of $C(\bar{\mathbb{F}}_q)$ and the action of the Frobenius, of the étale cohomology and of the unramified extensions $\mathbb{K} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ of $\mathbb{K}$.

Endomotives and Galois action

The adèle class space $X_{\mathbb{K}}$ of a global field admits a natural action of the idèle class group $C_{\mathbb{K}}$ and as such is on the adelic side of the class field theory isomorphism. In order to obtain a description of this space which is closer to geometry one needs to pass to the Galois side of class field theory. In the case $\mathbb{K} = \mathbb{Q}$, it is possible to present the adèle class space in a fairly simple manner not involving adèles, thanks to its intimate relation with the space of 1-dimensional $\mathbb{Q}$-lattices of Section 22. The direct interpretation of the action of the Galois group of $\bar{\mathbb{Q}}/\mathbb{Q}$ on the values of fabulous states for the BC-system (Section 22) then suggests that one should be able to construct directly the space $X_{\mathbb{Q}}$ with a canonical action of the Galois group of $\bar{\mathbb{Q}}/\mathbb{Q}$.

This was done in [69] thanks to an extension of the notion of Artin motives, called *endomotives*. Following Grothendieck, one can reformulate Galois theory over a field $\mathbb{K}$ as the equivalence of the category of reduced commutative finite dimensional algebras over $\mathbb{K}$ with the category of continuous actions of the Galois group G of $\bar{\mathbb{K}}/\mathbb{K}$ on finite sets. By construction the algebra of the BC-system is a crossed product of a commutative algebra A by a semigroup. When working over $\mathbb{K} = \mathbb{Q}$ which is essential in the definition of fabulous states, the algebra A is simply the group ring $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ of the torsion group $\mathbb{Q}/\mathbb{Z}$. Thus

$$A = \varinjlim A_n\,, \quad A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$$

and we are dealing with a projective limit of Artin motives. The key point then is to keep track of the corresponding action of the Galois group G of $\bar{\mathbb{K}}/\mathbb{K}$, $\mathbb{K} = \mathbb{Q}$. The Galois–Grothendieck correspondence associates to a reduced commutative finite dimensional algebra B over $\mathbb{K}$ the set of characters of B with values in $\bar{\mathbb{K}}$ together with the natural action of G. This action is non-trivial for the algebras $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$, where it corresponds to the cyclotomic theory.

One can then recover the Bost–Connes system with its natural Galois symmetry in a conceptual manner which extends to the general context of semigroup actions on projective systems of Artin motives. These typically arise from self-maps of algebraic varieties. Given a pointed algebraic variety (Y, y_0) over a field $\mathbb{K}$ and a countable unital abelian semigroup S of finite endomorphisms of (Y, y_0), unramified over $y_0 \in Y$, one constructs a projective system of Artin motives X_s over $\mathbb{K}$ from these data as follows. For $s \in S$, one sets

$$X_s = \{y \in Y \,:\, s(y) = y_0\}\,. \tag{26.1}$$

For a pair $s, s' \in S$, with $s' = sr$, the map $\xi_{s',s} : X_{sr} \to X_s$ is given by

$$X_{sr} \ni y \mapsto r(y) \in X_s \,. \tag{26.2}$$

This defines a projective system indexed by the semigroup S itself with partial order given by divisibility. We let $X = \varprojlim_s X_s$.

Since $s(y_0) = y_0$, the base point y_0 defines a component Z_s of X_s for all $s \in S$. Let $\xi_{s',s}^{-1}(Z_s)$ be the inverse image of Z_s in $X_{s'}$. It is a union of components of $X_{s'}$. This defines a projection e_s onto an open and closed subset X^{e_s} of the projective limit X. One then shows ([69]) that the semigroup S acts on the projective limit X by partial isomorphisms $\rho_s : X \to X^{e_s}$ defined by the property that

$$\xi_{su}(\rho_s(x)) = \xi_u(x),\, \forall\, u \in S, \forall\, x \in X \,. \tag{26.3}$$

The BC-system is obtained from the pointed algebraic variety $(\mathbb{G}_m(\mathbb{Q}), 1)$ where the affine group scheme $\mathbb{G}_m$ is the multiplicative group. The semigroup S is the semigroup of non-zero endomorphisms of $\mathbb{G}_m$. These correspond to maps of the form $u \mapsto u^n$ for some non-zero $n \in \mathbb{Z}$, and one restricts to $n \in \mathbb{N}^*$.

In this class of examples one has an "equidistribution" property, by which the uniform normalized counting measures μ_s on X_s are compatible with the projective system and define a probability measure on the limit X. Namely, one has

$$\xi_{s',s}\mu_s = \mu_{s'}, \qquad \forall\, s, s' \in S \,. \tag{26.4}$$

This follows from the fact that the number of preimages of a point under $s \in S$ is equal to $\deg s$. This provides exactly the data which makes it possible to perform the thermodynamical analysis of such endomotives. This gives a rather unexplored new territory since even the simplest examples beyond the BC-system remain to be investigated. For instance let Y be an elliptic curve defined over $\mathbb{K}$. Let S be the semigroup of non-zero endomorphisms of Y. This gives rise to an example in the general class described above. When the elliptic curve has complex multiplication, this gives rise to a system which, in the case of a maximal order, agrees with the one constructed in [81]. In the case without complex multiplication, this provides an example of a system where the Galois action does not factor through an abelian quotient.

Frobenius as dual of the time evolution

The Frobenius is such a universal symmetry in characteristic p, owing to the linearity of the map $x \mapsto x^p$, that it is very hard to find an analogue of such a far-reaching concept in characteristic zero. As we now explain, the classification of type III factors provides the basic ingredient which when combined with cyclic cohomology makes it possible to analyze the thermodynamics of a noncommutative space and to get an analog of the action of the Frobenius on étale cohomology.

The key ingredient is that noncommutativity generates a time evolution at the "measure theory" level. While it had long been known by operator algebraists that the theory of von Neumann algebras represents a far-reaching extension of measure theory, the main surprise which occurred at the beginning of the seventies in (Connes [53]) following Tomita's theory is that such an algebra M inherits from its noncommutativity a god-given time evolution:

$$\delta : \mathbb{R} \longrightarrow \mathrm{Out}\, M = \mathrm{Aut}\, M / \mathrm{Inn}\, M \,, \tag{26.5}$$

where $\mathrm{Out}\, M = \mathrm{Aut}\, M/\mathrm{Inn}\, M$ is the quotient of the group of automorphisms of M by the normal subgroup of inner automorphisms. This led in [53] to the reduction from type III to type II and their automorphisms and eventually to the classification of injective factors. They are classified by their module,

$$\mathrm{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^* \,, \tag{26.6}$$

which is a virtual closed subgroup of $\mathbb{R}_+^*$ in the sense of G. Mackey, *i.e.* an ergodic action of $\mathbb{R}_+^*$, called the flow of weights [93]. This invariant was first defined and used in [53], to show in particular the existence of hyperfinite factors which are not isomorphic to Araki-Woods factors.

The "measure theory" level, *i.e.* the set-up of von Neumann algebras, does not suffice to obtain the relevant cohomology theory and one needs to be given a weakly dense subalgebra $\mathcal{A} \subset M$ playing the role of smooth functions on the noncommutative space. This algebra will play a key role when cyclic cohomology is used at a later stage. At first one only uses its norm closure $A = \bar{\mathcal{A}}$ in M and assumes that it is globally invariant under the modular automorphism group σ_t^φ of a faithful normal state φ on M. One can then proceed with the thermodynamics of the C^* dynamical system (A, σ_t). By a very simple procedure assuming that KMS states at low temperature are of type I, one obtains a "cooling morphism" π which is a morphism of algebras from the crossed product $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$ to a type I algebra of compact operator valued functions on a canonical $\mathbb{R}_+^*$-principal bundle $\tilde{\Omega}_\beta$ over the space Ω_β of type I extremal KMS_β states fulfilling a suitable regularity condition (*cf.* [69]). Any $\varepsilon \in \Omega_\beta$ gives an irreducible representation π_ε of $\mathcal{A}$ and the choice of its essentially unique extension to $\hat{\mathcal{A}}$ determines the fiber of the $\mathbb{R}_+^*$-principal bundle $\tilde{\Omega}_\beta$. The cooling morphism is then given by,

$$\pi_{\varepsilon,H}\Big(\int x(t)\, U_t\, dt\Big) = \int \pi_\varepsilon(x(t))\, e^{itH}\, dt \,. \tag{26.7}$$

This morphism is equivariant for the dual action $\theta_\lambda \in \mathrm{Aut}(\hat{\mathcal{A}})$ of $\mathbb{R}_+^*$,

$$\theta_\lambda\Big(\int x(t)\, U_t\, dt\Big) = \int \lambda^{it}\, x(t)\, U_t\, dt\,. \tag{26.8}$$

The key point is that the range of the morphism π is contained in an algebra of functions on $\tilde{\Omega}_\beta$ with values in trace class operators. In other words, modulo *a* Morita equivalence one lands in the commutative world provided one lowers the temperature.

The interesting space is obtained by "distillation" and is simply given by the cokernel of the cooling morphism π, but this does not make sense in the category of algebras and algebra homomorphisms since the latter is not even an additive category. This is where cyclic cohomology enters the scene: the category of cyclic modules is an abelian category with a natural functor from the category of algebras and algebra homomorphisms.

Cyclic modules are modules of the cyclic category Λ, which is a small category, obtained by enriching with *cyclic morphisms* the familiar *simplicial category* Δ of totally ordered finite sets and increasing maps. Alternatively, Λ can be defined by means of its "cyclic covering", the category $E\Lambda$. The latter has one object $(\mathbb{Z}, n)$ for each $n \geq 0$ and the morphisms $f : (\mathbb{Z}, n) \to (\mathbb{Z}, m)$ are given by non-decreasing maps $f : \mathbb{Z} \to \mathbb{Z}$, such that $f(x+n) = f(x) + m\,, \quad \forall\, x \in \mathbb{Z}$. One has $\Lambda = E\Lambda/\mathbb{Z}$, with respect to the obvious action of $\mathbb{Z}$ by translations. To any algebra $\mathcal{A}$ one associates a module $\mathcal{A}^\natural$ over the category Λ by assigning to each n the $(n+1)$-st tensor power $\mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}$. The cyclic morphisms correspond to the cyclic permutations of the tensors while the face and degeneracy maps correspond to the algebra product of consecutive tensors and the insertion of the unit. The corresponding functor $\mathcal{A} \to \mathcal{A}^\natural$ gives a linearization of the category of associative algebras and cyclic cohomology appears as a derived functor.

One can thus define the *distilled* module $D(\mathcal{A}, \varphi)$ as the cokernel of the cooling morphism and consider the action of $\mathbb{R}_+^*$ (obtained from the above equivariance) in the cyclic homology group $HC_0(D(\mathcal{A}, \varphi))$. As shown in [69] this in the simplest case of the BC-system gives a cohomological interpretation of the above spectral realization of the zeros of the Riemann zeta function (and of Hecke L-functions).

One striking feature is that the KMS strip (*cf.* Figure 1) becomes canonically identified in the process with the critical strip of the zeta function (recall that $\beta > 1$) by multiplication by $i = \sqrt{-1}$.

This cohomological interpretation combines with the above theory of endomotives to give a natural action of the Galois group G of $\bar{\mathbb{Q}}/\mathbb{Q}$ on the above cohomology. This action factorizes through the abelianization G^{ab} and the corresponding decomposition according to characters of G^{ab} corresponds to the spectral realization of L-functions.

The role of the invariant $S(M)$ in the classification of factors or of the more refined flow of weights mentioned above, is very similar to the role of the module of local or global fields and the Brauer theory of central simple algebras. In fact there is a striking parallel (see [69]) between the lattice of unramified extensions $\mathbb{K} \to \mathbb{K} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ of a global field of characteristic p and the lattice of extensions of a factor M by the crossed product algebras $M \to M \rtimes_{\sigma_T} \mathbb{Z}$. Using the algebraic closure of $\mathbb{F}_q$, *i.e.* the operation $\mathbb{K} \to \mathbb{K} \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$, corresponds to passing to the dual algebra $M \to M \rtimes_\sigma \mathbb{R}$ and the dual action corresponds to the Frobenius automorphism when as above the appropriate cohomological operations (distillation and HC_0) are performed.

Global field $\mathbb{K}$	**Factor** M
$\mathrm{Mod}\,\mathbb{K} \subset \mathbb{R}_+^*$	$\mathrm{Mod}\,M \subset \mathbb{R}_+^*$
$\mathbb{K} \to \mathbb{K} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$	$M \to M \rtimes_{\sigma_T} \mathbb{Z}$
$K \to \mathbb{K} \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$	$M \to M \rtimes_\sigma \mathbb{R}$
Points $C(\bar{\mathbb{F}}_q)$	$\Xi \subset X_{\mathbb{Q}}$

A notable difference from the original Hilbert space theoretic spectral realization of [66] is that while in the latter case only the critical zeros were appearing directly (the possible non-critical ones appearing as resonances), in the cyclic homology set-up it is more natural to use everywhere the "rapid decay" framework (advocated in [166]) so that all zeros appear on the same footing. This eliminates the difficulty coming from the potential non-critical zeros, so that the trace formula is much easier to prove and reduces to the Riemann-Weil explicit formula. However, it was not obvious how to obtain a direct geometric proof of this formula from the S-local trace formula of [66]. This was done in [166], showing that the noncommutative geometry framework makes it possible to give a geometric interpretation of the Riemann–Weil explicit formula. While the spectral side of the trace formula is given by the action on the cyclic homology of the distilled space, the geometric side is given as follows [69].

Theorem 26.2. *Let $h \in S(C_{\mathbb{K}})$. Then the following holds*:

$$\mathrm{Tr}\,(\vartheta(h)|_{H^1}) = \widehat{h}(0)+\widehat{h}(1)- \Delta\bullet\Delta\; h(1)- \sum_v \int'_{(\mathbb{K}_v^*,e_{\mathbb{K}_v})} \frac{h(u^{-1})}{|1-u|}\, d^*u\,. \quad (26.9)$$

We refer to [69] for the detailed notations (in particular for the restricted Schwartz space $S(C_{\mathbb{K}})$) which are essentially those of [66]. The origin of the terms in the geometric side of the trace formula comes from the Lefschetz formula of Atiyah–Bott [11] and its adaptation by Guillemin–Sternberg (*cf.* [117]) to the distribution theoretic trace for flows on manifolds, which is a variation on the theme of [11]. For the action of $C_{\mathbb{K}}$ on the adèle class space $X_{\mathbb{K}}$ the relevant periodic orbits $X_{\mathbb{K},v}$ on which the computation concentrates turn out to be type III noncommutative spaces. Each of them admits classical points forming a subset $\Xi_{\mathbb{K},v} \subset X_{\mathbb{K},v}$. The union of these "classical periodic" points is given by

$$\Xi_{\mathbb{K}} = \cup\, \Xi_{\mathbb{K},v}\,, \quad \Xi_{\mathbb{K},v} = C_{\mathbb{K}}\,[v]\,, \quad v \in \Sigma_{\mathbb{K}}\,, \quad (26.10)$$

where for each place $v \in \Sigma_{\mathbb{K}}$, one lets $[v]$ be the adèle

$$[v]_w = 1\,, \quad \forall\, w \neq v\,, \quad [v]_v = 0\,. \quad (26.11)$$

In the function field case, one has a *non-canonical* isomorphism of the following form.

Proposition 26.3. *Let $\mathbb{K}$ be the function field of an algebraic curve C over $\mathbb{F}_q$. Then the action of the Frobenius on $Y = C(\bar{\mathbb{F}}_q)$ is isomorphic to the action of $q^{\mathbb{Z}}$ on the quotient*

$$\Xi_{\mathbb{K}}/C_{\mathbb{K},1}\,.$$

In the case $\mathbb{K} = \mathbb{Q}$ the space $\Xi_{\mathbb{Q}}/C_{\mathbb{Q},1}$ appears as the union of periodic orbits of period $\log p$ under the action of $C_{\mathbb{Q}}/C_{\mathbb{Q},1} \sim \mathbb{R}$ (*cf.* Figure 22). This gives a first approximation to the sought for space $Y = C(\bar{\mathbb{F}}_q)$ in characteristic zero. One important refinement is obtained from the subtle nuance between the adelic description of $X_{\mathbb{Q}}$ and the finer description in terms of the endomotive obtained from the pointed algebraic variety $(\mathbb{G}_m(\mathbb{Q}), 1)$. The second description keeps track of the Galois symmetry and as in Proposition 26.3 the isomorphism of the two descriptions is non-canonical.

At this point we have, in characteristic zero, several of the geometric notions which are the analogues of the ingredients of Weil's proof and it is natural to try and imitate the steps of his proof. The step (A), *i.e.* the explicit formula is taken care of by Theorem 26.2. What remains is to prove a positivity result. A well known result of A. Weil (*cf.* [24]) states that RH is equivalent to the positivity of the distribution entering in the explicit formulae. Thanks to the above H^1 obtained as the cyclic homology of the distilled module, Weil's reformulation can be stated as follows.

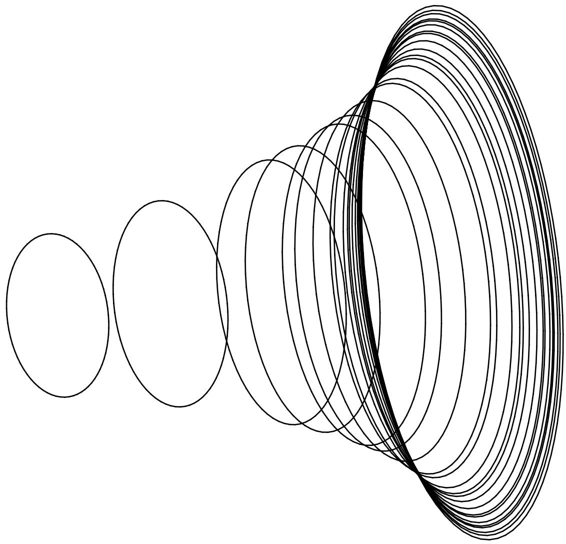


Figure 22. The classical points of the adèles class space.

We let $\vartheta(g)|_{H^1}$ denote the induced action of $g \in C_{\mathbb{K}}$ on the cokernel H^1 described above, We also write, as in [69], [70], $\vartheta(f)|_{H^1}$, with $f \in S(C_{\mathbb{K}})$, for the action of $\int_{C_{\mathbb{K}}} f(g)\vartheta(g)d^*g$. We then have the following result.

Theorem 26.4. *The following two conditions are equivalent.*

- *All L-functions with Grössencharakter on $\mathbb{K}$ satisfy the Riemann Hypothesis.*
- $\mathrm{Tr}\,\vartheta(f \star f^\sharp)|_{H^1} \geq 0$, *for all* $f \in S(C_{\mathbb{K}})$.

Here we used the notation

$$f = f_1 \star f_2\,, \quad \text{with} \ (f_1 \star f_2)(g) = \int f_1(k)\, f_2(k^{-1}\, g)\, d^*g \tag{26.12}$$

for the convolution of functions, using the multiplicative Haar measure d^*g, and for the adjoint

$$f \to f^\sharp\,, \quad f^\sharp(g) = |\,g|^{-1}\, \bar{f}(g^{-1})\,. \tag{26.13}$$

The role of the specific correspondences used in Weil's proof of RH in positive characteristic is played by the test functions $f \in S(C_{\mathbb{K}})$. More precisely the scaling map which replaces $f(x)$ by $f(g^{-1}x)$ has a graph, namely the set of pairs $(x, g^{-1}x) \in X_{\mathbb{K}} \times X_{\mathbb{K}}$, which we view as a correspondence Z_g. Then, given a test function f on the idèles classes, one assigns to f the linear combination

$$Z(f) = \int f(g) Z_g d^* g \tag{26.14}$$

of the above graphs, viewed as a “divisor” on $X_{\mathbb{K}} \times X_{\mathbb{K}}$.

The analogs of the degrees $d(Z)$ and codegrees $d^{'}(Z) = d(Z')$ of correspondences in the context of Weil's proof are given, for the degree, by

$$d(Z(h)) = \widehat{h}(1) = \int h(u)\, |u|\, d^*\, u\,, \tag{26.15}$$

so that the degree $d(Z_g)$ of the correspondence Z_g is equal to $|g|$. Similarly, for the codegree one has

$$d^{'}(Z(h)) = d(Z(\bar{h}^\sharp)) = \int h(u)\, d^*\, u = \widehat{h}(0)\,, \tag{26.16}$$

so that the codegree $d^{'}(Z_g)$ of the correspondence Z_g is equal to 1.

One of the major difficulties is to find the replacement for the principal divisors which in Weil's proof play a key role as an ideal in the algebra of correspondences on which the trace vanishes. At least already one can see that there is an interesting subspace $\mathcal{V}$ of the linear space of correspondences described above on which the trace also vanishes. It is given by the subspace

$$\mathcal{V} \subset S(C_{\mathbb{K}})\,, \quad \mathcal{V} = \{\, g(x) = \sum \xi(k\, x) \,|\, \xi \in \mathcal{S}(\mathbb{A}_{\mathbb{K}})_0 \}\,, \tag{26.17}$$

where the subspace $\mathcal{S}(\mathbb{A}_{\mathbb{K}})_0 \subset \mathcal{S}(\mathbb{A}_{\mathbb{K}})$ is defined by the two boundary conditions

$$\xi(0) = 0\,, \quad \int \xi(x)\, dx \,=\, 0\,.$$

Lemma 26.5. *For any $f \in \mathcal{V} \subset S(C_{\mathbb{K}})$, one has*

$$\vartheta(f)|_{H^1} \,=\, 0\,.$$

This shows that the Weil pairing of Theorem 26.4 admits a huge radical given by all functions which extend to adèles and gives another justification for working with the above cohomology H^1. In particular one can modify arbitrarily the degree and codegree of the correspondence $Z(h)$ by adding to h an element of the radical $\mathcal{V}$ using a subtle failure of Fubini's theorem. We show in [70] that several of the steps of Weil's proof can be transposed in the framework described above.

This constitutes a clear motivation to develop noncommutative geometry much further. One can write a very tentative form of a dictionary from the language of algebraic geometry (in the case of curves) and that of noncommutative geometry. The dictionary is summarized in the following table. It should be stressed that the main problem is to find the correct translation in the right column (noncommutative geometry) of the well established notion of principal divisor in the (algebraic geometry) left column. The table below is too crude in that respect, since one does not expect to be able to work in the usual "primary" theory which involves periodic cyclic homology and index theorems. Instead one expects that both the unstable cyclic homology and the finer invariants of spectral triples arising from transgression will play an important role. Thus the table below should be taken as a very rough first approximation, and a motivation for developing the missing finer notions in the right column.

Virtual correspondences	bivariant K-theory class Γ
Modulo torsion	$KK(A, B \otimes \mathrm{II}_1)$
Effective correspondences	Epimorphism of C^*-modules
Principal correspondences	Compact morphisms
Composition	cup product in KK-theory
Degree of correspondence	Pointwise index $d(\Gamma)$
$\deg D(P) \geq g \Rightarrow \sim$ effective	$d(\Gamma) > 0 \Rightarrow \exists\, K, \Gamma + K$ onto
Adjusting the degree by trivial correspondences	Fubini step on the test functions
Frobenius correspondence	Correspondence Z_g
Lefschetz formula	bivariant Chern of $Z(h)$ (localization on graph $Z(h)$)
Weil trace unchanged by principal divisors	bivariant Chern unchanged by compact perturbations

References

[1] J. Aastrup, R. Nest and E. Schrohe, *A Continuous field of C^*-algebras and the tangent groupoid for manifolds with boundary*, J. Funct. Anal. 237 (2006) no. 2, 482–506.

[2] B. Abadie and R. Exel, *Hilbert C^*-bimodules over commutative C^*-algebras and an isomorphism condition for quantum Heisenberg manifolds*, Rev. Math. Phys. 9 (1997) no. 4, 411–423.

[3] B. Abadie, S. Eilers and R. Exel, *Morita equivalence for crossed products by Hilbert C^*-bimodules*, Trans. Amer. Math. Soc. 350 (1998) no. 8, 3043–3054.

[4] J. E. Anderson and I. F. Putnam, *Topological invariants for substitution tilings and their associated C^*-algebras*, Ergod. Th. Dynam. Syst. 18 (1998) no. 3, 509–537.

[5] Anonymous Persian manuscript, *Fi tadakhul al-ashkal al-mutashabiha aw mutawafiqa*, Bibliothèque Nationale, Paris, Ancien fonds persan 169.

[6] C. Antonescu and E. Christensen, *Spectral triples for AF C^*-algebras and metrics on the Cantor set*, preprint math.OA/0309044. In print as E. Christensen, C. Ivan, *Spectral triples for AF C^*-algebras and metrics on the Cantor set*, J. Operator Theory 56 (2006) no. 1, 17–46.

[7] F. Ardalan and A. H. Fatollahi, *Point like structure in strings and noncommutative geometry*, Phys. Lett. B408 (1997) 157–163.

[8] F. Ardalan, H. Arfaei and N. Sadooghi, *On the anomalies and Schwinger terms in noncommutative gauge theories*, Internat. J. Mod. Phys. A 21 (2006) no. 19-20, 4161–4183.

[9] N. Ardalan and L. Bakhtiar, *The Sense of Unity: The Sufi Tradition in Persian Architecture*, The University of Chicago Press, 1973.

[10] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift. vol. I. Prog. Math., 86 (1990) 33–85.

[11] M.F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes: I*, Ann. Math. 86 (1967) 374–407.

[12] M. F. Atiyah, *Global theory of elliptic operators*, Proc. Internat. Conf. on *Functional Analysis and Related Topics* (Tokyo, 1969). University of Tokyo Press, Tokyo 1970, 21–30.

[13] M. Baake and R. V. Moody (Editors), *Directions in Mathematical Quasicrystals*, CRM Monograph Series, 13. American Mathematical Society, Providence, RI, 2000, viii+379 pp.

[14] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M theory as a Matrix model: A conjecture*, Phys. Rev. D55 (1997) 5112–5128.

[15] P. Baum and A. Connes, *Geometric K-theory for Lie groups and foliations*, Preprint IHES (M/82/), 1982, L'enseignement Mathematique, t.46 (2000), 1–35.

[16] P. Baum, A. Connes and N. Higson, *Classifying space for proper actions and K-theory of group C^*-algebras*, Contemp. Math. 167 (1994) 241–291.

[17] F. Bayen and C. Fronsdal, *Quantization on the sphere*, J. Math. Phys. 22 (1981) no. 7, 1345–1349.

[18] J. Bellissard, *Noncommutative geometry and quantum Hall effect*, in *Proceedings of the International Congress of Mathematicians*, Vol. 1, 2 (Zürich, 1994), pp. 1238–1246, Birkhäuser, Basel, 1995.

[19] J. Bellissard, A. van Elst and H. Schulz-Baldes, *The noncommutative geometry of the quantum Hall effect*, J. Math. Phys. 35 (1994) no. 10, 5373–5451.

[20] J. Bellissard, *The noncommutative geometry of aperiodic solids*, in *Geometric and Topological Methods for Quantum Field Theory* (Villa de Leyva, 2001), 86–156, World Scientific, 2003.

[21] M.-T. Benameur and T. Fack, *On von Neumann spectral triples*, math.KT/0012233.

[22] R. Berger, M. Dubois-Violette and M. Wambst, *Homogeneous algebras*, J. Algebra 261 (2003) no. 1, 172–185.

[23] C. G. Bollini and J. J. Giambiagi, *Lowest order divergent graphs in ν-dimensional space*, Phys. Lett. B40 (1972) 566–568.

[24] E. Bombieri, *Problems of the Millennium: The Riemann Hypothesis*, Clay Mathematical Institute, 2000.

[25] M. Born and P. Jordan, *Zur Quantenmechanik*, Z. Phys. 34 (1925) 858–888.

[26] D. Borthwick, S. Klimek, A. Lesniewski and M. Rinaldi, *Supersymmetry and Fredholm modules over quantized spaces*, Commun. Math. Phys. 166 (1994) no. 2, 397–415.

[27] J.-B. Bost and A. Connes, *Hecke Algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Mathematica, New Series 1, (1995) no. 3 411–457.

[28] P. Bouwknegt, K. Hannabuss and V. Mathai, *T-duality for principal torus bundles*, J. High Energy Phys. (2004) no. 3, 018 (electronic).

[29] P. Bouwknegt, J. Evslin and V. Mathai, *T-duality: Topology change from H-flux*, Commun. Math. Phys. 249 (2004) no. 2, 383–415.

[30] M. Boyle and D. Handelman, *Orbit equivalence, flow equivalence, and ordered cohomology*, Israel J. Math. 95 (1996) 169–210.

[31] O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. 171 (1972) 195–234.

[32] P. Breitenlohner and D. Maison, *Dimensional renormalization and the action principle*, Commun. Math. Phys. 52 (1977) no. 1, 11–38.

[33] J. Brüning and R. Seeley, *Regular singular asymptotics*, Adv. Math. 58 (1985) 133–148.

[34] J.-L. Brylinski and V. Nistor, *Cyclic Cohomology of Etale Groupoids*, K-theory 8 (1994) 341–365.

[35] M. S. Bulatov, *Geometricheskaya Garmonizatsiya v Arkhitekture Srednei Azii IX-XV vv.*, Nauka, 2nd ed., Moskva 1988, 360 pp.

[36] D. Burghelea, *The cyclic homology of the group rings*, Comment. Math. Helv. 60 (1985) 354–365.

[37] A. Carey, K. Hannabuss, V. Mathai and P. McCann, *Quantum Hall effect on the hyperbolic plane*, Commun. Math. Phys., 190, (1998) no. 3, 629–673.

[38] A. L. Carey, J. Phillips, A. Rennie and F. A. Sukochev, *The local index formula in semifinite von Neumann algebras I: Spectral flow*, math.OA/0411021.

[39] A. L. Carey, J. Phillips, A. Rennie and F. A. Sukochev, *The local index formula in semifinite von Neumann algebras II: The even case*, Adv. Math. 202 (2006) no. 2, 517–554.

[40] J. M. Castera, *Zellijs, Muqarnas and Quasicrystals*, ISAMA 99 Proceedings, pp. 99–104, 1999.

[41] J. M. Castera, *Arabesques*, ACR editions, 1999, 486 pp.

[42] P. S. Chakraborty and A. Pal, *Equivariant spectral triple on the quantum $SU(2)$-group*, K-Theory 28 (2003) no. 2, 107–126.

[43] T. Chakraborti and P. Pietilänen, *The Quantum Hall Effects*, 2nd ed., Springer 1995.

[44] A. Chamseddine and A. Connes, *Universal formulas for noncommutative geometry actions*, Phys. Rev. Lett. 77 (1996) no. 24, 4868–4871.
[45] A. Chamseddine and A. Connes, *The spectral action principle*, Commun. Math. Phys. 186 (1997) 731–750.
[46] A. Chamseddine and A. Connes, *Scale invariance in the spectral action*, J. Math. Phys. 47 (2006) no. 6, 063504.
[47] A. Chamseddine, A. Connes and M. Marcolli, *Gravity and the Standard Model with neutrino mixing*, hep-th/0610241.
[48] J. Cheeger, M. Gromov, *L^2-cohomology and group cohomology*, Topology 25 (1986) 189–215.
[49] A. Chou, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Trans. Am. Math. Soc. 289 (1985) no. 1, 1–40.
[50] E. Christensen and C. Ivan, *Sums of two dimensional spectral triples*, math.OA/0601024.
[51] E. Christensen, C. Ivan and M. Lapidus, *Dirac operators and spectral triples for some fractal sets built on curves*, math.MG/0610222.
[52] J. Collins, *Renormalization*, Cambridge Monographs in Math. Physics, Cambridge University Press, 1984.
[53] A. Connes, *Une classification des facteurs de type III*, Ann. Sci. Ecole Norm. Sup. 6 (1973) no. 4, 133–252.
[54] A. Connes, *Sur la théorie non commutative de l'intégration* Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978), pp. 19–143, Lecture Notes in Math., 725, Springer, Berlin, 1979.
[55] A. Connes, *Géométrie Noncommutative*, InterEditions, Paris, 1990.
[56] A. Connes, *Noncommutative geometry*, Academic Press, 1994.
[57] A. Connes, *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of* $\mathbb{R}$, Adv. in Math. 39 (1981) no. 1, 31–55.
[58] A. Connes, *Cyclic cohomology and the transverse fundamental class of a foliation*, in *Geometric Methods in Operator Algebras* (Kyoto, 1983). Pitman Res. Notes in Math., 123, Longman, Harlow 1986, 52–144.
[59] A. Connes, *C^* algèbres et géométrie differentielle*, C.R. Acad. Sci. Paris, Ser. A-B 290 (1980) 599–604.
[60] A. Connes, *Noncommutative differential geometry*, Inst. Hautes Etudes Sci. Publ. Math. 62 (1985) 257–360.
[61] A. Connes, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergod. Th. Dynam. Syst. 9 (1989) 207–220.
[62] A. Connes, *On the Chern character of θ-summable Fredholm modules*, Commun. Math. Phys. 139 (1991) no. 1, 171–181.
[63] A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, Commun. Math. Phys. 182 (1996) 155–176.
[64] A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36 (1995) no. 11, 6194–6231.
[65] A. Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. 34 (1995) no. 3, 203–238.
[66] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Mathematica. New Ser. 5 (1999) 29–106.
[67] A. Connes, *Cyclic cohomology, quantum group symmetries and the local index formula for* $\mathrm{SU}_q(2)$, J. Inst. Math. Jussieu 3 (2004) no. 1, 17–68.
[68] A. Connes, *Noncommutative Geometry and the standard model with neutrino mixing*, hep-th/0608226.

[69] A. Connes, C. Consani and M. Marcolli, *Noncommutative geometry and motives: The thermodynamics of endomotives*, math.QA/0512138.

[70] A. Connes, C. Consani, M. Marcolli, *The Weil proof and the geometry of the adeles class space*, math.NT/0703392.

[71] A. Connes, M. Douglas and A. Schwarz, *Noncommutative geometry and Matrix theory: Compactification on tori*, J. High Energy Phys. (1998) no. 2, Paper 3 (electronic).

[72] A. Connes and M. Dubois-Violette, *Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples*, Commun. Math. Phys. 230 (2002) no. 3, 539–579.

[73] A. Connes and M. Dubois-Violette, *Moduli space and structure of noncommutative 3-spheres*, Lett. Math. Phys. 66 (2003) no. 1–2, 91–121.

[74] A. Connes and M. Dubois-Violette, *Noncommutative finite-dimensional manifolds II. Moduli space and structure of noncommutative 3-spheres*, math.QA/0511337.

[75] A. Connes and G. Landi, *Noncommutative manifolds, the instanton algebra and isospectral deformations*, Commun. Math. Phys. 221 (2001) no. 1, 141–159.

[76] A. Connes and J. Lott, *Particle models and noncommutative geometry*, in *Recent Advances in Field Theory*, (Annecy-le-Vieux, 1990), Nucl. Phys. B Proc. Suppl. 18B (1990) 29–47.

[77] A. Connes and M. Marcolli, *From physics to number theory via noncommutative geometry. Part I: Quantum statistical mechanics of Q-lattices*, in *Frontiers in Number Theory Physics and Geometry*, pp. 269–347, Springer, Berlin, 2006.

[78] A. Connes and M. Marcolli, *$\mathbb{Q}$-lattices: Quantum statistical mechanics and Galois theory*, J. Geom. Phys. 56 (2005) no. 1, 2–23.

[79] A. Connes and M. Marcolli, *Anomalies, Dimensional Regularization and Noncommutative Geometry*, in preparation.

[80] A. Connes and M. Marcolli, *Noncommutative geometry, from quantum fields to motives* (tentative), book in preparation. To appear as a co-publication of the American Mathematical Society, Colloquium Publications Series, and Hindustan Book Agency, Texts and Readings in Mathematics Series.

[81] A. Connes, M. Marcolli and N. Ramachandran, *KMS states and complex multiplication*, Selecta Math. (N.S.) 11 (2005) no. 3-4, 325–347.

[82] A. Connes, M. Marcolli and N. Ramachandran, *KMS states and complex multiplication. Part II*, in *Operator Algebras - The Abel Symposium 200*, pp. 15–60, Springer Verlag, 2006.

[83] A. Connes and H. Moscovici, *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology 29 (1990) no. 3, 345–388.

[84] A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, GAFA 5 (1995) 174–243.

[85] A. Connes and H. Moscovici, *Hopf Algebras, cyclic cohomology and the transverse index theorem*, Commun. Math. Phys. 198 (1998) 199–246.

[86] A. Connes and H. Moscovici, *Differentiable cyclic cohomology and Hopf algebraic structures in transverse geometry*, Essays on geometry and related topics, Vol. 1, 2, 217–255, Monogr. Enseign. Math., 38, Enseignement Math., Geneva, 2001.

[87] A. Connes and H. Moscovici, *Modular Hecke algebras and their Hopf symmetry*, Mosc. Math. J. 4 (2004) no. 1, 67–109.

[88] A. Connes and H. Moscovici, *Rankin–Cohen brackets and the Hopf algebra of transverse geometry*, Mosc. Math. J. 4 (2004) no. 1, 111–130.

[89] A. Connes and H. Moscovici, *Background independent geometry and Hopf cyclic cohomology*, math.QA/0505475.

[90] A. Connes and H. Moscovici, *Spectral Geometry of Noncommutative Spaces* (tentative), book in preparation.

[91] A. Connes and M. Rieffel, *Yang–Mills for noncommutative two-tori*, Contemp. Math. Oper. Algebra Math. Phys. AMS (1987) 237–266.

[92] A. Connes and G. Skandalis, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. 20 (1984) no. 6, 1139–1183.

[93] A. Connes and M. Takesaki, *The flow of weights on factors of type III*, Tohoku Math. J. 29 (1977) 473–575.

[94] C. Consani and M. Marcolli, *Noncommutative geometry, dynamics, and ∞-adic Arakelov geometry*, Selecta Math. (N.S.) 10 (2004) no. 2, 167–251.

[95] C. Consani and M. Marcolli, *New perspectives in Arakelov geometry*, Number theory, 81–102, CRM Proc. Lecture Notes, 36, Amer. Math. Soc., Providence, RI, 2004.

[96] C. Consani and M. Marcolli, *Quantum statistical mechanics over function fields*, J. Number Th. 123 (2007) 487–528.

[97] M. Crainic, *Cyclic cohomology of étale groupoids, the general case*, K-theory 17 (1999) 319–362.

[98] J. Cuntz, *A class of C^*-algebras and topological Markov chains II: Reducible chains and the Ext–functor for C^*-algebras*, Invent. Math. 63 (1981) 25–40.

[99] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. 56 (1980) 251–268.

[100] L. Dabrowski, G. Landi, A. Sitarz, W. van Suijlekom and J.C. Várilly, *The Dirac operator on $SU_q(2)$*, Commun. Math. Phys. 259 (2005) no. 3, 729–759.

[101] K. R. Davidson, *C^*-algebras by Example*, Amer. Math. Soc., Providence, RI, 1996.

[102] J. Dixmier, *Existence de traces non normales*, C.R. Acad. Sci. Paris, Ser. A-B, 262 (1966) A1107–A1108.

[103] S. Doplicher, K. Fredenhagen and J. Roberts, *Spacetime quantization induced by classical gravity*, Phys. Lett. B 331 (1994) 33–44.

[104] M. Dubois-Violette, *Generalized differential spaces with $d^N = 0$ and the q-differential calculus*, Czech. J. Phys. 46 (1997) 1227–1233.

[105] M. Dubois-Violette, *$d^N = 0$: Generalized homology*, K-Theory 14 (1998) 371–404.

[106] M. Dubois-Violette, *Lectures on graded differential algebras and noncommutative geometry*, in Y. Maeda and H. Moriyoshi, (eds.), *Noncommutative Differential Geometry and Its Applications to Physics*, pp. 245–306. Shonan, Japan, 1999, Kluwer, 2001.

[107] M. Dubois-Violette and M. Henneaux, *Generalized cohomology for irreducible tensor fields of mixed Young symmetry type*, Lett. Math. Phys. 49 (1999) 245–252.

[108] M. Dubois-Violette and M. Henneaux, *Tensor fields of mixed Young symmetry type and N-complexes*, Commun. Math. Phys. 226 (2002) 393–418.

[109] H. Figueroa, J. M. Gracia-Bondía and J. Varilly, *Elements of Noncommutative Geometry*, Birkhäuser, 2000.

[110] D. Gaboriau, *Invariants ℓ^2 de relations d'équivalence et de groupes*, Publ. Math. Inst. Hautes Études Sci. No. 95 (2002) 93–150.

[111] D. Gaboriau, *Approximate dimension for equivalence relations and groups*, in preparation.

[112] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker and J. C. Várilly, *Moyal planes are spectral triples*, Commun. Math. Phys. 246 (2004) no. 3, 569–623.

[113] T. Giordano, I. F. Putnam and C. Fr. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. 469 (1995) 51–111.

[114] H. Grosse and R. Wulkenhaar, *Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base*, Commun. Math. Phys. 256 (2005) no. 2, 305–374.
[115] D. Guido and T. Isola, *Dimensions and singular traces for spectral triples, with applications to fractals*, J. Funct. Anal. 203 (2003) 362–400.
[116] D. Guido and T. Isola, *Dimensions and spectral triples for fractals in R^N*, in *Advances in Operator Algebras and Mathematical Physics*, pp. 89–108, Theta Ser. Adv. Math., 5, Theta, Bucharest, 2005.
[117] V. Guillemin, *Lectures on spectral theory of elliptic operators*, Duke Math. J. 44 (1977) no. 3, 485–517.
[118] D. Gurevich and L. Vainerman, *Non-commutative analogues of q-special polynomials and a q-integral on a quantum sphere*, J. Phys. A 31 (1998) 7, 1771–1780.
[119] E. Ha and F. Paugam, *Bost-Connes-Marcolli systems for Shimura varieties. I. Definitions and formal analytic properties*, IMRP 2005 5 (2005) 237–286.
[120] P. M. Hajac, M. Khalkhali, B. Rangipour and Y. Sommerhäuser, *Hopf-cyclic homology and cohomology with coefficients*, C.R. Math. Acad. Sci. Paris 338 (2004) no. 9, 667–672.
[121] E.H. Hall, *On a new action of the magnet on electric currents*, Amer. J. Math. 2 (1879) 287.
[122] E. Hawkins, *Hamiltonian gravity and noncommutative geometry*, Commun. Math. Phys. 187 (1997) no. 2, 471–489.
[123] W. Heisenberg, *Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen*, Z. Phys. 33 (1925) 879–893.
[124] G. Hochschild, B. Kostant and A. Rosenberg, *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc. 102 (1962) 383–408.
[125] G. 't Hooft and M. Veltman, *Regularization and renormalization of gauge fields*, Nucl. Phys. B 44 (1972) no. 1, 189–213.
[126] S. Hurder, *Secondary classes and the von Neumann algebra of a foliation*, Preprint MSRI Berkeley (1983).
[127] A. Jaffe, A. Lesniewski and K. Osterwalder, *Quantum K-theory: The Chern character*, Commun. Math. Phys. 118 (1988) 1–14.
[128] P. Julg, *Travaux de N. Higson et G. Kasparov sur la conjecture de Baum-Connes*, Seminaire Bourbaki, (1997-98), Expose 841 252 (1998) 151–183.
[129] W. Kalau and M. Walze, *Gravity, noncommutative geometry and the Wodzicki residue*, J. Geom. Phys. 16 (1995) 327–344.
[130] G. G. Kasparov, *The operator K-functor and extensions of C^* algebras*, Izv. Akad. Nauk SSSR, Ser. Mat. 44 (1980) 571–636; Math. USSR Izv. 16 (1981) 513–572.
[131] D. Kastler, *The Dirac operator and gravitation*, Commun. Math. Phys. 166 (1995) 633–643.
[132] D. Kastler, *A detailed account of Alain Connes' version of the standard model in noncommutative geometry. I, II*, Rev. Math. Phys. 5 (1993) no. 3, 477–532. *III*, Rev. Math. Phys. 8 (1996) no. 1, 103–165.
[133] M. Khalkhali, *Very basic noncommutative geometry*, math.KT/0408416. Lecture Notes Series IPM Tehran, Vol. 5, 2005.
[134] M. Khalkhali and B. Rangipour, *A new cyclic module for Hopf algebras*, K-Theory 27 (2002) no. 2, 111–131.
[135] M. Khalkhali and B. Rangipour, *Invariant cyclic homology*, K-Theory 28 (2003) no. 2, 183–205.
[136] M. Khalkhali and B. Rangipour, *Cyclic cohomology of (extended) Hopf algebras*, *Noncommutative Geometry and Quantum Groups*, (Warsaw, 2001), 59–89, Banach Center Publ., 61, Polish Acad. Sci., Warsaw, 2003.

[137] A. Klimyk and K. Schmüdgen, *Quantum Groups and Their Representations*, Springer Verlag, 1998.

[138] K. von Klitzing, G. Dorda and M. Pepper, *New method for high–accuracy determination of the fine–structure constant based on quantized Hall resistance*, Phys. Rev. Lett. 45 (1980) no. 6, 494–497.

[139] M. Kontsevich, *Deformation quantization of Poisson manifolds I*, IHES preprint M/97/72.

[140] M. Kontsevich, *Formal (non)commutative symplectic geometry*, in *The Gel'fand Mathematical Seminars, 1990–1992*, pp. 173–187, Birkhäuser Boston, Boston, MA, 1993.

[141] M. Kontsevich and A. L. Rosenberg, *Noncommutative smooth spaces*, in *The Gelfand Mathematical Seminars, 1996–1999*, pp. 85–108. Birkhäuser, Boston, MA, 2000.

[142] V. Lafforgue, *K-théorie bivariante pour les algèbres de Banach et conjecture de Baum–Connes*, Inv. Math. 149 (2002) 1–97.

[143] G. Landi, *An Introduction to Noncommutative Spaces and Their Geometry*, Lecture Notes in Physics: Monographs m51, Springer-Verlag, 1997.

[144] M. L. Lapidus and M. van Frankenhuysen, *Fractal Geometry and Number Theory. Complex Dimensions of Fractal Strings and Zeros of Zeta Functions*, Birkhäuser, Boston, MA, 2000.

[145] B. Laughlin, *Quantized Hall conductivity in two dimensions*, Phys. Rev. B 23 (1981) 5632.

[146] H. Laurent, *Sur la théorie des nombres premiers*, C.R. Acad. Sci. Paris 126 (1898) 809–810.

[147] M. Lesch, *Operators of Fuchs Type, Conical Singularities, and Asymptotic Methods*, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997.

[148] J.-M. Lescure, *Triplets Spectraux Pour les Variétés á Singularité Conique Isolée*, Bull. Soc. Math. France 129 (2001) no. 4, 593–623.

[149] P. J. Lu and P. J. Steinhardt, *Decagonal and quasi-crystalline tilings in medieval Islamic architecture*, Science 23 315 (2007) no. 5815, 1106–1110.

[150] F. Lizzi, G. Mangano, G. Miele and G. Sparano, *Fermion Hilbert space and fermion doubling in the noncommutative geometry approach to gauge theories*, Phys. Rev. D 55 (1997) 6357.

[151] Yu. I. Manin, *The notion of dimension in geometry and algebra*, Bull. Amer. Math. Soc. (N.S.) 43 (2006) no. 2, 139–161.

[152] Yu. I. Manin, *Real Multiplication and Noncommutative Geometry*, The legacy of Niels Henrik Abel, 685–727, Springer, Berlin, 2004.

[153] Yu. I. Manin, *Topics in Noncommutative Geometry*, Princeton University Press, Princeton, NJ, 1991.

[154] Yu. I. Manin, *Three-dimensional hyperbolic geometry as ∞-adic Arakelov geometry*, Invent. Math. 104 (1991) 223–244.

[155] Yu. I. Manin, *Quantum groups and noncommutative geometry*, Centre Recherche Math. Univ. Montréal 1988.

[156] Yu. I. Manin and M. Marcolli, *Continued fractions, modular symbols, and noncommutative geometry*, Selecta Math. (N.S.) 8 (2002) no. 3, 475–521.

[157] Yu. I. Manin and A. A. Panchishkin, *Introduction to Modern Number Theory. Fundamental Problems, Ideas and Theories*, 2nd ed. Encyclopaedia of Mathematical Sciences, 49. Springer-Verlag, 2005. xvi+514 pp.

[158] M. Marcolli, *Arithmetic noncommutative geometry*, University Lecture Series, 36. Amer. Math. Soc. 2005. xii+136 pp.

[159] M. Marcolli, *Limiting modular symbols and the Lyapunov spectrum*, J. Number Th. 98 (2003) no. 2, 348–376.

[160] M. Marcolli, *Modular curves, C^*-algebras, and chaotic cosmology*, in *Frontiers in Number Theory, Physics, and Geometry, II* pp. 361–372, Springer Verlag, 2006.

[161] M. Marcolli and V. Mathai, *Twisted index theory on good orbifolds, II: Fractional quantum numbers*, Commun. Math. Phys. 217 (2001) no. 1, 55–87.

[162] M. Marcolli and V. Mathai, *Twisted index theory on good orbifolds, I: Noncommutative Bloch theory*, Commun. Contemp. Math. 1 (1999) 553–587.

[163] M. Marcolli and V. Mathai, *Towards the fractional quantum Hall effect: A noncommutative geometry perspective*, in *Noncommutative Geometry and Number Theory*, pp. 235–262. Vieweg Verlag, 2006.

[164] V. Mathai and J. Rosenberg, *T-duality for torus bundles with H-fluxes via noncommutative topology*, Commun. Math. Phys. 253 (2005) no. 3, 705–721

[165] J. Mather, *Commutators of diffeomorphisms. II*, Comment. Math. Helv. 50 (1975) 33–40.

[166] R. Meyer, *On a representation of the idele class group related to primes and zeros of L-functions*, Duke Math. J. 127 (2005) no. 3, 519–595.

[167] J. S. Milne, *Shimura varieties: The geometric side of the zeta function*, preprint www.jmilne.org.

[168] J. Milnor and D. Stasheff, *Characteristic Classes*, Ann. Math. Stud., Princeton University Press, Princeton, NJ, 1974.

[169] B. Monthubert, *Pseudodifferential calculus on manifolds with corners and groupoids* Proc. Amer. Math. Soc. 127 (1999) no. 10, 2871–2881.

[170] B. Monthubert and V. Nistor, *A topological index theorem for manifolds with corners*, math/0507601.

[171] D. Nagy, *Visual mathematics: A missing link in a split culture*, Vis. Math. 1 (1999) no. 2, 20 pp. (electronic).

[172] N. Nekrasov and A. Schwarz, *Instantons in noncommutative $\mathbb{R}^4$ and $(2,0)$ superconformal six dimensional theory*, Commun. Math. Phys. 198 (1998) no. 3, 689–703.

[173] S. Neshveyev, L. Tuset, *The Dirac operator on compact quantum groups*, math/0703161v2.

[174] W. Parry and S. Tuncel, *Classification Problems in Ergodic Theory*, London Math. Soc. Lecture Notes Series 67, 1982.

[175] R. Penrose, *Pentaplexity: A class of nonperiodic tilings of the plane*, Math. Intelligencer 2 (1979/80) no. 1, 32–37.

[176] M. V. Pimsner, *A class of C^*-algebra generalizing both Cuntz–Krieger algebras and crossed products by Z*, Free probablility Theory, 189-212. Fields Inst. Comm. Vol. 12, Amer. Math. Soc., 1997.

[177] M. Pimsner and D. Voiculescu, *Exact sequences for K groups and Ext group of certain crossed product C^*-algebras*, J. Operator Th. 4 (1980) 93–118.

[178] A. Polishchuk, *Noncommutative two-tori with real multiplication as noncommutative projective varieties*, J. Geom. Phys. 50 (2004) no. 1–4, 162–187.

[179] M. Puschnigg, *Characters of Fredholm modules and a problem of Connes*, math.KT/0604528.

[180] I. Putnam, *C^*-algebras from Smale spaces*, Can. J. Math. 48 (1996) no. 1, 175–195.

[181] I. Putnam and J. Spielberg, *The structure of C^*-algebras associated with hyperbolic dynamical systems*, J. Funct. Anal. 163 (1999) 279–299.

[182] J.-P. Ramis, *Séries divergentes et théories asymptotiques*, Bull. Soc. Math. France, 121 (Panoramas et Syntheses, suppl.) 74, 1993.

[183] J. Renault, *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Math. 793, Springer, Berlin, 1980.
[184] A. Rennie and J. Várilly, *Reconstruction of manifolds in noncommutative geometry*, math.OA/0610418.
[185] M. A. Rieffel, *C^*-algebras associated with irrational rotations*, Pac. J. Math., 93 (1981) 415–429.
[186] M. A. Rieffel, *The cancellation theorem for projective modules over irrational rotation C^*-algebras*, Proc. London Math. Soc., 47 (1983) 285–302.
[187] M. A. Rieffel, *Deformation quantization of Heisenberg manifolds*, Commun. Math. Phys. 122 (1989) no. 4, 531–562.
[188] G. Robertson, *Boundary actions for affine buildings and higher rank Cuntz–Krieger algebras*, in J. Cuntz, S. Echterhoff (eds.), C*-*algebras*, Springer Verlag 2001, pp. 182–202.
[189] A. L. Rosenberg, *Noncommutative schemes*, Compositio Math. 112 (1998) no. 1, 93–125.
[190] A. L. Rosenberg, *Noncommutative Algebraic Geometry and Representations of Quantized Algebras*, Kluwer, Dordrecht, 1995.
[191] D. Ruelle, *Noncommutative algebras for hyperbolic diffeomorphisms*, Invent. Math. 93 (1988) 1–13.
[192] T. Schucker, *Spin group and almost commutative geometry*, hep-th/0007047.
[193] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, J. High Energy Phys. (1999) no. 9, Paper 32, 93 pp.
[194] M. M. Sheikh-Jabbari, *Tiny Graviton Matrix Theory: DLCQ of IIB Plane-Wave String Theory, A Conjecture*, J. High Energy Phys. 0409 (2004) 017.
[195] G. Shimura, *Arithmetic Theory of Automorphic Functions*, Iwanami Shoten and Princeton 1971.
[196] G. Skandalis, *Approche de la conjecture de Novikov par la cohomologie cyclique*, Seminaire Bourbaki, (1990-91), Expose 739, Astérisque No.201-203 (1992), 299–320.
[197] G. Skandalis, *Progrès récents sur la conjecture de Baum-Connes. Contribution de Vincent Lafforgue*, Séminaire Bourbaki, Vol.1999-2000, Expose 869, Astérisque No. 276 (2002), 105–135.
[198] C. Skau, *Orbit structure of topological dynamical systems and its invariants*, in *Operator Algebras and Quantum Field Theory* (Rome, 1996), 533–544, Internat. Press, Cambridge, MA, 1997.
[199] S. P. Smith and J. T. Stafford, *Regularity of the four dimensional Sklyanin algebra*, Compos. Math. 83 (1992) 259–289.
[200] Y. Soibelman, *Quantum tori, mirror symmetry and deformation theory*, Lett. Math. Phys. 56 (2001) 99–125.
[201] J. Spielberg, *Free–product groups, Cuntz–Krieger algebras, and covariant maps*, Int. J. Math. 2 (1991) no. 4, 457–476.
[202] W. van Suijlekom, L. Dabrowski, G. Landi, A. Sitarz and J. C. Värilly, *Local index formula for $SU_q(2)$*, K-Theory 35 (2005) no. 3–4, 375–394.
[203] D. Sullivan, *Geometric Periodicity and the Invariants of Manifolds*, Lecture Notes in Math., 197, Springer, 1971.
[204] T. Sunada, *A discrete analogue of periodic magnetic Schrödinger operators*, Contemp. Math. 173 (1994) 283–299.
[205] M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and Its Applications*, Lecture Notes in Math., 28, Springer, 1970.
[206] M. Veltman, *Diagrammatica*, Cambridge University Press, 1994.

[207] B. L. van der Waerden, *Sources of Quantum Mechanics*, Dover, NY, 1967.

[208] Abu'l-Wafa' al-Buzjani, *Kitab fima yahtaju ilayhi al-sani' min a'mal al-handasa*, Milan manuscript (Arabic): Biblioteca Ambrosiana, Arab. 68.

[209] A. Wassermann, *Ergodic actions of compact groups on operator algebras. III. Classification for* SU(2), Invent. Math. 93 (1988) 2, 309–354.

[210] H. E. Winkelnkemper, *The graph of a foliation*, Ann. Global Anal. Geom. 1 (1983) no 3, 51–75.

[211] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. B 460 (1996) 335–360.

[212] M. Wodzicki, *Noncommutative residue, Part I. Fundamentals*, in *K-Theory, Arithmetic and Geometry*. Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[213] D. Zagier, *Modular forms and differential operators*, Proc. Indian Acad. Sci. Math. Sci. (K. G. Ramanathan memorial issue) 104 (1994) no. 1, 57–75.

An Invitation to Noncommutative Geometry

RENORMALIZATION OF NONCOMMUTATIVE QUANTUM FIELD THEORY

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Contents

We recall some models for noncommutative space-time and discuss quantum field theories on these deformed spaces. We describe the IR/UV mixing disease, which implies nonrenormalizability for a large class of models. We review the power-counting analysis of field theories on the Moyal plane in momentum space and our recent renormalization proof of noncommutative ϕ^4-theory based on renormalization group techniques for dynamical matrix models. Some further developments of dynamical matrix models are mentioned.

Keywords: Noncommutative geometry; quantum field theory; renormalization.

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1. Introduction

Four-dimensional quantum field theory suffers from infrared and ultraviolet divergences as well as from the divergence of the renormalized perturbation expansion. Despite the impressive agreement between theory and experiments and despite many attempts, these problems are not settled and remain a big challenge for theoretical physics. Furthermore, attempts to formulate a quantum theory of gravity have not yet been fully successful. It is astonishing that the two pillars of modern physics, quantum field theory and general relativity, seem incompatible. This convinced physicists to look for more general descriptions: After the formulation of supersymmetry and supergravity string theory was developed, and anomaly cancellation forced the introduction of six additional dimensions. On the other hand, loop gravity was formulated, and led to spin networks and space-time foams. Both approaches are not fully satisfactory. A third impulse came from noncommutative geometry developed by Alain Connes, providing a natural interpretation of the Higgs effect at the classical level. This finally led to noncommutative quantum field theory, which is the subject of this contribution. It allows one to incorporate fluctuations of space into quantum field theory. There are of course relations among these three developments. In particular, the field theory limit of string theory leads to certain noncommutative field theory models, and some models defined over fuzzy spaces are related to spin networks.

We know from quantum mechanics that any measurement uncertainty (enforced by principles of Nature and not due to lack of experimental skills) goes hand in hand with noncommutativity. To the best of our knowledge, the possibility that geometry loses its meaning in quantum physics was first [a] considered by Schrödinger [2]. On the other hand, Heisenberg suggested to use coordinate uncertainty relations to ameliorate the short-distance singularities in the quantum theory of fields. His idea (which was first told to Ehrenfest in a letter in 1930 and appeared later [3]) inspired Peierls in the treatment of electrons in a strong external magnetic field [4]. Via Pauli and Oppenheimer the idea came to Snyder, who was the first to write down uncertainty relations between coordinates [5].

The uncertainty relations for coordinates were revived by Doplicher, Fredenhagen and Roberts [6] as a means to avoid gravitational collapse when localizing events with extreme precision.

1.1. *Noncommutative geometry*

Space-time differs from the quantum mechanical phase space. Field theory has to be defined over it. Thus, apart from only describing the algebra of space-time operators, we have to realize the geometry of gauge fields, fermions, differential calculi, Dirac operators and action functionals associated with this algebra. Fortunately for us, the relevant mathematical framework—*noncommutative geometry*—has been developed, foremost by Alain Connes [7, 8]. Related monographs are [9–12].

Noncommutative geometry is the reformulation of geometry in an algebraic and functional-analytic language, in this way permitting an enormous generalization. In physics, the most important achievement of noncommutative geometry is to overcome the distinction between *continuous* and *discrete* spaces, in the same way as quantum mechanics unified the concepts of waves and particles.

Eventually, noncommutative geometry achieved via the spectral action principle [13] a true unification of the Standard Model with general relativity on the level of classical field theories. Kinematically, Yang–Mills fields, Higgs fields and gravitons are all regarded as *fluctuations* of the free Dirac operator [14]. The spectral action

$$S = \mathrm{trace}\,\chi\Big(z\frac{\mathcal{D}^2}{\Lambda^2}\Big)\,, \tag{1}$$

(which is the weighted sum of the eigenvalues of $\mathcal{D}^2$ up to the cutoff Λ^2)

[a] Actually, Riemann himself speculated in his famous Habilitationsvortrag [1] about the possibility that the hypotheses of geometry lose their validity in the infinitesimally small regime.

of the single fluctuated Dirac operator $\mathcal{D}$ gives the complete bosonic action of the Standard Model, the Einstein–Hilbert action (with cosmological constant) and an additional Weyl action term in one stroke [13].

Of course, the unification of the standard model with general relativity via the spectral action is of limited value so long as it is not achieved at the level of quantum field theory. On the other hand, the arguments of [6] make clear that this will not be possible with almost commutative geometries (products of commutative geometries with matrices). Space-time has to be noncommutative itself. The complete problem of a gravitational dynamics of the noncommutative space-time being too difficult to treat, the first step is to consider field theory on noncommutative background spaces.

2. Some Models for Noncommutative Space(-Time)

2.1. *The Moyal plane*

The best-studied candidate for noncommutative space-time is the Moyal plane [15, 16], which was identified as a solution of the uncertainty conditions for coordinate operators [6]. The (D-dimensional) Moyal plane $\mathbb{R}^D_\theta$ is characterized by the *non-local* $\star$-product

$$(a\star b)(x) := \int d^D y \frac{d^D k}{(2\pi)^D} a\Big(x+\frac{1}{2}\theta\cdot k\Big) b(x+y)\, \mathrm{e}^{\mathrm{i}ky} , \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R} . \tag{2}$$

Here, $a, b \in \mathcal{S}(\mathbb{R}^D)$ are (complex-valued) Schwartz class functions of rapid decay. The entries $\theta^{\mu\nu}$ in (2) have the dimension of an area. Generalizations of (2) to deformations of C^*-algebras are considered in [17].

Using the identity $\int \frac{d^D k}{(2\pi)^D}\, \mathrm{e}^{\mathrm{i}k\cdot(x-y)} = \delta(x-y)$ it is not difficult to prove that the $\star$-product (2) is associative $((a\star b)\star c)(x) = (a\star(b\star c))(x)$ and noncommutative, $a\star b \neq b\star a$. Moreover, complex conjugation is an involution, $\overline{a\star b} = \overline{b}\star\overline{a}$. One has the important property

$$\int d^D x\ (a\star b)(x) = \int d^D x\ a(x) b(x) . \tag{3}$$

Partial derivatives are derivations, $\partial_\mu(a\star b) = (\partial_\mu a)\star b + a\star(\partial_\mu b)$. For various proofs (such as in [18]) one needs the fact that for each $f \in \mathbb{R}^D_\theta$ there exist $f_1, f_2 \in \mathbb{R}^D_\theta$ with $f = f_1 \star f_2$, see [19].

The Moyal product (2) has its origin in quantum mechanics, in particular in Weyl's operator calculus. Wigner introduced the useful concept of the phase space distribution function [20]. Then, Groenewold [15] and Moyal [16] showed that quantum mechanics can be formulated on classical phase space using the *twisted product* concept. In particular, Moyal proposed the "sine-Poisson bracket" (nowadays called Moyal bracket), which is the analogue of the quantum mechanical commutation relations. The twisted product was extended from Schwartz class functions to (appropriate) tempered distributions by Gracia-Bondía and Várilly. The programme of Groenewold and Moyal culminated in the axiomatic approach of *deformation quantization* [21, 22]. The problem of lifting a given Poisson structure to an associative $\star$-product was solved by Kontsevich [23]. Cattaneo and Felder [24] found a physical derivation of Kontsevich's formula in terms of a path integral quantization of a Poisson sigma model [25]. The Moyal plane is a spectral triple [18] and the spectral action has been computed [26, 27].

There is a (unfortunately more popular) different version of the $\star$-product,

$$(a \star b)(x) = \exp\Big(\mathrm{i}\theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\nu}\Big)a(y)b(z)\Big|_{y=z=x}\;, \tag{4}$$

which is obtained by the following steps from (2):

- Taylor expansion of $a(x+\frac{1}{2}\theta{\cdot}k)$ about $k=0$,
- repeated representation of $k_\mu \mathrm{e}^{\mathrm{i}k\cdot y} = -\mathrm{i}\frac{\partial}{\partial y^\mu}\mathrm{e}^{\mathrm{i}k\cdot y}$,
- integration by parts in y,
- k-integration yielding $\int \frac{d^Dk}{(2\pi)^D}\mathrm{e}^{\mathrm{i}k\cdot y} = \delta(y)$,
- y-integration.

Of course, as the Taylor expansion is involved, at least one of the functions a, b has to be analytic. Actually, the formula (4) is an *asymptotic expansion* of the $\star$-product (2) which becomes exact under the conditions given in [28]. We would like to stress that the most important property concerning physics is the *non-locality* of the $\star$-product (2), not its noncommutativity. To the value of $a \star b$ at the point x there are contributions of individual values of the functions a, b far away from x. This non-locality is hidden in (4): At first sight it seems to be local, as only the derivatives of a, b at x contribute to $(a \star b)(x)$. However, the point is that analyticity is required, where the information about a function is not localized at all.

A third version of the $\star$-product which is particularly useful for field theory in momentum space is obtained by expressing on the R.H.S. of (2) the functions by their Fourier transforms. [b] This yields

$$(a \star b)(x) = \int \frac{d^D p}{(2\pi)^D}\, \mathrm{e}^{-\mathrm{i}px} \int \frac{d^D q}{(2\pi)^D}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu} p_\mu q_\nu}\, \hat{a}(p-q)\hat{b}(q)\ . \tag{5}$$

Being a non-compact space, the algebra $\mathbb{R}^D_\theta$ cannot have a unit. For various reasons, the restriction of the $\star$-product to Schwartz class functions should be relaxed. That extension to tempered distributions was performed in [19]. A good summary is the appendix of [29]. Since (2) is smooth, for T being a tempered distribution and $f, g \in \mathcal{S}(\mathbb{R}^D)$ one defines the product $T \star f$ via

$$\langle T \star f, g\rangle := \langle T, f \star g\rangle\ , \tag{6}$$

and similarly for $f \star T$. Both $T \star f$ and $f \star T$ are smooth functions, but not necessarily of Schwartz class. The set of those T for which $T \star f$ is of Schwartz class is the left multiplier algebra $M_L(\mathbb{R}^D_\theta)$, and similarly for $M_R(\mathbb{R}^D_\theta)$ (which is different). Then, the *Moyal algebra* is defined as $M(\mathbb{R}^D_\theta) := M_L(\mathbb{R}^D_\theta) \cap M_R(\mathbb{R}^D_\theta)$. It is a unital algebra (in fact the largest compactification of $\mathbb{R}^D_\theta$) and contains also the coordinate functions x^μ and the "plane waves" $\mathrm{e}^{\mathrm{i}p_\mu x^\mu}$. In fact, the famous commutation relation $[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}$ holds in $M(\mathbb{R}^D_\theta)$ and not in $\mathbb{R}^D_\theta$. The Moyal algebra is huge so that for practical purposes appropriate subalgebras must be considered [19, 18]. There are several surprises on $M(\mathbb{R}^D_\theta)$: For instance, the Dirac δ-distribution belongs to $M(\mathbb{R}^D_\theta)$, with $\delta \star \delta = \frac{2^D}{\det\theta}1$. On the other hand, $\mathrm{e}^{\frac{2\mathrm{i}}{a}x^1x^2} \in M(\mathbb{R}^2_\theta)$ iff $|a| \neq \theta_1$, $\theta_1 := \theta^{12} = -\theta^{21}$. This proves, by the way, that for different θ the Moyal algebras $M(\mathbb{R}^D_\theta)$ are different.

Traditionally, physicists expand the algebra $\mathbb{R}^D_\theta$ into the Weyl basis (plane waves) $\mathrm{e}^{\mathrm{i}p_\mu x^\mu}$, which has the advantage that the resulting computations are similar to the usual treatment of commutative field theories in momentum space. For both mathematical investigations (see e.g. [19, 18]) and our recent renormalizability proof [30] it is, however, much more convenient to use the harmonic oscillator basis given by the eigenfunctions of

[b] We use the convention that $f(x) = \int \frac{d^D p}{(2\pi)^D}\, \mathrm{e}^{-\mathrm{i}px} \hat{f}(p)$ and $\hat{f}(p) = \int d^D x\, \mathrm{e}^{\mathrm{i}px} f(x)$.

the Hamiltonian $H = \frac{1}{2}x_\mu x^\mu$. In $D = 2$ dimensions one has [31, 19]

$$H \star f_{mn} = \theta_1\Big(m + \frac{1}{2}\Big) f_{mn} \;, \qquad f_{mn} \star H = \theta_1\Big(n + \frac{1}{2}\Big) f_{mn} \;. \tag{7}$$

$$f_{mn}(x) = \frac{2}{\sqrt{n!m!\,\theta_1^{m+n}}}\,\bar{a}^{\star m} \star \mathrm{e}^{-\frac{2H}{\theta_1}} \star a^{\star n} \;, \tag{8}$$

where $a = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2)$ and $\bar{a} = \frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2)$.

The eigenfunctions f_{mn} have the remarkable property that

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x) \;, \qquad \int d^2x\, f_{mn} = (2\pi)\sqrt{\det\theta}\,\delta_{mn} \;. \tag{9}$$

Thus, the f_{mn} behave like infinite standard matrices with entry 1 at the intersection of the $(m+1)^{\mathrm{th}}$ row with the $(n+1)^{\mathrm{th}}$ column, and with entry 0 everywhere else. In fact, the decomposition

$$\mathbb{R}^2_\theta \ni a(x) = \sum_{m,n=0}^{\infty} a_{mn} f_{mn}(x) \tag{10}$$

defines a Fréchet algebra isomorphism between $\mathbb{R}^2_\theta$ and the matrix algebra of rapidly decreasing double sequences $\{a_{mn}\}$ for which

$$r_k(a) := \left(\sum_{m,n=0}^{\infty} \theta_1^{2k} \Big(m + \frac{1}{2}\Big)^k (n + \tfrac{1}{2})^k |a_{mn}|^2 \right)^{\frac{1}{2}} \tag{11}$$

is finite for all $k \in \mathbb{N}$, see [19].

Both the $f_{mn}(x)$ and their Fourier transforms are given by Laguerre polynomials in the radial direction and Fourier modes in the angular direction. On one hand, this makes clear that the f_{mn} form a basis of the two-dimensional Moyal plane. On the other hand, restricting the matrix base to finite matrices f_{mn}, $n, m \leq N$, corresponds to a cutoff both in position space and momentum space.

Further, we note that the f_{mn} are also the common eigenfunctions of the Landau Hamiltonian

$$H_L^{\pm} = \frac{1}{2}(\mathrm{i}\partial_\mu \pm A_\mu)(\mathrm{i}\partial^\mu \pm A^\mu) \;, \qquad A_\mu = \frac{1}{2} B_{\mu\nu} x^\nu \;. \tag{12}$$

If $B_{\mu\nu} = 4(\theta^{-1})_{\mu\nu}$, and thus $B := \frac{4}{\theta_1}$, one has

$$H_L^+ f_{mn} = B\Big(m + \frac{1}{2}\Big) f_{mn} \;, \qquad H_L^- f_{mn} = B\Big(n + \frac{1}{2}\Big) f_{mn} \;. \tag{13}$$

Thus, the harmonic oscillator basis has the additional merit of diagonalizing the Landau Hamiltonian. This observation was the starting point of various exact solutions of quantum field theories on noncommutative phase space [32–34].

For more information about the noncommutative $\mathbb{R}^D$ we refer to [19, 35, 18].

2.2. *The noncommutative torus*

The Moyal plane is closely related to the noncommutative torus, which is the best-studied noncommutative space [36, 37]. A basis for the algebra $\mathbb{T}^D_\theta$ of the noncommutative D-torus is given by unitaries U^p labelled by $p = \{p_\mu\} \in \mathbb{Z}^D$, with $U^p(U^p)^* = (U^p)^*U^p = 1$. The multiplication is defined by

$$U^pU^q = \mathrm{e}^{\mathrm{i}\pi\theta^{\mu\nu}p_\mu q_\nu} U^{p+q} \;, \qquad \mu,\nu = 1,\dots,D \;, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R} \;. \tag{14}$$

Elements $a \in \mathbb{T}^d_\theta$ have the following form:

$$a = \sum_{p\in\mathbb{Z}^d} a_p U^p \;, \qquad a_p \in \mathbb{C} \;, \quad \|p\|^n |a_p| \to 0 \text{ for } \|p\| \to \infty \;. \tag{15}$$

If $\theta^{\mu\nu} \notin \mathbb{Q}$ (irrational case) one can define partial derivatives

$$\partial_\mu U^p := -\mathrm{i}p_\mu U^p \;, \tag{16}$$

which satisfy the Leibniz rule and Stokes' law with respect to the integral

$$\int a = a_0 \;, \tag{17}$$

where a is given by (15).

An excellent presentation of the noncommutative torus was given by Rieffel [38].

Other interesting noncommutative spectral triples are the Connes–Landi spheres [39] and the (mostly spherical) examples found by Connes and Dubois-Violette [40].

2.3. *Fuzzy spaces*

The fuzzy sphere [41] is one of the simplest noncommutative spaces. It is obtained by truncating the representations of $su(2)$. The algebra S^2_N is identified with the mappings from the representation space $\frac{N}{2}$ of $su(2)$ to itself, thus with the algebra $M_{N+1}(\mathbb{C})$. The fuzzy sphere S^2_N is generated by $\hat{X}_i$, $i = 1,2,3$, which form an $su(2)$-Lie algebra with suitable rescaling, identified by the requirement that the Casimir operator still fulfils the defining relation of the two-sphere as an operator:

$$[\hat{X}_i, \hat{X}_j] = \sum_{k=1}^{3} \mathrm{i}\lambda_N \epsilon_{ijk} \hat{X}_k \ , \quad \sum_{i=1}^{3} \hat{X}_i \hat{X}_i = R^2 \ , \quad \frac{R}{\lambda_N} = \sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)} \ . \tag{18}$$

One has to give a precise description of the embeddings of these algebras for different N. Then, for fixed radius R, one recovers the commutative algebra of the ordinary sphere, $\lambda = 0$, in the limit $N \to \infty$ [42].

The Lie algebra $su(2)$ generated by J_i, $i = 1,2,3$, acts on $a \in S^2_N$ by the adjoint action

$$J_i a = \frac{1}{\lambda}[\hat{X}_i, a] \ . \tag{19}$$

Thus, an element $a \in S^2_N$ can be represented by $a = \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm} \Psi_{lm}$, where

$$\sum_{i=1}^{3} J_i^2 \Psi_{lm} = l(l{+}1)\Psi_{lm}, \ J_3 \Psi_{lm} = m\Psi_{lm}, \ \frac{4\pi}{N{+}1}\mathrm{tr}\big(\overline{\Psi_{lm}}\Psi_{l'm'}\big) = \delta_{ll'}\delta_{mm'} \ . \tag{20}$$

For comments on field theoretical models, see Sec. 4. Other fuzzy spaces include the fuzzy $\mathbb{C}P^2$ [43, 44] and the q-deformed fuzzy sphere [45, 46].

3. Classical Field Theory on Noncommutative Spaces

Since classical field theories can be geometrically described, it is not difficult to write down classical action functionals on noncommutative spaces. The first example of this type was Yang–Mills theory on the noncommutative torus. Another example is the noncommutative geometrical description of the Standard Model recalled briefly in Sec. 1.1.

3.1. *Field theory on the noncommutative torus*

The noncommutative torus became popular to field theorists when Connes, Douglas and Schwarz [39] proposed to compactify M-theory on such a space. M-theory lives in higher dimensions so that some of them must be compactified to give a realistic model. Compactifying on a noncommutative instead of a commutative torus amounts to turning on a constant background 3-form C. An alternative interpretation based on D-branes on tori in the presence of a Neveu–Schwarz B-field was given by Douglas and Hull. Similar effects are obtained in boundary conformal field theory [47]. There are also other noncommutative spaces which arise as limiting cases of string theory [48].

Later, the appearance of noncommutative field theory in the zero-slope limit of type II string theory was thoroughly investigated by Seiberg and Witten [49]. Moreover, using the results of [50] about instantons on noncommutative $\mathbb{R}^4$, Seiberg and Witten argued that there is an equivalence between the Yang-Mills theories on standard $\mathbb{R}^4$ and noncommutative $\mathbb{R}^4$, which we comment on in Sec. 5.4.

It should be mentioned that matrix theories were studied long before M-theory was proposed, and that these matrix theories did contain certain noncommutative features. In the large-N limit of two-dimensional $SU(N)$ lattice gauge theory, the number of degrees of freedom is reduced and corresponds to a zero-dimensional model, under the condition that no spontaneous breakdown of the $[U(1)]^4$-symmetry appears. As shown in [51], a spontaneous symmetry breakdown does not appear when twisted boundary conditions [52] are used. In [53], the construction of the twisted Eguchi–Kawai model was extended to any even dimension. Here, the action can be rewritten in terms of noncommuting matrix derivatives $[\Gamma^{(j)},\ .\ ]$, with $[\Gamma^{(2j)},\Gamma^{(2j+1)}] = -2\pi\mathrm{i}/N$.

3.2. *Classical action functionals on the Moyal plane*

Here, we list for the example of the Moyal plane a few important action functionals for noncommutative field theories. In principle, these action functionals are related to connections on projective modules. To simplify the presentation, we restrict ourselves to trivial modules given by the algebra R^D_θ itself.

The most natural action from the point of view of noncommutative geometry is $U(N)$ Yang–Mills theory in four dimensions:

$$S_{\mathrm{YM}}[A] = \int d^4x \, \mathrm{tr}_{M_N(\mathbb{C})}\Big(\frac{1}{4g^2} F_{\mu\nu} \star F^{\mu\nu}\Big) \;, \tag{21}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}(A_\mu \star A_\nu - A_\nu \star A_\mu) \;, \tag{22}$$

where $A_\mu = A_\mu^* \in \mathbb{R}^4_\theta \otimes M_N(\mathbb{C})$. This action arises from the Connes–Lott action functional [54] and the spectral action principle [26, 27] as well as in the zero-slope limit of string theory [49]. For quantum field theory it has to be extended—as usual—by the ghost sector:

$$S_{\mathrm{gf}} = \int d^4x \, \mathrm{tr}_{M_N(\mathbb{C})}\Big(s\Big\{\bar{c} \star \partial_\mu A^\mu + \frac{\alpha}{2}\bar{c} \star B + \rho^\mu \star A_\mu + \sigma \star c\Big\}\Big) \;, \tag{23}$$

where α is the gauge parameter. The components of $\bar{c}, c, \rho^\mu$ are anticommuting fields and the graded BRST differential s [55] (which commutes with ∂_μ) is defined by

$$\begin{aligned} sA_\mu &= \partial_\mu c - \mathrm{i}(A_\mu \star c - c \star A_\mu) \;, \qquad & sc &= \mathrm{i}c \star c \;, \\ s\bar{c} &= B \;, & sB &= s\rho^\mu = s\sigma = 0 \;. \end{aligned} \tag{24}$$

The external fields ρ^μ and σ are the Batalin–Vilkovisky antifields [56] relative to A_μ and c, respectively.

There is always a reference frame in which the noncommutativity matrix θ takes the standard form where the only non-vanishing components are $\theta_{2i,2i+1} = -\theta_{2i+1,2i} \equiv \theta_i$. Each of the two-dimensional blocks is invariant under two-dimensional rotations. This means that action functionals which involve the $\star$-product like (21) are invariant under the subgroup $\big(SO(2)\big)^{\frac{D}{2}}$ of the D-dimensional rotation group $SO(D)$.

The Yang–Mills action (21) suggests that action functionals for field theories on $\mathbb{R}^D_\theta$ are simply obtained by replacing the ordinary (commutative) product of functions on Euclidean space by the $\star$-product (2). This procedure leads to the following action for noncommutative ϕ^4-theory:

$$S[\phi] := \int d^Dx \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}\mu^2 \phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\Big)(x) \;. \tag{25}$$

It must be stressed, however, that this is a formal procedure and that—in contrast to the Yang–Mills action (21)—the scalar field action (25) does not directly follow from noncommutative geometry or the scaling limit of string theory [49]. In fact, we have proven in [30] that it has to be extended by a harmonic oscillator term.

It was pointed out by Langmann and Szabo [57] that the $\star$-product interaction is (up to rescaling) invariant under a duality transformation between positions and momenta. Indeed, using a modified Fourier transform $\hat{\phi}(p_a) = \int d^4x\, \mathrm{e}^{(-1)^a \mathrm{i} p_{a,\mu} x_a^\mu} \phi(x_a)$, where the subscript a refers to the cyclic order in the $\star$-product, one obtains from the definitions (2) and (5) and the reality condition $\phi(x) = \overline{\phi(x)}$ the representation

$$\begin{aligned} S_{\mathrm{int}}[\phi;\lambda] &= \int d^4x\, \frac{\lambda}{4!}(\phi\star\phi\star\phi\star\phi)(x) \\ &= \int \Big(\prod_{a=1}^4 d^4x_a\Big)\, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\, V(x_1,x_2,x_3,x_4) \\ &= \int \Big(\prod_{a=1}^4 \frac{d^4p_a}{(2\pi)^4}\Big)\, \hat{\phi}(p_1)\hat{\phi}(p_2)\hat{\phi}(p_3)\hat{\phi}(p_4)\, \hat{V}(p_1,p_2,p_3,p_4)\;, \end{aligned} \tag{26}$$

with

$$\hat{V}(p_1,p_2,p_3,p_4) = \frac{\lambda}{4!}(2\pi)^4\delta^4(p_1-p_2+p_3-p_4)\, \cos\Big(\tfrac{1}{2}\theta^{\mu\nu}(p_{1,\mu}p_{2,\nu}+p_{3,\mu}p_{4,\nu})\Big)\;, \tag{27a}$$

$$V(x_1,x_2,x_3,x_4) = \frac{\lambda}{4!}\frac{1}{\pi^4\det\theta}\,\delta^4(x_1-x_2+x_3-x_4)\, \cos\Big(2(\theta^{-1})_{\mu\nu}(x_1^\mu x_2^\nu+x_3^\mu x_4^\nu)\Big)\,. \tag{27b}$$

Thus, the replacements

$$\hat{\phi}(p) \leftrightarrow \pi^2\sqrt{|\det\theta|}\,\phi(x)\;, \qquad p_\mu \leftrightarrow \tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}x^\nu\;, \tag{28}$$

exchange the a,b-versions of (26) and (27).

On the other hand, the usual free scalar field action given by $\lambda = 0$ in (25) is not invariant under this duality transformation. In order to achieve this we have to extend the free scalar field action by a harmonic oscillator potential:

$$S_{\text{free}}[\phi;\mu,\Omega_0]=\int d^4x\;\Big(\frac{1}{2}(\partial_\mu\phi)\star(\partial^\mu\phi)+\frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)+\frac{\mu^2}{2}\phi\star\phi\Big)(x)\;. \tag{29}$$

Of course, the oscillator potential breaks translation invariance. For complex scalar fields φ of electric charge Ω, another possibility is given by a constant external magnetic field $B_{\mu\nu}=4(\theta^{-1})_{\mu\nu}$ via the covariant derivative $D_\mu\varphi:=\partial_\mu\varphi+\mathrm{i}\Omega A_\mu\varphi$, with $A_\mu=\frac{1}{2}B_{\mu\nu}x^\nu$:

$$S^B_{\text{free}}[\varphi;\mu,\Omega]=\int d^4x\;\Big(\frac{1}{2}(D_\mu\varphi)^*\star(D^\mu\varphi)+\frac{\mu^2}{2}\varphi^*\star\varphi\Big)(x)\;. \tag{30}$$

Adding the interaction term $S_{\text{int}}[\varphi;\lambda]=\frac{\lambda}{4!}\int d^4x\,\varphi\star\varphi^*\star\varphi\star\varphi^*$, the quantum field theory associated with the magnetic field action (30) was analyzed and for $\Omega=1$ exactly solved in [33, 34]. Note that

$$\begin{aligned}&S_{\text{free}}[\phi_1;\mu,\Omega]+S_{\text{free}}[\phi_2;\mu,\Omega]\\&\qquad=\frac{1}{2}S^B_{\text{free}}[\phi_1+\mathrm{i}\phi_2;\mu,\Omega]+\frac{1}{2}S^B_{\text{free}}[\phi_1+\mathrm{i}\phi_2;\mu,-\Omega]\;.\end{aligned} \tag{31}$$

The interaction mixes ϕ_1,ϕ_2, though.

Now, under the transformation (28) one has for the total action $S=S_{\text{free}}+S_{\text{int}}$

$$S\big[\phi;\mu,\lambda,\Omega\big]\mapsto\Omega^2 S\Big[\phi;\frac{\mu}{\Omega},\frac{\lambda}{\Omega^2},\frac{1}{\Omega}\Big]\;, \tag{32}$$

and accordingly for $S^B_{\text{free}}[\varphi,\Omega]+S_{\text{int}}[\varphi,\lambda]$. In the special case $\Omega=1$ the action $S[\phi;\mu,\lambda,1]$ is invariant under the duality (28) and can be written as a standard matrix model.

4. Regularization

The philosophy that space-time noncommutativity regularizes quantum field theories was made explicit in fuzzy noncommutative spaces [41, 9]. The partition function relative to the scalar field action on the fuzzy sphere,

$$
\begin{aligned}
Z[j] &= \int \mathcal{D}[\phi]\, \mathrm{e}^{-S[\phi]-\frac{4\pi}{N+1}\mathrm{tr}(j\phi)} , \\
S[\phi] &= \frac{4\pi}{N+1}\mathrm{tr}\left(\frac{1}{2}\sum_{i=1}^{3}\phi J_i^2\phi + V[\phi]\right) , \qquad \phi \in S_N^2 ,
\end{aligned}
\tag{33}
$$

where $V[\phi]$ is some polynomial in ϕ, leads to an automatic UV-regularization [58, 59, 43] of the resulting Feynman graphs. See also [60].

Of course, the standard divergences of the ϕ^4-model on the commutative sphere S^2 will reappear in the limit $N \to \infty$. This limit was investigated in [61]. For the one-loop self-energy in the ϕ^4-model, a finite but non-local difference between the $N \to \infty$ limit of the fuzzy sphere and the ordinary sphere was found. See [62] for similar calculations.

The construction of gauge models on the fuzzy sphere is less obvious. See e.g. [63, 64].

We would also like to mention another construction of finite quantum field theories on noncommutative spaces which is based on point-splitting via tensor products [65, 66].

5. Renormalization

It is not difficult to write down classical action functionals on noncommutative spaces (see Sec. 3.), but it is not clear that quantum field theories [67–69] can be defined consistently.[c] As locality is so important in quantum field theory, it is perfectly possible that quantum field theories are implicitly built upon the assumption that the action functional has to live on a (commutative) manifold.

The first results on noncommutative quantum field theories (with an infinite number of degrees of freedom) are due to Filk [70] who showed that the planar graphs of a field theory on the Moyal plane[d] are identical to the commutative theory (and thus have the same divergences). An achievement in [70] which turned out to be important for later work was the definition of the *intersection matrix* of a graph which is read off from its

[c] This refers to infinite-dimensional quantum field theories. There is no problem with finite-dimensional examples [58, 59].

[d] Filk's model refers to [6] but is formulated in the $\star$-product formalism. It is certainly inspired by the twisted Eguchi–Kawai model [51, 53] discussed in Sec. 3.1.

reduction to a rosette. In [71] the persistence of divergences was rephrased in the framework of noncommutative geometry, based on the general definition of a dimension and the noncommutative formulation of external field quantization. See also [72].

Knowing that divergences persist in quantum field theories on the Moyal plane, the question arises whether these models are renormalizable. It was, therefore, an important step to prove that Yang–Mills theory is one-loop renormalizable on the Moyal plane and on the noncommutative torus [73–75] This means that these models are divergent [70], but the one-loop divergences are absorbable in a multiplicative renormalization of the initial action such that the Ward identities are fulfilled.

In this line of success, it was somewhat surprising when Minwalla, Van Raamsdonk and Seiberg [76] pointed out that there is a new type of infrared-like divergences which makes the renormalization of scalar field theories on the Moyal plane very unlikely. Non-planar graphs are regulated by the phase factors in the $\star$-product (5), but only if the external momenta of the graph are non-exceptional. Inserting non-planar graphs (declared as regular) as subgraphs into bigger graphs, external momenta of the subgraph are internal momenta for the total graph. As such, exceptional external momenta for the subgraph are realized in the loop integration, resulting in a divergent integral for the total graph. This is the so-called *UV/IR-mixing* problem [76].

5.1. *Quantum field theory on the noncommutative torus*

The paper [77] inspired many activities on the interface between string/M-theory and noncommutative geometry (we come back to that in Sec. 5.2). Among others the question was raised whether Yang–Mills theory on the noncommutative torus is renormalizable. See also [78]. We have confirmed one-loop renormalizability in [75]: Using ζ-function techniques and cocycle identities we have extracted the pole parts related to the Feynman graphs and proved that they can be removed by multiplicative renormalization of the initial action. In particular, the Ward identities are satisfied. See also [79].

Based on ideas developed in [80] on type IIB matrix models, it was shown in [81] that, imposing a natural constraint for the (finite) matrices, the twisted Eguchi–Kawai construction [51] can be generalized to noncommutative Yang–Mills theory on a toroidal lattice. The appearing gauge-invariant operators are the analogues of Wilson loops. This formulation enabled numerical simulations [82, 83] of the various limiting procedures which confirmed conjectures [84] about striped and disordered patterns in the phase diagram of spontaneously broken noncommutative ϕ^4-theory. On the other hand, the limit $N \to \infty$ of the matrix size is mathematically delicate [85]. To deal with that problem, a new formulation [86, 87] of matrix models approximating field theories on the noncommutative torus has been proposed which is based on noncommutative solitons [88].

An important development is the exact (non-perturbative) solution of Yang–Mills theory on the two-dimensional noncommutative torus [89, 90]. This solution is in the same spirit as the original Connes–Rieffel analysis [36, 37], but expands it to completely solve the quantum theory.

5.2. *Quantum field theories on the Moyal plane*

With the motivation of the Moyal plane in [6], the proof that UV-divergences in quantum field theories persist [70], and the relationship of the noncommutative torus to M-theory [77] and the noncommutative $\mathbb{R}^D$ to type II string theory [91, 50]. The time was ready in 1998 to investigate the renormalization of quantum field theories on the noncommutative torus and the noncommutative $\mathbb{R}^D$. It is, therefore, not surprising that this question was addressed by different groups at about the same time [73–75].

Martín and Sánchez-Ruiz [73] investigated $U(1)$ Yang–Mills theory on the noncommutative $\mathbb{R}^4$ at the one-loop level. They found that all one-loop pole terms of this model in dimensional regularization [e] can be removed by multiplicative renormalization (minimal subtraction) in a way preserving the BRST symmetry. This is completely analogous to the situation on the noncommutative 4-torus [75]. Shortly after there also appeared an investigation of $(2+1)$-dimensional super-Yang–Mills theory with the two-dimensional space being the noncommutative torus [74].

The paper [49] of Seiberg and Witten from August 1999 made the interface between string theory and noncommutative geometry extremely popular. Thousands of papers on this subject appeared, making it impossible to give an adequate overview. We restrict ourselves to the renormalization question and refer to the following reviews for further information:

[e] There is of course a problem extending θ to complex dimensions; this is however discussed in [73].

- by Konechny and Schwarz with focus on compactifications of M-theory on noncommutative tori [92] as well as on instantons and solitons on noncommutative $\mathbb{R}^D$ [93],
- by Douglas and Nekrasov [94] and by Szabo [95], both with focus on field theory on noncommutative spaces in relation to string theory,
- by Aref'eva, Belov, Giryavets, Koshelev and Medvedev [96] with focus on string field theory.

A systematic analysis of field theories on noncommutative $\mathbb{R}^D$, to any loop order, was first performed by Chepelev and Roiban [97]. The essential technique is the representation of Feynman graphs as ribbon graphs, drawn on an (oriented) Riemann surface with boundary, to which the external legs of the graph are attached. Using sophisticated mathematical tools (which we review in Sec. 5.3), Chepelev and Roiban were able to relate the power-counting behavior to the topology of the graph. Their first conclusion was that a noncommutative field theory is renormalizable iff its commutative counterpart is renormalizable. Then, by computing the non-planar one-loop graphs explicitly, Minwalla, Van Raamsdonk and Seiberg pointed out a serious problem in the renormalization of ϕ^4-theory on noncommutative $\mathbb{R}^4$ and ϕ^3-theory on noncommutative $\mathbb{R}^6$ [76]. It turned out that this problem was simply overlooked in the first version of [97], with the power-counting analysis being correct. A refined proof of the power-counting theorem was given in [98].

Anyway, the problem discovered in [76] made the subject of noncommutative field theories extremely popular. In the following months, an enormous number of articles doing (mostly) one-loop computations of all kinds of models appeared. We do not want to give an overview about these activities and mention only a few papers: the two-loop calculation of ϕ^4-theory [99]; the renormalization of complex $\phi \star \phi^* \star \phi \star \phi^*$ theory [100], later explained by a topological analysis [98]; computations in noncommutative QED [101]; the calculation of noncommutative $U(1)$ Yang–Mills theory [102], with an outlook to super-Yang–Mills theory; the one-loop analysis of noncommutative $U(N)$ Yang–Mills theory [103].

5.3. *The power-counting analysis of Chepelev and Roiban*

The previously mentioned one-loop calculations are superseded by the power-counting theorem of Chepelev and Roiban [98] which decides the renormalizability question of (massive, Euclidean) quantum field theories on the Moyal plane to all orders. Roughly speaking, quantum field theories with only logarithmic divergences are renormalizable[f] on the Moyal plane. Still, the 1PI Green's functions do not exist pointwise (at exceptional momenta) so that multiplication with IR-smoothing test functions is necessary. Apart from some exceptional cases such as the $\phi \star \phi^* \star \phi \star \phi^*$ interaction, models with quadratic divergences are not perturbatively renormalizable.

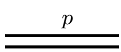
As we have the impression that the work of Chepelev and Roiban is not sufficiently known, we would like to review the main steps for the example of the noncommutative ϕ^4-model arising from the action (25). As usual, the Euclidean quantum field theory is (formally) defined via the partition function,

$$Z[J] := \int \mathcal{D}[\phi]\, \mathrm{e}^{-S[\phi] - \int d^D x\, J(x)\phi(x)} \,. \tag{34}$$

We suppose here that the fields are expanded in the Weyl basis $\phi(x) = \int \frac{d^D p}{(2\pi)^D}\, \phi(p) \mathrm{e}^{\mathrm{i}px}$, where $\phi(p)$ are commuting amplitudes of rapid decay in $\|p\|$ and $\mathrm{e}^{\mathrm{i}px}$ is the base of an appropriate subalgebra of the Moyal algebra $M(\mathbb{R}^D_\theta)$. Then, the "measure" of the functional integration is formally defined as $\mathcal{D}[\phi] = \prod_{p \in \mathbb{R}^D} d\phi(p)$.

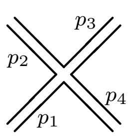
As usual, the integral (34) is solved perturbatively about the solution of the free theory given by $\lambda = 0$. The solution is conveniently organized by *Feynman graphs* built according to Feynman rules out of propagators and vertices. For the noncommutative scalar field action (25), the representation (5) leads to the following rules:

- Due to (3), the propagator is unchanged compared with commutative ϕ^4-theory, but for later purposes is written in double line notation:



$$= \frac{1}{p^2 + m^2} \,. \tag{35}$$

- The vertices receive phase factors [70] which depend on the cyclic order of the legs:



$$= \frac{\lambda}{4!}\, \mathrm{e}^{-\frac{\mathrm{i}}{2} \sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu}} \,. \tag{36}$$

[f] The reason is that logarithms are integrable, see [104] for an explicit construction of the estimations.

There is momentum conservation $p_1+p_2+p_3+p_4=0$ at each vertex (due to translation invariance of (25)).

The double line notation reflects the fact that the vertex (36) is invariant only under cyclic permutations of the legs (using momentum conservation). The resulting Feynman graphs are *ribbon graphs* [62, 97] which depend crucially on how the valences of the vertices are connected. For *planar graphs* the total phase factor of the integrand is independent of internal momenta, whereas *non-planar graphs* have a total phase factor which involves internal momenta. Planar graphs are integrated as usual and give (up to symmetry factors) the same divergences as commutative ϕ^4-theory [70]. One would remove these divergences as usual by appropriate normalization conditions for physical correlation functions. Non-planar graphs require a separate treatment.

There is a closed formula for the integral associated to a noncommutative Feynman graph in terms of the intersection matrices I, J, K (which encode the phase factors) and the incidence matrix $\mathcal{E}$. We give an orientation to each inner line l and let k_l be the momentum flowing through the line l. For each vertex v we define [g]

$$\mathcal{E}_{vl} = \begin{cases} 1 & \text{if } l \text{ emerges from } v\ , \\ -1 & \text{if } l \text{ arrives at } v\ , \\ 0 & \text{if } l \text{ is not attached to } v\ . \end{cases} \tag{37}$$

We let P_v be the total external momentum flowing into the vertex v. Restricting ourselves to four dimensions, a 1PI (one-particle irreducible) Feynman graph $\mathcal{G}$ with I internal lines and V vertices gives rise to the integral

$$\begin{aligned}\mathcal{I}_{\mathcal{G}}(P) = \int \prod_{l=1}^{I} \frac{d^4 k_l}{(k_l^2+m^2)} \prod_{v=1}^{V} (2\pi)^4 \delta\Big(P_v - \sum_{l=1}^{l} \mathcal{E}_{vl} k_l\Big) \\ \times \exp \mathrm{i}\theta_{\mu\nu}\Big(\sum_{m,n=1}^{I} I^{mn} k_m^\mu k_n^\nu + \sum_{m=1}^{I}\sum_{v=1}^{V} J^{mv} k_m^\mu P_v^\nu + \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu\Big).\end{aligned} \tag{38}$$

One can show that $I^{mn}, J^{mv}, K^{vw} \in \{1,-1,0\}$ after use of momentum conservation [70].

[g] We assume that tadpoles (a line starting and ending at the same vertex) are absent. In the final formula they can be taken into account [98].

Next, one introduces Schwinger parameters $\frac{1}{k^2+m^2} = \int d\alpha\, \mathrm{e}^{-\alpha(k^2+m^2)}$ and the identity $(2\pi)^4\delta(q_v) = \int d^4y_v\ \mathrm{e}^{\mathrm{i}y_v q_v}$ for each vertex in (38), then completes the squares in k and performs the Gaussian k-integrations.[h] Writing $y_{\bar{v}} = y_V + z_{\bar{v}}$ for $\bar{v} = 1,\dots,V{-}1$ one has $\sum_{v=1}^{V} y_v\mathcal{E}_{vl} = \sum_{\bar{v}=1}^{V-1} z_{\bar{v}}\bar{\mathcal{E}}^{\bar{v}l}$. The y_V-integration yields the overall momentum conservation. It remains to complete the squares for $z_{\bar{v}}$ and finally to evaluate the Gaussian z_v-integrations. The result is [97]

$$
\begin{aligned}
\mathcal{I}_{\mathcal{G}}(P) &= (2\pi)^4\delta\Big(\sum_{v=1}^{V} P_v\Big)\frac{1}{16^I\pi^{2L}}\exp\Big(\mathrm{i}\theta_{\mu\nu}\sum_{v,w=1}^{V} K^{vw}P_v^\mu P_w^\nu\Big) \\
&\qquad\times\int_0^\infty \prod_{l=1}^{I} d\alpha_l\ \frac{\mathrm{e}^{-\sum_{l=1}^{I}\alpha_l m^2}}{\sqrt{\det\mathcal{A}\det\mathcal{B}}}\ \exp\Big(-\frac{1}{4}(J\tilde{P})^T\mathcal{A}^{-1}(J\tilde{P}) \\
&\qquad+\frac{1}{4}\big(\bar{\mathcal{E}}\mathcal{A}^{-1}(J\tilde{P})+2\mathrm{i}P'\big)^T\mathcal{B}^{-1}\big(\bar{\mathcal{E}}\mathcal{A}^{-1}(J\tilde{P})+2\mathrm{i}P'\big)\Big)\,, \qquad (39)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}^{mn}_{\mu\nu} &:= \alpha_m\delta^{mn}\delta_{\mu\nu} - \mathrm{i}I^{mn}\theta_{\mu\nu}\,, &\qquad (J\tilde{P})^m_\mu &:= \sum_{v=1}^{V} J^{mv}\theta_{\mu\nu}P^\nu_v\,, \\
\bar{\mathcal{E}}^{\bar{v}l} &:= \mathcal{E}_{\bar{v}l}\quad \text{for } \bar{v}=1,\dots,V{-}1\,, &\qquad P'^{\bar{v}}_\mu &:= P^\mu_{\bar{v}}\quad \text{for } \bar{v}=1,\dots,V{-}1\,, \\
\mathcal{B}^{\bar{v}\bar{w}}_{\mu\nu} &:= \sum_{m,n=1}^{I}\bar{\mathcal{E}}^{\bar{v}m}(\mathcal{A}^{-1})^{\mu\nu}_{mn}\bar{\mathcal{E}}^{\bar{w}n}\,. && \qquad (40)
\end{aligned}
$$

The formula (39) is referred to as the parametric integral representation of a noncommutative Feynman graph. See also [76]. Actually, [98] treats a more general case where also derivative couplings are admitted.

Possible divergences of (39) show up in the $\alpha_i \to 0$ behavior.[i] In order to analyze them one reparametrizes the integration domain in (39), similar to the usual procedure described in [69]. For each Hepp sector [106]

$$
\alpha_{\pi_1} \leq \alpha_{\pi_2} \leq \cdots \leq \alpha_{\pi_I} \qquad \text{related to a permutation } \pi \text{ of } 1,\dots,I \qquad (41)
$$

[h] This means that the order of integrations is exchanged in an integral which is in general not absolutely convergent. Thus, the result (39) is based on a certain limiting procedure, which is not necessarily unique. That leaves the possibility of circumventing the UV/IR-problems arising from (39) by different limiting procedures.

[i] The mass term regularizes the $\alpha \to \infty$ behavior of (39). It should be possible to proceed accordingly for massless models using Lowenstein's trick of auxiliary masses [105].

one defines $\alpha_{\pi_i} = \prod_{j=i}^{I} \beta_j^2$, with $0 \leq \beta_I < \infty$ and $0 \leq \beta_j \leq 1$ for $j \neq I$. The leading contribution for small β_j has a topological interpretation.

A ribbon graph can be drawn on a genus-g Riemann surface with possibly several holes to which the external legs are attached [97, 98]. We will say more on ribbon graphs on Riemann surfaces in Sec. 6.2. We will explain, in particular, how a ribbon graph $\mathcal{G}$ defines a Riemann surface. On such a Riemann surface one considers *cycles*, i.e. equivalence classes of closed paths which cannot be contracted to a point. According to homological algebra, one actually factorizes with respect to commutants, i.e. one considers the path $aba^{-1}b^{-1}$ involving two cycles a, b as trivial. We let $c_{\mathcal{G}}(\mathcal{G}_i)$ be the number of non-trivial cycles of the ribbon graph $\mathcal{G}$ wrapped by the subgraph $\mathcal{G}_i$. Next, there may exist external lines m, n such that the graph obtained by connecting m, n has to be drawn on a Riemann surface of genus $g_{mn} > g$. If this happens one defines an index $j(\mathcal{G}) = 1$, otherwise $j(\mathcal{G}) = 0$. The index extends to subgraphs by defining $j_{\mathcal{G}}(\mathcal{G}_i) = 1$ if there are external lines m, n of $\mathcal{G}$ which are already attached to $\mathcal{G}_i$ so that the line connecting m, n wraps a cycle of the additional genus $g \to g_{mn}$ of $\mathcal{G}$.

Now we can formulate the relation between the parametric integral representation and the topology of the ribbon graph. Each sector (41) of the α-parameters defines a sequence of (possibly disconnected) subgraphs $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_I = \mathcal{G}$, where $\mathcal{G}_i$ is made up of the i doublelines $\pi_1, \ldots, \pi_i$ and the vertices to which these lines are attached. If $\mathcal{G}_i$ forms L_i loops it has a power-counting degree of divergence $\omega_i = 4L_i - 2i$. Using sophisticated mathematical techniques on determinants (e.g. the Cauchy–Binet theorem and Jacobi ratio theorem), Chepelev and Roiban have derived in [98] the following leading contribution to the integral:

$$
\begin{aligned}
\mathcal{I}_{\mathcal{G}}(P) &= (2\pi)^4 \delta\left(\sum_{v=1}^{V} P_v\right) \frac{1}{8^I \pi^{2L} (\det\theta)^g} \exp\left(\mathrm{i}\theta_{\mu\nu} \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu\right) \\
&\quad \times \sum_{\text{Hepp sectors}} \int_0^\infty \frac{d\beta_I\, \mathrm{e}^{-\beta_I^2 m^2}}{\beta_I^{1+\omega_I - 4c_{\mathcal{G}}(\mathcal{G})}} \int_0^1 \left(\prod_{i=1}^{I-1} \frac{d\beta_i}{\beta_i^{1+\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i)}}\right) \\
&\quad \times \exp\left(-f_\pi(P) \prod_{i=1}^{I} \frac{1}{\beta^{2j_{\mathcal{G}}(\mathcal{G}_i)}}\right) \left(1 + \mathcal{O}(\beta^2)\right),
\end{aligned} \tag{42}
$$

where $f_\pi(P) \geq 0$, with equality for *exceptional momenta*. In order to obtain a finite integral $\mathcal{I}_{\mathcal{G}}$, one obviously needs

(a) $\omega_i - 4c_\mathcal{G}(\mathcal{G}_i) < 0$ for all i if $j(\mathcal{G}) = 0$ or $j(\mathcal{G}) = 1$ but the external momenta are exceptional, or
(b) $\omega_i - 4c_\mathcal{G}(\mathcal{G}_i) < 0$ or $j_\mathcal{G}(\mathcal{G}_i) = 1$ for all i if $j(\mathcal{G}) = 1$ and the external momenta are non-exceptional.

There are two types of divergences where these conditions are violated.

First let the non-planarity be due to internal lines only, $j(\mathcal{G}) = 0$. Since the total graph $\mathcal{G}$ is non-planar, one has $c_\mathcal{G}(\mathcal{G}) > 0$ and therefore no superficial divergence. However, there might exist subgraphs $\mathcal{G}_i$ related to a Hepp sector of integration (41) where $\omega_i - 4c_\mathcal{G}(\mathcal{G}_i) \geq 0$. Such a situation requires *disconnected* [j] loops wrapping the same handle of the Riemann surface. In this case the integral (39) does not exist unless one introduces a regulator. The problem is that such a subdivergence may appear in graphs with an arbitrary number of external lines. In the commutative theory this also happens, but in that case the subdivergence is already renormalized. This procedure is based on normalization conditions, which can only be imposed for *local* divergences. Since a non-planar graph wrapping a handle of a Riemann surface is clearly a non-local object (it cannot be reduced to a point, i.e. a counterterm vertex), it is not possible in the noncommutative case to remove that subdivergence. We are thus forced to use normalization conditions for the total graph, but as the problem is independent of the number of external legs of the total graph, we finally need an infinite number of normalization conditions. Hence, the model is not renormalizable in the standard way. This is the UV/IR-mixing problem.

The proposal to treat the UV/IR-mixing problem is a reordering of the perturbation series [76]. More details of this idea are given in [98]. The procedure is promising, but a renormalization proof based on the resummation of non-planar graphs is still missing. [k] Clearly, the problem is absent in theories with only logarithmic divergences.

The second class of problems is found in graphs where the non-planarity is at least partly due to the external legs, $j(\mathcal{G}) = 1$. This means that there

[j] We have the impression that the problem with disconnected graphs as discovered by Chepelev and Roiban is completely ignored in the recent literature. Therefore, we have to underline that, in renormalization schemes for noncommutative quantum field theories which are based on the forest formula, it is not possible to restrict oneself to connected graphs. The reason is that, in contrast to the commutative situation, disconnected subgraphs can be coupled in the noncommutative case via the topology of the Riemann surface defined by the total graph.

[k] We conjecture that the result of such a reordering and resummation procedure would be equivalent to the duality-covariant ϕ^4-action (29)+(26), but we cannot prove this idea.

is no way to remove possible divergences in these graphs by normalization conditions. Fortunately, these graphs are superficially finite as long as the external momenta are non-exceptional. Subdivergences are supposed to be treated by a resummation. However, since the non-exceptional external momenta can become arbitrarily close to exceptional ones, these graphs are unbounded: For every $\delta > 0$ one finds non-exceptional momenta $\{p_n\}$ such that $\big|\langle\phi(p_1)\dots\phi(p_n)\rangle\big| > \frac{1}{\delta}$. This problem also arises in models with only logarithmic divergences.

5.4. *θ-expanded field theories*

The only way to circumvent the power-counting theorem of [98] is a different limiting procedure of the loop calculations. Namely, in intermediate steps one changes the order of integrations of integrals which are not absolutely convergent. One possibility is the use [107] of the Seiberg–Witten map [49] which, however, does not help [108].

In their famous paper on type II string theory in the presence of a Neveu–Schwarz B-field [49], Seiberg and Witten noticed that passing to the zero-slope limit in two different regularization schemes (point-splitting and Pauli–Villars) gives rise to a Yang–Mills theory either on noncommutative or on commutative $\mathbb{R}^D$. Since the regularization scheme cannot matter, Seiberg and Witten argued that the two theories must be gauge-equivalent. More generally, under an infinitesimal transformation of θ, which can be related to deformation quantization as in [49] or simply to a coordinate rotation [109], one has to require that gauge-invariant quantities remain gauge-invariant. This requirement leads to the Seiberg–Witten differential equation

$$\frac{dA_\mu}{d\theta_{\rho\sigma}} = -\frac{1}{8}\big\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\big\}_\star + \frac{1}{8}\big\{A_\sigma, \partial_\rho A_\mu + F_{\rho\mu}\big\}_\star\,, \tag{43}$$

where $\{a,b\}_\star = a\star b + b\star a$.

The differential equation (43) is usually solved by integrating it from an initial condition $A^{(0)}$ at $\theta = 0$ in the spirit of deformation quantization [21, 22]. Then, A becomes a formal power series in θ and the initial condition $A^{(0)}$. The solution depends on the path of integration, but the difference between paths is a field redefinition [110]. The solution to all orders in θ and lowest order in $A^{(0)}$ was given in [111]. A generating functional for the complete solution of (43) was derived in [112]. The Seiberg–Witten approach was made popular in [113] where it was argued that this is the only way to obtain a finite number of degrees of freedom in non-Abelian noncommutative Yang–Mills theory.

Inserting the solution of the Seiberg–Witten differential equation (43) into the noncommutative Yang–Mills action $\int d^D x F_{\mu\nu} F^{\mu\nu}$ leads to the so-called θ-expanded field theories. It must be stressed, however, that unless a complete solution to all orders in θ and $A^{(0)}$ is known (which is not the case), the θ-expansion of the noncommutative Yang-Mills action describes a *local* field theory. As such, θ-expanded field theories lose the interesting features of the original field theory on the Moyal plane.

The quantum field theoretical treatment of θ-expanded field theories was initiated in [107]. We have shown that the one-loop divergences to the θ-expanded Maxwell action in second order in θ are gauge-invariant, independent of a linear or a non-linear gauge fixing and independent of the gauge parameter. There is no UV/IR-problem in that approach. We have shown in [114] that these one-loop divergences can be removed by a field redefinition related to the freedom in the Seiberg–Witten map. In fact, the superficial divergences in the photon self-energy are field redefinitions to all orders in θ and any loop order [114]. However, we have shown in [108] that θ-expanded field theories are not renormalizable as regards to more complicated graphs than the self-energy. On the other hand, one of us has found in [108] striking evidence for new symmetries in the θ-expanded action which eliminates several divergences expected from the counting of allowed divergences modulo field redefinitions. Finally, we have shown in [115] that the use of the θ-expanded $\star$-product (4) without application of the Seiberg–Witten map leads (up to field redefinitions) to exactly the same result. Thus, the Seiberg–Witten map is merely an unphysical (but convenient) change of variables [116].

Recently, phenomenological investigations of θ-expanded field theories became popular [117, 118]. However, quantitative statements are delicate because in the presence of a new field $\theta^{\mu\nu}$, many new terms in the action are not only possible but in fact required by renormalizability [115] or the desire to cure the UV/IR-problem [119]. Moreover, deformed spaces are too rigid to be a realistic model [120].

5.5. *Noncommutative space-time*

We have to stress that all mentioned contributions refer to a Euclidean space and a definition of the quantum field theory via the partition function (the Euclidean analogue of the path integral). It was pointed out in [121] that a simple Wick rotation does *not* give a meaningful theory on Minkowskian space-time, first of all because unitarity may be lost [122–124]. The original proposal [6] of a quantum field theory on noncommutative space-time stayed within the Minkowskian framework, but later work started from Feynman graphs, the admissibility of a Wick rotation taken (erroneously) for granted. To obtain a consistent Minkowskian quantum field theory, it was proposed in [121] to iteratively solve the field equations à la Yang–Feldman [125]. See also [126]. Other possibilities are a functional formalism for the S-matrix [127] and time-ordered perturbation theory [128, 129]. See also [130, 131]. Unfortunately, the resulting Feynman rules become so complicated that apart from tadpole-like diagrams [130] it seems impossible to perform perturbative calculations in time-ordered perturbation theory. Moreover, it seems impossible to preserve Ward identities [132].

On the other hand, the rôle of time in noncommutative geometry is not completely clear. Time should be established around the ideas presented in [133]. For general approaches to Minkowskian noncommutative spaces we refer to [134–136]. There is a recent proposal [137] to combine spectral geometry with local covariant quantum field theory.

6. Renormalization of Noncommutative ϕ^4-theory to All Orders

After the previous unsuccessful attempts to renormalize noncommutative quantum field theories, the last resort is a more careful way of performing the limits in the spirit of Wilson [138] and Polchinski [139]. Early attempts [140, 141] did not notice the new effects in higher-genus graphs of noncommutative field theories, which are not visible in one-loop calculations. A rigorous treatment exists for the large-θ limit [142, 143]. Eventually, the Wilson-Polchinski programme for noncommutative ϕ^4-theory was realized in the series of papers [144, 145, 30]. The main ideas are summarized in [146]. We achieved the remarkable balance of proving renormalizability of the ϕ^4-model to all orders and reconfirming the UV/IR-duality of [76]. Our proof rests on two concepts:

- the use of the harmonic oscillator base of the Moyal plane, which avoids the phase factors appearing in momentum space,
- the renormalization by flow equations.

The renormalized ϕ^4-model corresponds to the classical action

$$S=\int d^4x\Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi+\frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)+\frac{\mu^2}{2}\phi\star\phi+\frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x), \tag{44}$$

with $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}\,x^\nu$. The appearance of the harmonic oscillator term $\frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)$ in the action (44) is a result of the renormalization proof.

6.1. *The ϕ^4-action in the matrix base*

We assume for simplicity that $\theta_{12}=-\theta_{21}=\theta_{34}=-\theta_{43}$ are the only non-vanishing components. Expanding the fields in the harmonic oscillator base (8) of the Moyal plane, $\phi(x)=\sum_{m^1,n^1,m^2,n^2\in\mathbb{N}}\phi_{\substack{m^1\\m^2}\substack{n^1\\n^2}}\times f_{m^1n^1}(x_1,x_2)f_{m^2n^2}(x_3,x_4)$, the action (44) takes the form

$$S[\phi]=(2\pi\theta)^2\sum_{m,n,k,l\in\mathbb{N}^2}\left(\frac{1}{2}\phi_{mn}G_{mn;kl}\phi_{kl}+\frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\right), \tag{45}$$

$$\begin{aligned}
G_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} &= \Big(\mu^2+\frac{2+2\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2)\Big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2}\\
&\quad-\frac{2-2\Omega^2}{\theta}\big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1}\\
&\qquad+\sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2}\\
&\quad-\frac{2-2\Omega^2}{\theta}\big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2}\\
&\qquad+\sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1}.
\end{aligned} \tag{46}$$

The quantum field theory is constructed as a perturbative expansion about the free theory, which is solved by the propagator $\Delta_{mn;kl}$, the inverse of $G_{mn;kl}$. After diagonalization of $G_{mn;kl}$ (which leads to orthogonal Meixner polynomials, see [147]) and the use of identities for hypergeometric functions one arrives at

$$
\begin{aligned}
&\Delta_{\substack{m^1 \\ m^2}\substack{n^1 \\ n^2};\substack{k^1 \\ k^2}\substack{l^1 \\ l^2}} \\
&\quad = \frac{\theta}{2(1+\Omega)^2}\delta_{m^1+k^1,n^1+l^1}\delta_{m^2+k^2,n^2+l^2} \times \sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{m^1+l^1}{2}} \sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{m^2+l^2}{2}} \\
&\quad\quad \times B\big(1+\frac{\mu^2\theta}{8\Omega}+\frac{1}{2}(m^1+k^1+m^2+k^2)-v^1-v^2, 1+2v^1+2v^2\big) \\
&\quad\quad \times {}_2F_1\left(\begin{matrix} 1+2v^1+2v^2\,,\ \frac{\mu^2\theta}{8\Omega}-\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2 \\ 2+\frac{\mu^2\theta}{8\Omega}+\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2 \end{matrix}\,\middle|\, \frac{(1-\Omega)^2}{(1+\Omega)^2}\right) \\
&\quad\quad \times \left(\frac{1-\Omega}{1+\Omega}\right)^{2v^1+2v^2} \\
&\quad\quad \times \prod_{i=1}^{2}\sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\,.
\end{aligned} \tag{47}
$$

It is important that the sums in (47) are finite.

6.2. *Renormalization group approach to dynamical matrix models*

The (Euclidean) quantum field theory is defined by the partition function

$$
Z[J] = \int \mathcal{D}[\phi]\, \exp\Big(-S[\phi]-(2\pi\theta)^2\sum_{m,n}\phi_{mn}J_{nm}\Big)\,. \tag{48}
$$

The idea inspired by Polchinski's renormalization proof [139] of commutative ϕ^4-theory is to change the weights of the matrix indices in the kinetic part of $S[\phi]$ as a smooth function of an energy scale Λ and to compensate this by a careful adaptation of the effective action $L[\phi,\Lambda]$ such that $Z[J]$ becomes independent of the scale Λ. If the modification of the weights of a matrix index $m\in\mathbb{N}$ is described by a function $K\big(\frac{m}{\theta\Lambda^2}\big)$, then the required Λ-dependence of the effective action is given by the matrix Polchinski equation

$$
\Lambda \frac{\partial L[\phi,\Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2}\big(2\pi\theta Q_{nm;lk}(\Lambda)\big) \times \Big(\frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} - \frac{1}{(2\pi\theta)^2} \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{mn}\,\partial \phi_{kl}} \Big)\,, \tag{49}
$$

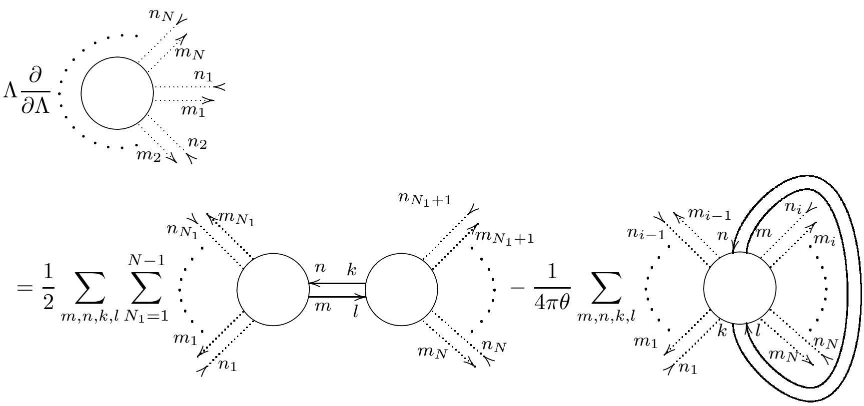
where

$$
2\pi\theta Q_{mn;kl}(\Lambda) := \Lambda \frac{\partial}{\partial \Lambda} \Bigg(\prod_{i \in m^1, m^2, \dots, l^1, l^2} K\Big(\frac{i}{\theta\Lambda^2}\Big) \Delta_{mn;kl}(\Lambda) \Bigg)\,. \tag{50}
$$

We look for a perturbative solution of the matrix Polchinski equation (49). In terms of the expansion coefficients

$$
L[\phi,\Lambda] = \sum_{V=1}^{\infty} \lambda^V \sum_{N=2}^{2V+2} \frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!} \sum_{m_1, n_i \in \mathbb{N}^2} A^{(V)}_{m_1 n_1; \dots; m_N n_N}[\Lambda] \phi_{m_1 n_1} \cdots \phi_{m_N n_N} \tag{51}
$$

of the effective action, the matrix Polchinski equation (49) is represented by *ribbon graphs*:



(52)

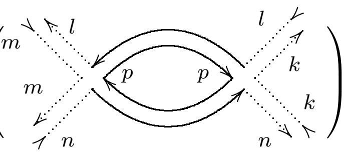
An internal double line $\overset{n \quad k}{\underset{m \quad l}{\rightleftarrows}}$ symbolizes the propagator $Q_{mn;kl}(\Lambda)$. In this way, very complicated ribbon graphs can be produced which cannot be drawn any longer in a plane. A ribbon graph represents a simplicial complex for a Riemann surface and thus defines the topology of the Riemann surface on which it can be drawn. The Riemann surface is characterized by its genus g computable via the Euler characteristic of the graph, $g = 1 - \frac{1}{2}(L - I + V)$, and the number B of holes. Here, L is the number of single-line loops if we close the external lines of the graph, I is the number of double-line propagators and V the number of vertices. The number B of holes coincides with the number of single-line cycles which carry external legs. Accordingly, we also label the expansion coefficients in (51) by the topology, $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}$.

We have proven in [144] a power-counting estimation for these coefficients which relates the Λ-scaling of a ribbon graph to the topology of the graph and to two asymptotic scaling dimensions of the differentiated cut-off propagator $Q_{mn;kl}(\Lambda)$. As a result, if these scaling dimensions coincide with the classical momentum space dimensions, then all non-planar graphs are suppressed by the renormalization flow. This is a necessary requirement for the renormalizability of a model. On the other hand, as the expansion coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ carry an infinite number of matrix indices, the general power-counting estimation proven in [144] leaves, *a priori*, an infinite number of divergent planar graphs. These planar graphs require a separate analysis.

6.3. *Power-counting behavior of the noncommutative ϕ^4-model*

The key is the integration procedure of the Polchinski equation (52), which involves the entire magic of renormalization. We consider the example of the planar one-particle irreducible four-point function with two vertices, $A^{(2,1,0)1\mathrm{PI}}_{m_1n_1;\dots;m_Nn_N}$. The Polchinski equation (52) provides the Λ-derivative of that function:

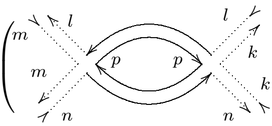
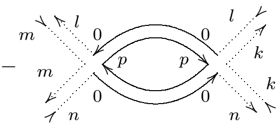
$$\Lambda\frac{\partial}{\partial\Lambda}A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] = \sum_{p\in\mathbb{N}^2}\Bigg(\quad\Bigg)(\Lambda)$$

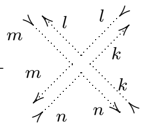
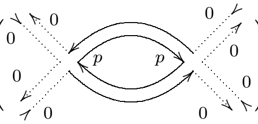


$$+\ \text{permutations.} \tag{53}$$

Performing the Λ-integration of (53) from some initial scale Λ_0 (sent to ∞ at the end) down to Λ, we obtain $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln\frac{\Lambda_0}{\Lambda}$, which diverges for $\Lambda_0 \to \infty$. Renormalization can be understood as the change of the boundary condition for the integration. Thus, integrating (53) from a renormalization scale Λ_R up to Λ, we have $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln\frac{\Lambda}{\Lambda_R}$, and there would be no problem for $\Lambda_0 \to \infty$. However, since there is an *infinite number* of matrix indices and there is no symmetry which could relate the amplitudes, that integration procedure entails an infinite number of initial conditions $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda_R]$. To have a renormalizable model, we can only afford a finite number of integrations from Λ_R up to Λ. Thus, the correct choice is

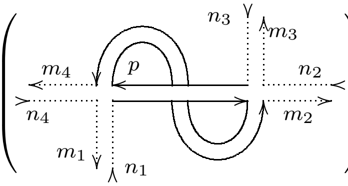
$$A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda]$$

$$= -\int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$

$$-$$

$$\Bigg)[\Lambda']$$

$$+$$

$$\Bigg[\int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$

$$\Bigg)[\Lambda'] + A^{(2,1,0)1\mathrm{PI}}_{00;00;00;00}[\Lambda_R]\Bigg]\,. \qquad (54)$$

The second graph in the first line on the R.H.S. and the graph in brackets in the last line are identical, because only the indices on the propagators determine the value of the graph. Moreover, the vertex in the last line in front of the bracket equals 1. Thus, differentiating (54) with respect to Λ we indeed obtain (53). As a further check one can consider (54) for $m=n=k=l=0$. Finally, the independence of $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda_0]$ on the indices m,n,k,l is built-in. This property is, for $\Lambda_0 \to \infty$, dynamically generated by the model.

There is a similar Λ_0-Λ_R-mixed integration procedure for the planar 1PI two-point functions $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ and all other $A^{(V,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}$. These involve in total four different sub-integrations from Λ_R up to Λ. We refer to [30] for details. All other graphs are integrated from Λ_0 down to Λ, e.g.

$$A^{(2,2,0)1\mathrm{PI}}_{m_1n_1;\dots;m_4n_4}[\Lambda] = -\int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$

$$\Bigg)[\Lambda']\,. \qquad (55)$$

Theorem 1. The previous integration procedure yields

$$
\begin{aligned}
&\big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\
&\qquad \leq \big(\sqrt{\theta}\Lambda\big)^{(4-N)+4(1-B-2g)} P^{4V-N}\Big[\frac{\max(\|m_1\|,\|n_1\|,\dots\|n_N\|)}{\theta\Lambda^2}\Big] \\
&\qquad\quad \times P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big],
\end{aligned}
\tag{56}
$$

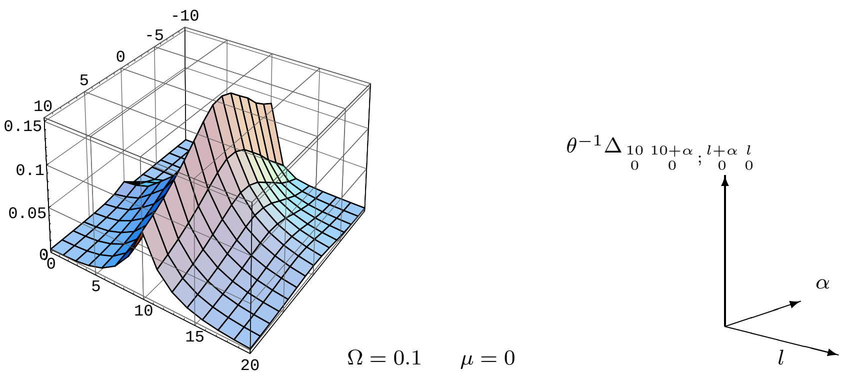
where $P^q[X]$ stands for a polynomial of degree q in X.

Idea of the Proof. For the choice $K(x)=1$ for $0\leq x\leq 1$ and $K(x)=0$ for $x\geq 2$ of the cutoff function in (50) one has

$$
|Q_{mn;kl}(\Lambda)| < \frac{C_0}{\Omega\theta\Lambda^2}\delta_{m+k,n+l}\ . \tag{57}
$$

Thus, the propagator and the volume of a loop summation have the same power-counting dimensions as a commutative ϕ^4-model in momentum space, giving the total power-counting degree $4-N$ for an N-point function.

This is (more or less) correct for planar graphs. The scaling behavior of non-planar graphs is considerably improved by the *quasi-locality* of the propagator:



(58)

As a consequence, for given index m of the propagator $Q_{mn;kl}(\Lambda) = \underset{m\quad l}{\overset{n\quad k}{\rightleftarrows}}$, the contribution to a graph is strongly suppressed unless the other index l

on the trajectory through m is close to m. Thus, the sum over l for given m converges and does not alter (apart from a factor Ω^{-1}) the power-counting behavior of (57):

$$\sum_{l\in\mathbb{N}^2}\Big(\max_{n,k}|Q_{mn;kl}(\Lambda)|\Big) < \frac{C_1}{\theta\Omega^2\Lambda^2}\,. \tag{59}$$

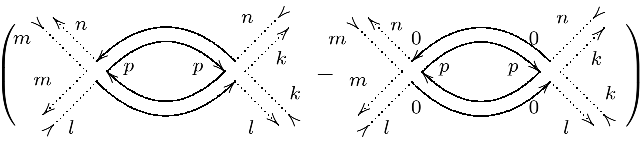
In a non-planar graph like the one in (55), the index n_3—fixed as an external index—localizes the summation index $p \approx n_3$. Thus, we save one volume factor $\theta^2\Lambda^4$ compared with a true loop summation as in (54). In general, each hole in the Riemann surface saves one volume factor, and each handle even saves two.

A more careful analysis of (47) shows that also planar graphs get suppressed with $\Big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)\Big| < \frac{C_2}{\Omega\theta\Lambda^2}\prod_{i=1}^2\Big(\frac{\max(m^i,l^i)+1}{\theta\Lambda^2}\Big)^{\frac{|m^i-l^i|}{2}}$, for $m^i \leq n^i$, if the index along a trajectory jumps. This leaves the functions $A^{(V,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ and $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ as the only relevant or marginal ones. In these functions one has to use a discrete version of the Taylor expansion such as

$$\Big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}(\Lambda) - Q_{\substack{0\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\0}}(\Lambda)\Big| < \frac{C_3}{\Omega\theta\Lambda^2}\Big(\frac{\max(m^1,m^2)}{\theta\Lambda^2}\Big)\,, \tag{60}$$

which can be traced back to the Meixner polynomials. The discrete Taylor subtractions are used in the integration from Λ_0 down to Λ in prescriptions like (54):

$$-\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Bigg($$



$$\Bigg)[\Lambda']$$

$$= \int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\int_{\Lambda'}^{\Lambda_0}\frac{d\Lambda''}{\Lambda''}\sum_{p\in\mathbb{N}^2}\Big((Q_{np;pn}-Q_{0p;p0})(\Lambda')Q_{lp;pl}(\Lambda'') + Q_{0p;p0}(\Lambda')(Q_{lp;pl}-Q_{0p;p0})(\Lambda'')\Big) \sim \frac{C(\|n\|+\|l\|)}{\theta\Omega^2\Lambda^2}\,. \tag{61}$$

This explains the polynomial in fractions like $\frac{\|m\|}{\theta\Lambda^2}$ in (56). □

As the estimation (56) is achieved by a finite number of initial conditions at Λ_R (see (54)), the noncommutative ϕ^4-model with oscillator term is renormalizable to all orders in perturbation theory. These initial conditions correspond to normalization experiments for the mass, the field amplitude, the coupling constant and the oscillator frequency in the bare action related to (44). The resulting one-loop β-functions are computed in [148].

We have also proven renormalizability of the two-dimensional case in [145], where the oscillator frequency required in intermediate steps can be switched off at the end.

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References

[1] B. Riemann, "Ueber die Hypothesen, welche der Geometrie zugrunde liegen," in *Bernhard Riemann's Gesammelte Mathematische Werke und Wissenschaftlicher Nachlaß*, eds. H. Weber and B. G. Teubner, Leipzig, 1892.

[2] E. Schrödinger, "Über die Unanwendbarkeit der Geometrie im Kleinen," *Naturwiss.* **31** (1934) 342–344.

[3] W. Heisenberg, "Die Grenzen der Anwendbarkeit der bisherigen Quantentheorie," *Z. Phys.* **110** (1938) 251–266.

[4] R. Peierls, "Zur Theorie des Diamagnetismus von Leitungselektronen," *Z. Phys.* **80** (1933) 763.

[5] H. S. Snyder, "Quantized space-time," *Phys. Rev.* **71** (1947) 38–41.

[6] S. Doplicher, K. Fredenhagen and J. E. Roberts, "The quantum structure of space-time at the Planck scale and quantum fields," *Commun. Math. Phys.* **172** (1995) 187–220.

[7] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[8] A. Connes, "Noncommutative geometry: Year 2000," `math.qa/0011193`.

[9] J. Madore, *An Introduction to Noncommutative Differential Geometry and Its Physical Applications*, Vol. 257 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2nd ed., 2000.

[10] G. Landi, *An Introduction to Noncommutative Spaces and Their Geometries*, Vol. m51 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, 1997.

[11] J. C. Várilly, "An introduction to noncommutative geometry," `physics/9709045`.

[12] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*. Birkhäuser advanced texts. Birkhäuser, Boston, 2001.

[13] A. H. Chamseddine and A. Connes, "The spectral action principle," *Commun. Math. Phys.* **186** (1997) 731–750.

[14] A. Connes, "Gravity coupled with matter and the foundation of noncommutative geometry," *Commun. Math. Phys.* **182** (1996) 155–176.

[15] H. J. Groenewold, "On the principles of elementary quantum mechanics," *Physica* **12** (1946) 405–460.

[16] J. E. Moyal, "Quantum mechanics as a statistical theory," *Proc. Cambridge Phil. Soc.* **45** (1949) 99–124.

[17] M. A. Rieffel, *Deformation quantization for actions of* $\mathbb{R}^d$, Vol. 106. Mem. Amer. Math. Soc., Providence, 1993.

[18] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker and J. C. Várilly, "Moyal planes are spectral triples," *Commun. Math. Phys.* **246** (2004) 569–623.

[19] J. M. Gracia-Bondía and J. C. Várilly, "Algebras of distributions suitable for phase space quantum mechanics. I," *J. Math. Phys.* **29** (1988) 869–879.

[20] E. P. Wigner, "On the quantum correction for thermodynamic equilibrium," *Phys. Rev.* **40** (1932) 749–760.

[21] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, "Deformation theory and quantization. I. Deformations of symplectic structures," *Ann. Phys.* **111** (1978) 61–110.

[22] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer "Deformation theory and quantization. II. Physical applications," *Ann. Phys.* **111** (1978) 111–151.

[23] M. Kontsevich, "Deformation quantization of Poisson manifolds, I," `q-alg/9709040`.

[24] A. S. Cattaneo and G. Felder, "A path integral approach to the Kontsevich quantization formula," *Commun. Math. Phys.* **212** (2000) 591–611.

[25] P. Schaller and T. Strobl, "Poisson structure induced (topological) field theories," *Mod. Phys. Lett. A* **9** (1994) 3129–3136.

[26] D. V. Vassilevich, "Non-commutative heat kernel," *Lett. Math. Phys.* **67** (2004) 185–194.

[27] V. Gayral and B. Iochum, "The spectral action for Moyal planes," *J. Math. Phys.* **46** (2005) 043503.

[28] R. Estrada, J. M. Gracia-Bondía and J. C. Várilly, "On asymptotic expansions of twisted products," *J. Math. Phys.* **30** (1989) 2789–2796.

[29] J. M. Gracia-Bondía, F. Lizzi, G. Marmo, and P. Vitale, "Infinitely many star products to play with," *JHEP* **04** (2002) 026.

[30] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," *Commun. Math. Phys.* **256** (2005) 305–374.

[31] M. S. Bartlett and J. E. Moyal, "The exact transition probabilities of quantum-mechanical oscillators calculated by the phase-space method," *Proc. Cambridge Phil. Soc.* **45** (1949) 545–553.

[32] E. Langmann, "Interacting fermions on noncommutative spaces: Exactly solvable quantum field theories in $2n+1$ dimensions," *Nucl. Phys. B* **654** (2003) 404–426.

[33] E. Langmann, R. J. Szabo and K. Zarembo, "Exact solution of noncommutative field theory in background magnetic fields," *Phys. Lett. B* **569** (2003) 95–101.
[34] E. Langmann, R. J. Szabo and K. Zarembo, "Exact solution of quantum field theory on noncommutative phase spaces," *JHEP* **01** (2004) 017, `hep-th/0308043`.
[35] J. C. Várilly and J. M. Gracia-Bondía, "Algebras of distributions suitable for phase space quantum mechanics. II. Topologies on the Moyal algebra," *J. Math. Phys.* **29** (1988) 880–887.
[36] A. Connes, "C^* algèbres et géométrie différentielle," *C. R. Acad. Sci. Paris Sér. A-B* **290** (1980) 599–604.
[37] M. A. Rieffel, "C^*-algebras associated with irrational rotations," *Pacific J. Math.* **93** (1981) 415–429.
[38] M. A. Rieffel, "Noncommutative tori—a case study of noncommutative differentiable manifolds," *Contemp. Math.* **105** (1990) 191–211.
[39] A. Connes and G. Landi, "Noncommutative manifolds: The instanton algebra and isospectral deformations," *Commun. Math. Phys.* **221** (2001) 141–159.
[40] A. Connes and M. Dubois-Violette, "Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples," *Commun. Math. Phys.* **230** (2002) 539–579.
[41] J. Madore, "The fuzzy sphere," *Class. Quant. Grav.* **9** (1992) 69–88.
[42] M. A. Rieffel, "Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance," *Mem. Amer. Math. Soc.* **168** (2004) 67–91.
[43] H. Grosse and A. Strohmaier, "Noncommutative geometry and the regularization problem of 4D quantum field theory," *Lett. Math. Phys.* **48** (1999) 163–179.
[44] G. Alexanian, A. P. Balachandran, G. Immirzi, and B. Ydri, "Fuzzy $\mathbb{C}P^2$," *J. Geom. Phys.* **42** (2002) 28–53.
[45] H. Grosse, J. Madore, and H. Steinacker, "Field theory on the q-deformed fuzzy sphere. I," *J. Geom. Phys.* **38** (2001) 308–342.
[46] H. Grosse, J. Madore and H. Steinacker, "Field theory on the q-deformed fuzzy sphere. II: Quantization," *J. Geom. Phys.* **43** (2002) 205–240.
[47] V. Schomerus, "D-branes and deformation quantization," *JHEP* **06** (1999) 030.
[48] A. Yu. Alekseev, A. Recknagel and V. Schomerus, "Non-commutative world-volume geometries: Branes on SU(2) and fuzzy spheres," *JHEP* **09** (1999) 023.
[49] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *JHEP* **09** (1999) 032.
[50] N. Nekrasov and A. Schwarz, "Instantons on noncommutative $\mathbb{R}^4$ and $(2,0)$ superconformal six dimensional theory," *Commun. Math. Phys.* **198** (1998) 689–703.
[51] A. Gonzalez-Arroyo and M. Okawa, "The twisted Eguchi–Kawai model: A

reduced model for large N lattice gauge theory," *Phys. Rev. D* **27** (1983) 2397.
[52] G. 't Hooft, "Some twisted selfdual solutions for the Yang–Mills equations on a hypertorus," *Commun. Math. Phys.* **81** (1981) 267–275.
[53] A. Gonzalez-Arroyo and C. P. Korthals Altes, "Reduced model for large N continuum field theories," *Phys. Lett. B* **131** (1983) 396.
[54] V. Gayral, "The action functional for Moyal planes," *Lett. Math. Phys.* **65** (2003) 147–157.
[55] C. Becchi, A. Rouet, and R. Stora, "Renormalization of gauge theories," *Ann. Phys.* **98** (1976) 287–321.
[56] I. A. Batalin and G. A. Vilkovisky, "Gauge algebra and quantization," *Phys. Lett. B* **102** (1981) 27–31.
[57] E. Langmann and R. J. Szabo, "Duality in scalar field theory on noncommutative phase spaces," *Phys. Lett. B* **533** (2002) 168–177.
[58] H. Grosse, C. Klimčík and P. Prešnajder, "Towards finite quantum field theory in noncommutative geometry," *Int. J. Theor. Phys.* **35** (1996) 231–244.
[59] H. Grosse, C. Klimčík, and P. Prešnajder, "On finite 4D quantum field theory in noncommutative geometry," *Commun. Math. Phys.* **180** (1996) 429–438.
[60] B. Ydri, *Fuzzy physics.* PhD thesis, Syracuse University, Syracuse, 2001. `hep-th/0110006`.
[61] C.-S. Chu, J. Madore, and H. Steinacker, "Scaling limits of the fuzzy sphere at one loop," *JHEP* **08** (2001) 038.
[62] E. Hawkins, "Noncommutative regularization for the practical man," `hep-th/9908052`.
[63] S. Iso, Y. Kimura, K. Tanaka, and K. Wakatsuki, "Noncommutative gauge theory on fuzzy sphere from matrix model," *Nucl. Phys. B* **604** (2001) 121–147.
[64] H. Steinacker, "Quantized gauge theory on the fuzzy sphere as random matrix model," *Nucl. Phys. B* **679** (2004) 66–98.
[65] S. Cho, R. Hinterding, J. Madore and H. Steinacker, "Finite field theory on noncommutative geometries," *Int. J. Mod. Phys. D* **9** (2000) 161–199.
[66] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, "Ultraviolet finite quantum field theory on quantum spacetime," *Commun. Math. Phys.* **237** (2003) 221–241.
[67] N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Vol. 3 of *Interscience Monographs in Physics and Astronomy.* Interscience, New York, 1959.
[68] G. Velo and A. S. Wightman, eds., *Renormalization Theory*, Proc. Erice 1975. D. Reidel Publishing, Dordrecht, 1976.
[69] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, *International Series in Pure and Applied Physics*, McGraw-Hill, New York, 1980.
[70] T. Filk, "Divergencies in a field theory on quantum space," *Phys. Lett. B* **376** (1996) 53–58.
[71] J. C. Várilly and J. M. Gracia-Bondía, "On the ultraviolet behaviour of

quantum fields over noncommutative manifolds," *Int. J. Mod. Phys. A* **14** (1999) 1305.

[72] M. Chaichian, A. Demichev and P. Prešnajder, "Quantum field theory on noncommutative space-times and the persistence of ultraviolet divergences," *Nucl. Phys. B* **567** (2000) 360–390.

[73] C. P. Martín and D. Sánchez-Ruiz, "The one-loop UV divergent structure of U(1) Yang-Mills theory on noncommutative $\mathbb{R}^4$," *Phys. Rev. Lett.* **83** (1999) 476–479.

[74] M. M. Sheikh-Jabbari, "Renormalizability of the supersymmetric Yang-Mills theories on the noncommutative torus," *JHEP* **06** (1999) 015.

[75] T. Krajewski and R. Wulkenhaar, "Perturbative quantum gauge fields on the noncommutative torus," *Int. J. Mod. Phys.* **A15** (2000) 1011–1030.

[76] S. Minwalla, M. Van Raamsdonk and N. Seiberg, "Noncommutative perturbative dynamics," *JHEP* **02** (2000) 020.

[77] A. Connes, M. R. Douglas, and A. Schwarz, "Noncommutative geometry and matrix theory: Compactification on tori," *JHEP* **02** (1998) 003.

[78] M. R. Douglas, "Two lectures on D-geometry and noncommutative geometry," in *Nonperturbative Aspects of Strings, Branes and Supersymmetry*, M. Duff *et al.*, eds., pp. 131–156. World Scientific, Singapore, 1999.

[79] T. Krajewski, *Géométrie Non Commutative et Interactions Fondamentales*, PhD thesis, Université de Provence, Marseille, 1998. `math-ph/9903047`.

[80] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, "Noncommutative Yang–Mills in IIB matrix model," *Nucl. Phys. B* **565** (2000) 176–192.

[81] J. Ambjørn, Y. M. Makeenko, J. Nishimura and R. J. Szabo, "Finite N matrix models of noncommutative gauge theory," *JHEP* **11** (1999) 029.

[82] W. Bietenholz, F. Hofheinz and J. Nishimura, "A non-perturbative study of gauge theory on a non-commutative plane," *JHEP* **09** (2002) 009.

[83] W. Bietenholz, F. Hofheinz and J. Nishimura, "Non-commutative field theories beyond perturbation theory," *Fortsch. Phys.* **51** (2003) 745–752.

[84] S. S. Gubser and S. L. Sondhi, "Phase structure of non-commutative scalar field theories," *Nucl. Phys. B* **605** (2001) 395–424.

[85] G. Landi, F. Lizzi, and R. J. Szabo, "From large N matrices to the noncommutative torus," *Commun. Math. Phys.* **217** (2001) 181–201.

[86] G. Landi, F. Lizzi, and R. J. Szabo, "A new matrix model for noncommutative field theory," *Phys. Lett. B* **578** (2004) 449–458.

[87] G. Landi, F. Lizzi, and R. J. Szabo, "Matrix quantum mechanics and soliton regularization of noncommutative field theory," *Adv. Theor. Math. Phys.* **8** (2004) 1–82.

[88] R. Gopakumar, S. Minwalla, and A. Strominger, "Noncommutative solitons," *JHEP* **05** (2000) 020.

[89] L. D. Paniak and R. J. Szabo, "Instanton expansion of noncommutative gauge theory in two dimensions," *Commun. Math. Phys.* **243** (2003) 343–387.

[90] L. D. Paniak and R. J. Szabo, "Lectures on two-dimensional noncommutative gauge theory. 2: Quantization," `hep-th/0304268`.

[91] M. R. Douglas and C. M. Hull, "D-branes and the noncommutative torus," *JHEP* **02** (1998) 008.
[92] A. Konechny and A. Schwarz, "Introduction to M(atrix) theory and noncommutative geometry," *Phys. Rep.* **360** (2002) 353–465 (Parts I and II).
[93] A. Konechny and A. Schwarz, "Introduction to M(atrix) theory and noncommutative geometry. II," *Phys. Rep.* **360** (2002) 353–465 (Parts I and II).
[94] M. R. Douglas and N. A. Nekrasov, "Noncommutative field theory," *Rev. Mod. Phys.* **73** (2001) 977–1029.
[95] R. J. Szabo, "Quantum field theory on noncommutative spaces," *Phys. Rep.* **378** (2003) 207–299.
[96] I. Ya. Aref'eva, D. M. Belov, A. A. Giryavets, A. S. Koshelev and P. B. Medvedev, "Noncommutative field theories and (super)string field theories," in *Particles and Fields*, G. A. Alves *et al.*, eds., pp. 1–163. World Scientific, Singapore, 2001.
[97] I. Chepelev and R. Roiban, "Renormalization of quantum field theories on noncommutative $\mathbb{R}^d$. I: Scalars," *JHEP* **05** (2000) 037.
[98] I. Chepelev and R. Roiban, "Convergence theorem for non-commutative Feynman graphs and renormalization," *JHEP* **03** (2001) 001.
[99] I. Ya. Aref'eva, D. M. Belov, and A. S. Koshelev, "Two-loop diagrams in noncommutative ϕ^4_4 theory," *Phys. Lett. B* **476** (2000) 431–436.
[100] I. Ya. Aref'eva, D. M. Belov, and A. S. Koshelev, "A note on UV/IR for noncommutative complex scalar field," `hep-th/0001215`.
[101] M. Hayakawa, "Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on $\mathbb{R}^4$," `hep-th/9912167`.
[102] A. Matusis, L. Susskind, and N. Toumbas, "The IR/UV connection in the non-commutative gauge theories," *JHEP* **12** (2000) 002, `hep-th/0002075`.
[103] L. Bonora and M. Salizzoni, "Renormalization of noncommutative U(N) gauge theories," *Phys. Lett. B* **504** (2001) 80–88.
[104] H. Grosse, T. Krajewski and R. Wulkenhaar, "Renormalization of noncommutative Yang–Mills theories: A simple example," `hep-th/0001182`.
[105] J. H. Lowenstein, "Convergence theorems for renormalized Feynman integrals with zero-mass propagators," *Commun. Math. Phys.* **47** (1976) 53–68.
[106] K. Hepp, "Proof of the Bogolyubov–Parasiuk theorem on renormalization," *Commun. Math. Phys.* **2** (1966) 301–326.
[107] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda and R. Wulkenhaar, "Perturbative analysis of the Seiberg–Witten map," *Int. J. Mod. Phys. A* **17** (2002) 2219–2232.
[108] R. Wulkenhaar, "Non-renormalizability of θ-expanded noncommutative QED," *JHEP* **03** (2002) 024.
[109] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, "Noncommutative Lorentz symmetry and the origin of the Seiberg-Witten map," *Eur. Phys. J. C* **24** (2002) 165–176.
[110] T. Asakawa and I. Kishimoto, "Comments on gauge equivalence in noncommutative geometry," *JHEP* **11** (1999) 024.

[111] M. R. Garousi, "Non-commutative world-volume interactions on D-branes and Dirac-Born-Infeld action," *Nucl. Phys. B* **579** (2000) 209–228.

[112] B. Jurčo, P. Schupp and J. Wess, "Nonabelian noncommutative gauge theory via noncommutative extra dimensions," *Nucl. Phys. B* **604** (2001) 148–180.

[113] B. Jurčo, S. Schraml, P. Schupp and J. Wess, "Enveloping algebra valued gauge transformations for non-Abelian gauge groups on non-commutative spaces," *Eur. Phys. J. C* **17** (2000) 521–526.

[114] A. A. Bichl, J. M. Grimstrup, H. Grosse, L. Popp, M. Schweda and R. Wulkenhaar, "Renormalization of the noncommutative photon self-energy to all orders via Seiberg–Witten map," *JHEP* **06** (2001) 013.

[115] J. M. Grimstrup and R. Wulkenhaar, "Quantisation of θ-expanded non-commutative QED," *Eur. Phys. J. C* **26** (2002) 139–151.

[116] S. Kamefuchi, L. O'Raifeartaigh and A. Salam, "Change of variables and equivalence theorems in quantum field theories," *Nucl. Phys.* **28** (1961) 529–549.

[117] X. Calmet, B. Jurčo, P. Schupp, J. Wess and M. Wohlgenannt, "The standard model on non-commutative space-time," *Eur. Phys. J. C* **23** (2002) 363–376.

[118] W. Behr, N. G. Deshpande, G. Duplančić, P. Schupp, J. Trampetić and J. Wess, "The $Z \to \gamma\gamma,\ gg$ decays in the noncommutative standard model," *Eur. Phys. J. C* **29** (2003) 441–446.

[119] A. A. Slavnov, "Consistent noncommutative quantum gauge theories?," *Phys. Lett. B* **565** (2003) 246–252.

[120] E. Hawkins, "Noncommutative rigidity," `math.qa/0211203`.

[121] D. Bahns, S. Doplicher, K. Fredenhagen, and G. Piacitelli, "On the unitarity problem in space/time noncommutative theories," *Phys. Lett. B* **533** (2002) 178–181.

[122] J. Gomis and T. Mehen, "Space-time noncommutative field theories and unitarity," *Nucl. Phys. B* **591** (2000) 265–276.

[123] L. Alvarez-Gaumé, J. L. F. Barbón and R. Zwicky, "Remarks on time-space noncommutative field theories," *JHEP* **05** (2001) 057.

[124] C.-S. Chu, J. Lukierski and W. J. Zakrzewski, "Hermitian analyticity, IR/UV mixing and unitarity of noncommutative field theories," *Nucl. Phys. B* **632** (2002) 219–239.

[125] C.-N. Yang and D. Feldman, "The S matrix in the Heisenberg representation," *Phys. Rev.* **79** (1950) 972–978.

[126] D. Bahns, "Unitary quantum field theory on the noncommutative Minkowski space," *Fortsch. Phys.* **51** (2003) 658–663.

[127] C.-h. Rim and J. H. Yee, "Unitarity in space-time noncommutative field theories," *Phys. Lett. B* **574** (2003) 111–120, `hep-th/0205193`.

[128] Y. Liao and K. Sibold, "Time-ordered perturbation theory on noncommutative spacetime: Basic rules," *Eur. Phys. J. C* **25** (2002) 469–477.

[129] Y. Liao and K. Sibold, "Time-ordered perturbation theory on noncommutative spacetime. II. Unitarity," *Eur. Phys. J. C* **25** (2002) 479–486.

[130] H. Bozkaya, P. Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda

and R. Wulkenhaar, "Space/time noncommutative field theories and causality," *Eur. Phys. J. C* **29** (2003) 133–141.
[131] S. Denk and M. Schweda, "Time ordered perturbation theory for non-local interactions: Applications to NCQFT," *JHEP* **09** (2003) 032.
[132] T. Ohl, R. Rückl and J. Zeiner, "Unitarity of time-like noncommutative gauge theories: The violation of Ward identities in time-ordered perturbation theory," *Nucl. Phys. B* **676** (2004) 229–242.
[133] A. Connes and C. Rovelli, "Von Neumann algebra automorphisms and time thermodynamics relation in general covariant quantum theories," *Class. Quant. Grav.* **11** (1994) 2899–2918.
[134] E. Hawkins, "Hamiltonian gravity and noncommutative geometry," *Commun. Math. Phys.* **187** (1997) 471–489.
[135] A. Strohmaier, "On noncommutative and semi-Riemannian geometry," `math-ph/0110001`.
[136] T. Kopf and M. Paschke, "A spectral quadruple for de Sitter space," *J. Math. Phys.* **43** (2002) 818–846.
[137] M. Paschke and R. Verch, "Local covariant quantum field theory over spectral geometries," *Class. Quant. Grav.* **21** (2004) 5299–5316.
[138] K. G. Wilson and J. B. Kogut, "The renormalization group and the epsilon expansion," *Phys. Rep.* **12** (1974) 75–200.
[139] J. Polchinski, "Renormalization and effective Lagrangians," *Nucl. Phys. B* **231** (1984) 269–295.
[140] L. Griguolo and M. Pietroni, "Wilsonian renormalization group and the non-commutative IR/UV connection," *JHEP* **05** (2001) 032.
[141] S. Sarkar, "On the UV renormalizability of noncommutative field theories," *JHEP* **06** (2002) 003.
[142] C. Becchi, S. Giusto and C. Imbimbo, "The Wilson-Polchinski renormalization group equation in the planar limit," *Nucl. Phys. B* **633** (2002) 250–270.
[143] C. Becchi, S. Giusto, and C. Imbimbo, "The renormalization of non-commutative field theories in the limit of large non-commutativity," *Nucl. Phys. B* **664** (2003) 371–399.
[144] H. Grosse and R. Wulkenhaar, "Power-counting theorem for non-local matrix models and renormalisation," *Commun. Math. Phys.* **254** (2005) 91–127.
[145] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^2$ in the matrix base," *JHEP* **12** (2003) 019.
[146] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ to all orders," *Lett. Math. Phys.* **71** (2005) 13–26.
[147] R. Koekoek and R. F. Swarttouw, "The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue," `math.CA/9602214`.
[148] H. Grosse and R. Wulkenhaar, "The β-function in duality-covariant noncommutative ϕ^4-theory," *Eur. Phys. J. C* **35** (2004) 277–282.

An Invitation to Noncommutative Geometry
 (pp. 169–273)

LECTURES ON NONCOMMUTATIVE GEOMETRY

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Contents

This text is an introduction to a few selected areas of Alain Connes' noncommutative geometry written for the volume of the school/conference "Noncommutative Geometry 2005" held at IPM Tehran. It is an expanded version of my lectures which was directed at graduate students and novices in the subject.

1. Introduction

David Hilbert once said "No one will drive us from the paradise which Cantor created for us" [70]. He was of course referring to set theory and the vast new areas of mathematics which were made possible through it. This was particularly relevant for geometry where our ultimate geometric intuition of *space* so far has been a set endowed with some extra structure (e.g. a topology, a measure, a smooth structure, a sheaf, etc.). With Alain Connes' noncommutative geometry [18, 22, 24], however, we are now gradually moving into a new "paradise", a "paradise" which contains the "Hilbertian paradise" as one of its old small neighborhoods. Interestingly enough though, and I hasten to say this, methods of functional analysis, operator algebras, and spectral theory, pioneered by Hilbert and his disciples, play a big role in Connes' noncommutative geometry.

The following is a greatly expanded version of talks I gave during the conference on noncommutative geometry at IPM, Tehran. The talks were directed at graduate students, mathematicians, and physicists with no background in noncommutative geometry. It inevitably covers only certain selected parts of the subject and many important topics such as: metric and spectral aspects of noncommutative geometry, the local index formula, connections with number theory, and interactions with physics, are left out. For an insightful and comprehensive introduction to the current state of the art, I refer to Connes and Marcolli's article "A Walk in the Noncommutative Garden" [34] in this volume. For a deeper plunge into the subject, one cannot do better than directly going to Connes' book [24] and original articles.

I would like to thank Professors Alain Connes and Matilde Marcolli for their support and encouragement over a long period of time. Without their kind support and advice this project would have taken much longer to come to its conclusion, if ever. My sincere thanks go also to Arthur Greenspoon who took a keen interest in the text and kindly and carefully edited the entire manuscript. Arthur's superb skills resulted in substantial improvments in the original text.

2. From C^*-algebras to Noncommutative Spaces

Our working definition of a *noncommutative space* is a noncommutative algebra, possibly endowed with some extra structure. Operator algebras, i.e. algebras of bounded linear operators on a Hilbert space, provided the first really deep insights into this noncommutative realm. It is generally agreed that the classic series of papers of Murray and von Neumann starting with [102], and Gelfand and Naimark [61] are the foundations upon which the theory of operator algebras is built. The first is the birthplace of von Neumann algebras as the noncommutative counterpart of measure theory, while in the second C^*-algebras were shown to be the noncommutative analogues of locally compact spaces. For lack of space we shall say nothing about von Neumann algebras and their place in noncommutative geometry (cf. [24] for the general theory as well as links with noncommutative geometry). We start with the definition of C^*-algebras and results of Gelfand and Naimark. References include [7, 24, 49, 57].

2.1. *Gelfand–Naimark theorems*

By an *algebra* in these notes we shall mean an *associative algebra* over the field of complex numbers $\mathbb{C}$. Algebras are not assumed to be commutative or unital, unless explicitly specified so. An *involution* on an algebra A is a conjugate linear map $a \mapsto a^*$ satisfying

$$(ab)^* = b^* a^* \quad \text{and} \quad (a^*)^* = a$$

for all a and b in A. By a *normed algebra* we mean an algebra A such that A is a normed vector space and

$$\|ab\| \leq \|a\|\|b\|,$$

for all a, b in A. If A is unital, we assume that $\|1\| = 1$. A *Banach algebra* is a normed algebra which is *complete* as a metric space. One of the main consequences of completeness is that norm convergent series are convergent; in particular if $\|a\| < 1$ then the geometric series $\sum_{n=1}^{\infty} a^n$ is convergent. From this it easily follows that the group of invertible elements of a unital Banach algebra is open in the norm topology.

Definition 2.1. *A C^*-algebra is an involutive Banach algebra A such that for all $a \in A$ the C^*-identity*

$$\|aa^*\| = \|a\|^2 \tag{1}$$

holds.

A *morphism* of C^*-algebras is an algebra homomorphism $f : A \longrightarrow B$ which preserves the $*$ structure, i.e.

$$f(a^*) = f(a)^*, \quad \text{for all } a \in A.$$

The C^*-identity (1) puts C^*-algebras in a unique place among all Banach algebras, comparable to the unique position enjoyed by Hilbert spaces among all Banach spaces. Many facts which are true for C^*-algebras are not necessarily true for an arbitrary Banach algebra. For example, one can show, using the spectral radius formula $\rho(a) = \text{Lim}\, \|a^n\|^{\frac{1}{n}}$ coupled with the C^*-identity, that the norm of a C^*-algebra is unique. In fact it can be shown that a morphism of C^*-algebras is automatically *contractive* in the sense that for all $a \in A$, $\|f(a)\| \leq \|a\|$. In particular they are always continuous. It follows that if $(A, \| \|_1)$ and $(A, \| \|_2)$ are both C^*-algebras then

$$\|a\|_1 = \|a\|_2,$$

for all $a \in A$. Note also that a morphism of C^*-algebras is an *isomorphism* if and only if it is one-to-one and onto. Isomorphisms of C^*-algebras are necessarily *isometric.*

Example 2.1. Let X be a locally compact Hausdorff space and let $A = C_0(X)$ denote the algebra of continuous complex valued functions on X vanishing at infinity. Equipped with the sup norm and the involution defined by complex conjugation, A is easily seen to be a commutative C^*-algebra. It is unital if and only if X is compact, in which case A will be denoted by $C(X)$. By a fundamental theorem of Gelfand and Naimark, to be recalled below, any commutative C^*-algebra is of the form $C_0(X)$ for a canonically defined locally compact Hausdorff space X.

Example 2.2. The algebra $A = \mathcal{L}(H)$ of all bounded linear operators on a complex Hilbert space H endowed with the operator norm and the usual adjoint operation is a C^*-algebra. The crucial C^*-identity $\|aa^*\| = \|a\|^2$ is easily checked. When H is finite dimensional of dimension n we obtain the matrix algebra $A = M_n(\mathbb{C})$. A direct sum of matrix algebras

$$A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

is a C^*-algebra as well. It can be shown that any finite dimensional C^*-algebra is unital and is a direct sum of matrix algebras [7, 49]. In other words, finite dimensional C^*-algebras are semisimple.

Any norm closed subalgebra of $\mathcal{L}(H)$ which is also closed under the adjoint map is clearly a C^*-algebra. A nice example is the algebra $\mathcal{K}(H)$ of *compact operators* on H. By definition, a bounded operator $T : H \to H$ is called compact if it is the norm limit of a sequence of finite rank operators. By the second fundamental theorem of Gelfand and Naimark, to be discussed below, any C^*-algebra is realized as a subalgebra of $\mathcal{L}(H)$ for some Hilbert space H.

Let A be an algebra. A *character* of A is a nonzero multiplicative linear map

$$\chi : A \to \mathbb{C}.$$

If A is unital then necessarily $\chi(1) = 1$. Let $\widehat{A}$ denote the set of characters of A. It is also known as the *maximal spectrum* of A. If A is a Banach algebra it can be shown that any character of A is automatically continuous and has norm one. We can thus endow $\widehat{A}$ with the weak* topology inherited from A^*, the continuous dual of A. The unit ball of A^* is compact in the weak* topology and one can deduce from this fact that $\widehat{A}$ is a locally compact Hausdorff space. It is compact if and only if A is unital.

When A is a unital Banach algebra there is a one to one correspondence between characters of A and the set of maximal ideals of A: to a character χ we associate its kernel, which is a maximal ideal, and to a maximal ideal I one associates the character $\chi : A \to A/I \simeq \mathbb{C}$. If A is a C^*-algebra a character is necessarily a C^* morphism.

An arbitrary C^*-algebra may well have no characters at all. This happens, for example, for simple C^*-algebras, i.e. C^*-algebras with no non-trivial closed two-sided ideal; a simple example of these are matrix algebras $M_n(\mathbb{C})$. A commutative C^*-algebra, however, has plenty of characters to the extent that characters separate points of A and in fact completely characterize it, as we shall see.

Example 2.3. Let X be a locally compact Hausdorff space. For any $x \in X$ we have the *evaluation character*

$$\chi = \mathrm{ev}_x : C_0(X) \longrightarrow \mathbb{C}, \quad \mathrm{ev}_x(f) = f(x).$$

It is easy to see that all characters of $C_0(X)$ are of this form and that the map

$$X \to \widehat{C_0(X)}, \qquad x \mapsto \mathrm{ev}_x$$

is a homeomorphism.

For any commutative Banach algebra A, the *Gelfand transform*

$$\Gamma : A \to C_0(\widehat{A})$$

is defined by $\Gamma(a) = \hat{a}$, where

$$\hat{a}(\chi) = \chi(a).$$

It is a norm contractive algebra homomorphism, as can be easily seen. In general Γ need not be injective or surjective, though its image separates the points of the spectrum. The kernel of Γ is the *nilradical* of A consisting of nilpotent elements of A.

The paper of Gelfand and Naimark [61] is the birthplace of the theory of C^*-algebras. Together with Murray-von Neumann's series of papers on von Neumann algebras [102], they form the foundation stone of operator algebras. The following two fundamental results on the structure of C^*-algebras are proved in this paper. The first result is the foundation for the belief that noncommutative C^*-algebras can be regarded as noncommutative locally compact Hausdorff spaces.

Theorem 2.1. (Gelfand–Naimark [61]) (a) *For any commutative C^*-algebra A with spectrum $\widehat{A}$ the Gelfand transform*

$$A \longrightarrow C_0(\widehat{A}), \quad a \mapsto \hat{a},$$

defines an isomorphism of C^*-algebras.
(b) *Any C^*-algebra is isomorphic to a C^*-subalgebra of the algebra $\mathcal{L}(H)$ of bounded operators on a Hilbert space H.*

Using part (a), it is easy to see that the functors

$$X \rightsquigarrow C_0(X), \qquad A \rightsquigarrow \widehat{A}$$

define an equivalence between the category of locally compact Hausdorff spaces and proper continuous maps and the opposite of the category of commutative C^*-algebras and proper C^*-morphisms. Under this correspondence compact Hausdorff spaces correspond to unital commutative C^*-algebras. We can therefore think of the opposite of the category of C^*-algebras as the category of locally compact noncommutative spaces.

Exercise 2.1. *Let X be a compact Hausdorff space and $x_0 \in X$. To test your understanding of the Gelfand–Naimark theorem, give a completely C^*-algebraic definition of the fundamental group $\pi_1(X, x_0)$.*

2.2. *GNS, KMS, and the flow of time*

We briefly indicate the proof of part (b) of Theorem 2.1 in the unital case. It is based on the notion of *state* of a C^*-algebra and the accompanying "left regular representation", called the GNS (Gelfand–Naimark–Segal) construction. We then look at the KMS (Kubo–Martin–Schwinger) condition characterizing the equilibrium states in quantum statistical mechanics and the time evolution defined by a state. In the context of von Neumann algebras, a fundamental result of Connes states that the time evolution is unique up to inner automorphisms.

The concept of state is the noncommutative analogue of Borel probability measure. A state of a unital C^*-algebra A is a positive normalized linear functional $\varphi : A \mapsto \mathbb{C}$:

$$\varphi(a^*a) \geq 0 \quad \forall\, a \in A, \quad \text{and} \quad \varphi(1) = 1.$$

The *expectation value* of an element (an "observable") $a \in A$, when the system is in the state φ, is defined by $\varphi(a)$. This terminology is motivated by states in statistical mechanics where one abandons the idea of describing the state of a system by a point in the phase space. Instead, the only reasonable question to ask is the probability of finding the system within a certain region in the phase space. This probability is of course given by a probability measure μ. Then the expected value of an observable $f : M \to \mathbb{R}$, if the system is in the state μ, is $\int f d\mu$.

Similarly, in quantum statistical mechanics the idea of describing the quantum states of a system by a vector (or ray) in a Hilbert space is abandoned and one instead uses a *density matrix*, i.e. a trace class positive operator p with $\mathrm{Tr}(p) = 1$. The expectation value of an observable a, if the system is in the state p, is given by $\mathrm{Tr}(ap)$.

A state is called *pure* if it is not a non-trivial convex combination of two other states. This corresponds to point masses in the classical case and to vector states in the quantum case.

Example 2.4. By the Riesz representation theorem, there is a one to one correspondence between states on $A = C(X)$ and Borel probability measures on X, given by

$$\varphi(f) = \int_X f d\mu.$$

φ is pure if and only if μ is a Dirac mass at some point $x \in X$.

Example 2.5. For $A = M_n(\mathbb{C})$, there is a one-to-one correspondence between states φ of A and positive matrices p with $\mathrm{Tr}(p) = 1$ (p is called a *density matrix*). It is given by

$$\varphi(a) = \mathrm{Tr}\,(ap).$$

φ is pure if and only if p is of rank 1.

Example 2.6. For a different example, assume $\pi : A \to \mathcal{L}(H)$ is a *representation* of A on a Hilbert space H. This simply means π is a morphism of C^*-algebras. Given any unit vector $v \in H$, we can define a state on A by

$$\varphi(a) = \langle \pi(a)v,\, v\rangle.$$

Such states are called *vector states.* As a consequence of the GNS construction, to be described next, one knows that any state is a vector state in the corresponding GNS representation.

Let φ be a positive linear functional on A. Then

$$\langle a,\, b\rangle := \varphi(b^*a)$$

is a positive semi-definite bilinear form A and hence satisfies the *Cauchy-Schwarz* inequality. That is, for all a, b in A we have

$$|\varphi(b^*a)|^2 \leq \varphi(a^*a)\varphi(b^*b).$$

Let

$$N = \{a \in A; \quad \varphi(a^*a) = 0\}.$$

It is easy to see, using the above Cauchy–Schwarz inequality, that N is a closed left ideal of A and

$$\langle a + N,\; b + N\rangle := \langle a,\, b\rangle,$$

is a positive definite inner product on the quotient space A/N. Let H_φ denote the Hilbert space completion of A/N. The *GNS representation*

$$\pi_\varphi : A \longrightarrow \mathcal{L}(H_\varphi)$$

is, by definition, the unique extension of the left regular representation $A \times A/N \to A/N$, $(a,\; b + N) \mapsto ab + N$. Let $v = 1 + N$. Notice that we have

$$\varphi(a) = \langle(\pi_\varphi a)(v),\; v\rangle,$$

for all $a \in A$. This shows that the state φ can be recovered from the GNS representation as a vector state.

Example 2.7. For $A = C(X)$ and $\varphi(f) = \int_X f d\mu$, where μ is a Borel probability measure on X, we have $H_\varphi = L^2(X, \mu)$. The GNS representation is the representation of $C(X)$ by multiplication operators on $L^2(X, \mu)$. In particular, when μ is the Dirac mass at a point $x \in X$, we have $H_\varphi \simeq \mathbb{C}$ and $\pi_\varphi(f) = f(x)$.

The GNS representation (π_φ, H_φ) may fail to be faithful. It can be shown that it is irreducible if and only if φ is a pure state [7, 49]. To construct a faithful representation, and hence an embedding of A into the algebra of bounded operators on a Hilbert space, one first shows that there are enough pure states on A. The proof of the following result is based on the Hahn–Banach and Krein–Milman theorems.

Lemma 2.1. *For any positive element a of A, there exists a pure state φ on A such that $\varphi(a) = \|a\|$.*

Using the GNS representation associated to φ, we can then construct, for any $a \in A$, an irreducible representation π of A such that $\|\pi(a)\| = |\varphi(a)| = \|a\|$.

We can now prove the second theorem of Gelfand and Naimark.

Theorem 2.2. *Every C^*-algebra is isomorphic to a C^*-subalgebra of the algebra of bounded operators on a Hilbert space.*

Proof. Let $\pi = \sum_{\varphi \in \mathcal{S}(A)} \pi_\varphi$ denote the direct sum of all GNS representations for all states of A. By the above remark π is faithful. $\square$

Exercise 2.2. *Identify the GNS representation in the case where $A = M_n(\mathbb{C})$ and φ is the normalized trace on A. For which state is the standard representation on $\mathbb{C}^n$ obtained?*

In the remainder of this section we look at the Kubo–Martin–Schwinger (KMS) equilibrium condition for states and some of its consequences. KMS states replace the Gibbs equilibrium states for interacting systems with infinite degrees of freedom. See [12] for an introduction to quantum statistical mechanics; see also [33] and the forthcoming book of Connes and Marcolli [36] for relations between quantum statistical mechanics, number theory and noncommutative geometry. For relations with Tomita-Takesaki theory and Connes' classification of factors the best reference is Connes' book [24].

A *C^*-dynamical system* is a triple (A, G, σ) consisting of a C^*-algebra A, a locally compact group G and a continuous action

$$\sigma : G \longrightarrow \mathrm{Aut}\,(A)$$

of G on A, where $\mathrm{Aut}\,(A)$ denotes the group of C^*-automorphisms of A. The correct continuity condition for σ is *strong continuity* in the sense that for all $a \in A$ the map $g \mapsto \sigma_g(a)$ from $G \to A$ should be continuous. Of particular interest is the case $G = \mathbb{R}$ representing a quantum mechanical system evolving in time. For example, by Stone's theorem one knows that one- parameter groups of automorphisms of $A = \mathcal{L}(\mathcal{H})$ are of the form

$$\sigma_t(a) = e^{itH} a e^{-itH},$$

where H, the *Hamiltonian* of the system, is a self-adjoint, in general unbounded, operator on $\mathcal{H}$. Assuming the operator $e^{-\beta H}$ is trace class, the corresponding *Gibbs equilibrium state* at inverse temperature $\beta = \frac{1}{kT} > 0$ is the state

$$\varphi(a) = \frac{1}{Z(\beta)} \,\mathrm{Tr}\,(a e^{-\beta H}), \tag{2}$$

where the *partition function* Z is defined by

$$Z(\beta) = \mathrm{Tr}\,(e^{-\beta H}).$$

According to Feynman [56], formula (2) for the Gibbs equilibrium state (and its classical analogue) is the apex of statistical mechanics. It should however be added that (2) is not powerful enough to deal with interacting systems with an infinite number of degrees of freedom (cf. the first chapter of Connes' book [24] for an example), and in general should be replaced by the KMS equilibrium condition.

Let $(A,\ \sigma_t)$ be a C^*-dynamical system evolving in time. A state $\varphi : A \to \mathbb{C}$ is called a *KMS state at inverse temperature* $\beta > 0$ if for all $a, b \in A$ there exists a function $F_{a,b}(z)$ which is continuous and bounded on the closed strip $0 \le \mathrm{Im}\, z \le \beta$ in the complex plane and holomorphic in the interior such that for all $t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \qquad \text{and} \qquad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a).$$

Let $\mathcal{A} \subset A$ denote the set of *analytic vectors* of σ_t consisting of those elements $a \in A$ such that $t \mapsto \sigma_t(a)$ extends to a holomorphic function on $\mathbb{C}$. One shows that $\mathcal{A}$ is a dense $*$-subalgebra of A. Now the KMS condition is equivalent to a *twisted trace property* for φ: for all analytic vectors $a, b \in \mathcal{A}$ we have

$$\varphi(ba) = \varphi(a\sigma_{i\beta}(b)).$$

Notice that the automorphism $\sigma_{i\beta}$ obtained by analytically continuing σ_t to imaginary time (in fact imaginary temperature!) is only densely defined.

Example 2.8. Any Gibbs state is a KMS state as can be easily checked.

Example 2.9. (Hecke algebras, Bost–Connes and Connes–Marcolli systems [9, 33]) A subgroup Γ_0 of a group Γ is called *almost normal* if every left coset $\gamma\Gamma_0$ is a *finite* union of right cosets. In this case we say (Γ, Γ_0) is a *Hecke pair*. Let $L(\gamma)$ denote the number of distinct right cosets $\Gamma_0\gamma_i$ in the decomposition

$$\gamma\Gamma_0 = \cup_i \Gamma_0\gamma_i,$$

and let $R(\gamma) = L(\gamma^{-1})$.

The rational *Hecke algebra* $\mathcal{A}_{\mathbb{Q}} = \mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ of a Hecke pair (Γ, Γ_0) consists of functions with finite support

$$f : \Gamma_0 \setminus \Gamma \to \mathbb{Q}$$

which are right Γ_0-invariant, i.e. $f(\gamma\gamma_0) = f(\gamma)$ for all $\gamma \in \Gamma$ and $\gamma_0 \in \Gamma_0$. Under the convolution product

$$(f_1 * f_2)(\gamma) := \sum_{\Gamma_0\setminus\Gamma} f_1(\gamma\gamma_1^{-1})f_2(\gamma_1),$$

$\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ is an associative unital algebra. Its complexification

$$\mathcal{A}_{\mathbb{C}} = \mathcal{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$$

is a $*$-algebra with an involution given by

$$f^*(\gamma) := \overline{f(\gamma^{-1})}.$$

Notice that if Γ_0 is normal in Γ then one obtains the group algebra of the quotient group Γ/Γ_0. We refer to [9, 33] for the C^*-completion of $\mathcal{A}_{\mathbb{C}}$, which is similar to the C^*-completion of group algebras. There is a one-parameter group of automorphisms of this Hecke algebra (and its C^*-completion) defined by

$$(\sigma_t f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)}\right)^{-it} f(\gamma).$$

Let P^+ denote the subgroup of the "$ax + b$" group with $a > 0$. The corresponding C^*-algebra for the Hecke pair (Γ_0, Γ) where $\Gamma = P^+_{\mathbb{Q}}$ and $\Gamma_0 = P^+_{\mathbb{Z}}$ is the Bost–Connes C^*-algebra. We refer to [33] for a description of the Connes–Marcolli system. One feature of these systems is that their partition functions are expressible in terms of zeta and L-functions of number fields.

Given a state φ on a C^*-algebra A one may ask if there is a one-parameter group of automorphisms of A for which φ is a KMS state at inverse temperature $\beta = 1$. Thanks to Tomita's theory (cf. [24, 7]) one knows that the answer is positive if A is a von Neumann algebra which we will denote by M now. The corresponding automorphism group σ_t^{φ}, called the *modular automorphism group*, is uniquely defined subject to the condition $\varphi\sigma_t^{\varphi} = \varphi$ for all $t \in \mathbb{R}$.

A von Neumann algebra typically carries many states. One of the first achievements of Connes, which set his grand classification program of von Neumann algebras in motion, was his proof that the modular automorphism group is unique up to inner automorphisms. More precisely he showed that for any other state ψ on M there is a continuous map u from $\mathbb{R}$ to the group of unitaries of M such that

$$\sigma_t^{\varphi}(x) = u_t \sigma_t^{\psi}(x) u_t^{-1} \quad \text{and} \quad u_{t+s} = u_t \sigma_s^{\varphi} u_s.$$

It follows that the modular automorphism group is independent, up to inner automorphisms, of the state (or weight) and if $\mathrm{Out}\,(M)$ denotes the quotient of the group of automorphisms of M by inner automorphisms, any von Neumann algebra has a god-given dynamical system

$$\sigma : \mathbb{R} \to \mathrm{Out}\,(M)$$

attached to it. This is a purely non-abelian phenomenon as the modular automorphism group is trivial for abelian von Neumann algebras as well for type II factors. For type III factors it turns out the the modular automorphism group possesses a complete set of invariants for the isomorphism type of the algebra in the injective case. This is the beginning of Connes' grand classification theorems for von Neumann algebras, for which we refer the reader to his book [24] and references therein.

2.3. *From groups to noncommutative spaces*

Many interesting C^*-algebras are defined as group C^*-algebras or as crossed product C^*-algebras. Group C^*-algebras are completions of group algebras with respect to certain pre C^*-norms. To illustrate some of the general ideas of noncommutative geometry and noncommutative index theory, we shall sketch Connes' proof of "connectedness" of the group C^*-algebra of free groups.

Example 2.10. (Group C^*-algebras) To any locally compact topological group G one can associate two C^*-algebras, the *full* and the *reduced* group C^*-algebras of G, denoted by $C^*(G)$ and $C_r^*(G)$, respectively. There is a 1-1 correspondence between unitary representations of G and the representations of $C^*(G)$ and a 1-1 correspondence between unitary representations of G which are equivalent to a sub-representation of its left regular representation and representations of the reduced group C^*-algebra $C_r^*(G)$. Both algebras are completions of the convolution algebra of G (under different norms). There is always a surjective C^*-morphism $C^*(G) \to C_r^*(G)$ which is injective if and only if the group G is *amenable*. It is known that abelian, solvable, as well as compact groups are amenable while, for example, non-abelian free groups are non-amenable. Here we consider only discrete groups.

Let Γ be a discrete group and let $H = \ell^2(\Gamma)$ denote the Hilbert space of square summable functions on Γ. It has an orthonormal basis consisting of delta functions $\{\delta_g\}$, $g \in \Gamma$. The *left regular representation* of Γ is the unitary representation

$$\pi : \Gamma \longrightarrow \mathcal{L}(\ell^2(\Gamma))$$

defined by

$$(\pi g)f(h) = f(g^{-1}h).$$

Let $\mathbb{C}\Gamma$ be the group algebra of Γ. There is a unique linear extension of π to an (injective) $*$-algebra homomorphism

$$\pi : \mathbb{C}\Gamma \longrightarrow \mathcal{L}(H), \qquad \pi\left(\sum a_g g\right) = \sum a_g \pi(g).$$

The *reduced group C^*-algebra* of Γ, denoted by $C_r^*\Gamma$, is the norm closure of $\pi(\mathbb{C}\Gamma)$ in $\mathcal{L}(H)$. It is obviously a unital C^*-algebra. The *canonical trace* τ on $C_r^*\Gamma$ is defined by

$$\tau(a) = \langle a\delta_e,\ \delta_e\rangle,$$

for all $a \in C_r^*\Gamma$. Notice that on $\mathbb{C}\Gamma$ we have $\tau(\sum a_g g) = a_e$. It is easily seen that τ is *positive* and *faithful* in the sense that $\tau(a^*a) \geq 0$ for all a with equality holding only for $a = 0$.

The *full* group C^*-algebra of Γ is the norm completion of $\mathbb{C}\Gamma$ under the norm

$$\|f\| = \sup\{\|\pi(f)\|; \pi \text{ is a } *\text{-representation of } \mathbb{C}\Gamma\},$$

where by a $*$-representation we mean a $*$-representation on a Hilbert space. Note that $\|f\|$ is finite since for $f = \sum_{g\in\Gamma} a_g g$ (finite sum) and any $*$-representation π we have

$$\|\pi(f)\| \leq \sum \|\pi(a_g g)\| \leq \sum |a_g|\|\pi(g)\| \leq \sum |a_g|.$$

By its very definition it is clear that there is a 1-1 correspondence between unitary representations of Γ and C^* representations of $C^*\Gamma$. Since the identity map id : $(\mathbb{C}\Gamma, \| \|) \to (\mathbb{C}\Gamma, \| \|_r)$ is continuous, we obtain a surjective C^*-algebra homomorphism

$$C^*\Gamma \longrightarrow C_r^*\Gamma.$$

It is known that this map is an isomorphism if and only if Γ is an amenable group [7, 49, 57].

Example 2.11. By Fourier transform, or the Gelfand–Naimark theorem, we have an algebra isomorphism

$$C_r^*\mathbb{Z}^n \simeq C(\mathbb{T}^n).$$

Under this isomorphism, the canonical trace τ is identified with the Haar measure on the torus $\mathbb{T}^n$. More generally, for any abelian group Γ let $\widehat{\Gamma} = \mathrm{Hom}\,(\Gamma, \mathbb{T})$ be the group of unitary characters of Γ. It is a compact group which is in fact homeomorphic to the space of characters of the commutative C^*-algebra $C_r^*\Gamma$. Thus the Gelfand transform defines an algebra isomorphism

$$C_r^*\Gamma \simeq C(\widehat{\Gamma}). \tag{3}$$

Again the canonical trace τ on the left is identified with the Haar measure on $C(\widehat{\Gamma})$.

In general one should think of the group C^*-algebra of a group Γ as the "algebra of coordinates" on the noncommutative space representing the unitary dual of Γ. Note that by the above example this is fully justified in the commutative case. In the noncommutative case, the unitary dual is a badly behaved space in general but the noncommutative dual is a perfectly legitimate noncommutative space (see the unitary dual of the infinite dihedral group in [24, 34] and its noncommutative replacement).

Example 2.12. For a finite group Γ the group C^*-algebra coincides with the group algebra of Γ. From basic representation theory we know that the group algebra $\mathbb{C}\Gamma$ decomposes as a sum of matrix algebras

$$C^*\Gamma \simeq \mathbb{C}\Gamma \simeq \oplus M_{n_i}(\mathbb{C}),$$

where the summation is over the set of conjugacy classes of Γ.

Example 2.13. (A noncommutative "connected" space) A *projection* in an $*$-algebra is a selfadjoint idempotent, i.e. an element e satisfying

$$e^2 = e = e^*.$$

It is clear that a compact space X is connected if and only if $C(X)$ has no non-trivial projections. Let us agree to call a noncommutative space represented by a C^*-algebra A "connected" if A has no non-trivial projections. The *Kadison conjecture* states that the reduced group C^*-algebra of a torsion-free discrete group is connected. This conjecture is still open although it has now been verified for various classes of groups [119]. The validity of the conjecture is known to follow from the surjectivity of the *Baum–Connes assembly map*, which is an equivariant index map

$$\mu : K_*^\Gamma(\underline{E\Gamma}) \longrightarrow K_*(C_r^*(\Gamma)).$$

This means that if there are enough "elliptic operators" on the classifying space for proper actions of Γ then one can prove an integrality theorem for values of $\tau(e)$, which then immediately implies the conjecture (e is a projection and τ is the canonical trace on $C_r^*(\Gamma)$). This principle is best described in the example below, due to Connes, where the validity of the conjecture for free groups F_n is established [22]. Notice that the conjecture is obviously true for finitely generated torsion-free abelian groups $\mathbb{Z}^n$ since by Pontryagin duality $C^*(\mathbb{Z}^n) \simeq C(\mathbb{T}^n)$ and $\mathbb{T}^n$ is clearly connected.

Let $\tau : C_r^*(F_2) \to \mathbb{C}$ be the canonical trace. Since τ is positive and faithful, if we can show that, for a projection e, $\tau(e)$ is an integer then we can deduce that $e = 0$ or $e = 1$. The proof of the *integrality* of $\tau(e)$ is remarkably similar to integrality theorems for characteristic numbers in topology proved through index theory. In fact we will show that there is a Fredholm operator F_e^+ with the property that

$$\tau(e) = \text{index}\,(F_e^+),$$

which clearly implies the integrality of $\tau(e)$. For $p \in [1, , \infty)$, let $\mathcal{L}^p(H) \subset \mathcal{L}(H)$ denote the *Schatten ideal* of p-summable compact operators on H.

The proper context for noncommutative index theory is the following [22]:

Definition 2.2. *A p-summable Fredholm module over an algebra $\mathcal{A}$ is a pair (H, F) where*
1. *$H = H^+ \oplus H^-$ is a $\mathbb{Z}/2$-graded Hilbert space with grading operator ε,*
2. *H is a left even A-module,*
3. *$F \in \mathcal{L}(H)$ is an odd operator with $F^2 = I$ and for all $a \in \mathcal{A}$ one has*

$$[F, a] = Fa - aF \in \mathcal{L}^p(H). \tag{4}$$

We say that (H, F) is a Fredholm module over $\mathcal{A}$ if instead of (4) *we have*:

$$[F,\ a] \in \mathcal{K}(H) \tag{5}$$

for all $a \in \mathcal{A}$.

The p-summability condition (4) singles out "smooth subalgebras" of $\mathcal{A}$: the higher the summability order p is the smoother $a \in \mathcal{A}$ is. This principle is easily corroborated in the commutative case. The smoother a function is, the more rapidly decreasing its Fourier coefficients are. That is why if $\mathcal{A}$ is a C^*-algebra, the natural condition to consider is the "compact resolvent" condition (5) instead of (4). In general any Fredholm module over a C^*-algebra A defines a series of subalgebras of "smooth functions" in A which are closed under holomorphic functional calculus and have the same K-theory as A (cf. Section 3.1 for an explanation of these terms).

For $A = C^\infty(M)$, M a closed n-dimensional smooth manifold, p-summable Fredholm modules for $p > n$ are defined using elliptic operators D acting between sections of vector bundles E^+ and E^- on M. Then one lets $F = \frac{D}{|D|}$ be the phase of D (assuming D is injective), and H^+ and H^- the Hilbert spaces of square integrable sections of E^+ and E^-, respectively. The algebra of continuous (respec. smooth) functions on M acts by multiplication on sections and the resulting pair (H, F) is a Fredholm module (resp. p-summable Fredholm module for $p > n$).

Now let (H, F) be a 1-summable Fredholm module over an algebra $\mathcal{A}$. Its *character* is the linear functional $\mathrm{Ch}\,(H, F) : \mathcal{A} \to \mathbb{C}$ defined by

$$\mathrm{Ch}\,(H, F)(a) = \frac{1}{2}\mathrm{Tr}\,(\varepsilon F[F, a]),$$

which is finite for all $a \in \mathcal{A}$ thanks to the 1-summability condition on (H, F). As a good exercise the reader should check that $\mathrm{Ch}\,(H, F)$ is in fact a trace on $\mathcal{A}$.

The second ingredient that we need is an index formula. Let $e \in \mathcal{A}$ be an idempotent. Let $F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$, $H_1 = eH^+$, $H_2 = eH^{-1}$ and $P' : H_1 \to H_2$ and let $Q' : H_2 \to H_1$ be the restrictions of P and Q to H_1 and H_2, respectively. The trace condition $[F, e] \in \mathcal{L}^1(H)$ is equivalent to $P'Q' - 1_{H_2} \in \mathcal{L}^1(H_2)$ and $Q'P' - 1_{H_1} \in \mathcal{L}^1(H_1)$, which of course imply that the operator

$$F_e^+ := P' \tag{6}$$

is Fredholm and its Fredholm index is given by

$$\begin{aligned}\mathrm{Index} F_e^+ &= \mathrm{Tr}(I - Q'P') - \mathrm{Tr}(I - P'Q') = \frac{1}{2}\mathrm{Tr}(\varepsilon F[F, e]) \\ &= \mathrm{Ch}\,(H, F)(e).\end{aligned}$$

We see that we are done provided we can construct a 1-summable Fredholm module over a dense and closed under holomorphic functional calculus subalgebra $\mathcal{A}$ of $A = C_r^*(F_2)$ such that for any projection $e \in \mathcal{A}$,

$$\tau(e) = \mathrm{Ch}(H, F)(e).$$

It is known that a group is free if and only if it acts freely on a tree. Let then T be a tree with a free action of F_2 and let T^0 and T^1 denote the set of vertices and edges of T respectively. Let $H^+ = \ell^2(T^0)$, $H^- = \ell^2(T^1) \oplus \mathbb{C}$ with orthonormal basis denoted by ε_q. The action of F_2 on T induce an action of $C_r^*(F_2)$ on H^+ and H^- respectively. Fixing a vertex $p \in T^0$ we can define a one to one correspondence $\varphi : T^0 - p \to T^1$ by sending $q \in T^0 - p$ to the edge containing q and lying between p and q. This defines a unitary operator $P : H^+ \to H^-$ by

$$P(\varepsilon_q) = \varepsilon_{\varphi(q)} \quad \text{and} \quad P(\varepsilon_p) = (0, 1).$$

The compact resolvent condition (36) is a consequence of *almost equivariance* of φ in the sense that for any $q \in T^0 - p$, $\varphi(gq) = g\varphi(q)$ for all but a finite number of g, which is not difficult to prove.

Exercise 2.3. *Prove the last statement. To see what is going on start with $\Gamma = \mathbb{Z}$ and go through all the steps in the proof.*

Let $\mathcal{A} \subset A$ denote the subalgebra of all $a \in A$ such that

$$[F, a] \in \mathcal{L}^1(H).$$

Clearly (H, F) is a 1-summable Fredholm module over $\mathcal{A}$. It can be shown that $\mathcal{A}$ is stable under holomorphic functional calculus in A (see Section 3.1). It still remains to be checked that for all $a \in \mathcal{A}$ we have $\mathrm{Ch}\,(H, F)(a) = \tau(a)$, which we leave as an exercise.

Example 2.14. (Crossed product algebras) Let $\alpha : G \to \mathrm{Aut}\,(A)$ be an action of a group G by algebra automorphisms on an algebra A. The action of g on a will be denoted by $g(a)$. The (algebraic) *crossed product algebra* $A \rtimes G$ is the algebra generated by the two subalgebras A and $\mathbb{C}G$ subject to relations

$$gag^{-1} = g(a),$$

for all g in G and $a \in A$. Formally we define $A \rtimes G = A \otimes \mathbb{C}G$ with product given by

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh.$$

One checks that this is an associative product and that the two definitions are in fact the same.

In many cases $A = C(X)$ or $A = C^\infty(X)$ is an algebra of functions on a space X and G acts on X by homeomorphisms or diffeomorphisms. Then there is an induced action of G on A defined by $(gf)(x) = f(g^{-1}(x))$. One of the key ideas of noncommutative geometry is Connes' dictum that in such situations the crossed product algebra $C(X) \rtimes G$ replaces the algebra of functions on the quotient space X/G (see Section 4.1 for more on this).

Example 2.15. Let $G = \mathbb{Z}_n$ be the cyclic group of order n acting by translations on $X = \{1, 2, \ldots, n\}$. The crossed product algebra $C(X) \rtimes G$ is the algebra generated by elements U and V subject to the relations

$$U^n = 1, \quad V^n = 1, \quad UVU^{-1} = \lambda V,$$

where $\lambda = e^{\frac{2\pi i}{n}}$. Here U is a generator of $\mathbb{Z}_n$, V is a generator of $\hat{\mathbb{Z}}_n$ and we have used the isomorphism $C(X) \simeq \mathbb{C}\hat{\mathbb{Z}}_n$.

We have an isomorphism

$$C(X) \rtimes \mathbb{Z}_n \simeq M_n(\mathbb{C}). \tag{7}$$

To see this consider the $n \times n$ matrices

$$u = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & \bar{\lambda} & 0 & & 0 \\ 0 & 0 & \bar{\lambda}^2 & & 0 \\ & & & & \\ 0 & \cdots & \cdots & 0 & \bar{\lambda}^{n-1} \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & \cdots & 0 \\ & & 0 & 1 & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

They clearly satisfy the relations

$$u^n = 1, \quad v^n = 1, \quad uvu^{-1} = \lambda v.$$

Moreover one checks that the matrices $u^p v^q$ for $1 \leq p, q \leq n$ are linearly independent, which shows that the algebra generated by u and v in $M_n(\mathbb{C})$ is all of $M_n(\mathbb{C})$. The isomorphism (7) is now defined by

$$U \mapsto u, \quad V \mapsto v.$$

This isomorphism can be easily generalized:

Exercise 2.4. *Let $\mathbb{Z}_n$ act on an algebra A. Show that the dual group $\hat{\mathbb{Z}}_n$ acts on $A \rtimes \mathbb{Z}_n$ and that we have an isomorphism*

$$A \rtimes \mathbb{Z}_n \rtimes \hat{\mathbb{Z}}_n \simeq M_n(A).$$

Extend this to actions of finite abelian groups.

The above isomorphism is a discrete analogue of Takai duality for C^*-crossed products of actions of locally compact abelian groups (cf. [7])

$$A \rtimes G \rtimes \hat{G} \simeq A \otimes \mathcal{K}(L^2(G)),$$

where $\mathcal{K}$ is the algebra of compact operators.

The above purely algebraic theory can be extended to a C^*-algebraic context. For any locally compact topological group G acting by C^*-automorphisms on a C^*-algebra A, one defines the reduced $A \rtimes_r G$ and the full $A \rtimes G$ crossed product C^*-algebras. Assuming $G = \Gamma$ is a discrete group, to define the crossed product let $\pi : A \to \mathcal{L}(H)$ be a faithful representation of A (the definition turns out to be independent of the choice of π). Consider the Hilbert space $\ell^2(\Gamma, H)$ of square summable functions $\xi : \Gamma \to H$ and define a representation $\rho : A \rtimes \Gamma \to \mathcal{L}(H)$ by

$$(\rho(x)\xi)(g) = \sum_h \pi(g^{-1}(x(h)))\xi(h^{-1}g)$$

for all $x \in A \rtimes \Gamma$, $\xi \in H$, and $g \in \Gamma$. It is an injective $*$-algebra homomorphism and the reduced C^*-crossed product algebra $A \rtimes_r \Gamma$ is defined to be the norm completion of the image of ρ.

Example 2.16. The crossed product C^*-algebra

$$A_\theta = C(S^1) \rtimes \mathbb{Z}$$

is known as the *noncommutative torus*. Here $\mathbb{Z}$ acts on S^1 by rotation through the angle $2\pi\theta$.

2.4. *Continuous fields of C^*-algebras*

Let X be a locally compact Hausdorff topological space and for each $x \in X$ let a C^*-algebra A_x be given. Let also $\Gamma \subset \coprod A_x$ be a $*$-subalgebra. We say this data defines a *continuous field* of C^*-algebras over X if

(1) for all $x \in X$, the set $\{s(x);\, s \in \Gamma\}$ is dense in A_x,
(2) for all $s \in \Gamma$, $x \to \|s(x)\|$ is continuous on X,
(3) Γ is locally uniformly closed, i.e. for any section $s \in \coprod A_x$, if for any $x \in X$ we can approximate s around x arbitrarily closely by elements of Γ, then $s \in \Gamma$.

To any continuous field of C^*-algebras over a locally compact space X we associate the C^* algebra of its *continuous sections* vanishing at infinity (a section $s : X \to \coprod A_x$ is called continuous if $x \to \|s(x)\|$ is continuous).

Examples 2.1. (i) A rather trivial example is the C^*-algebra

$$A = \Gamma(X, \mathrm{End}\,(E))$$

of continuous sections of the endomorphism bundle of a vector bundle E over a compact space X. A nice example of this is the noncommutative torus A_θ for $\theta = \frac{p}{q}$ a rational number. As we shall see

$$A_{\frac{p}{q}} \simeq \Gamma(\mathbb{T}^2, \mathrm{End}\,(E))$$

is the algebra of continuous sections of the endomorphism bundle of a vector bundle of rank q over the torus $\mathbb{T}^2$. Notice that in general $C(X) \subset Z(A)$ is a subalgebra of the center of A. As we shall see $Z(A_{\frac{p}{q}}) \simeq C(\mathbb{T}^2)$.
(ii) A less trivial example is the field over the interval $[0,1]$, where $A_x = M_2(\mathbb{C})$ for $0 < x < 1$ and $A_0 = A_1 = \mathbb{C} \oplus \mathbb{C}$.
(iii) The noncommutative tori A_θ, $0 \leq \theta \leq 1$, can be put together to form a continuous field of C^*-algebras over the circle (see [7]).

Exercise 2.5. *An Azumaya algebra is the algebra of continuous sections of a (locally trivial) bundle of finite dimensional full matrix algebras over a space X. Give an example of an Azumaya algebra which is not of the type $\Gamma(End(E))$ for some vector bundle E.*

Of particular interest are C^*-algebras of continuous sections of a locally trivial bundle of algebras with fibers the algebra $\mathcal{K}(H)$ of compact operators and structure group $\mathrm{Aut}\,\mathcal{K}(H)$. When H is infinite dimensional such bundles can be completely classified in terms of their *Dixmier–Douady invariant* $\delta(E) \in H^3(X, \mathbb{Z})$ [51]. (cf. [107] for a modern detailed account.) The main reason for this is that any automorphism of $\mathcal{K}(H)$ is inner. In fact for any Hilbert space H there is an exact sequence of topological groups

$$1 \longrightarrow U(1) \longrightarrow U(H) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}\,\mathcal{K}(H) \longrightarrow 1, \tag{8}$$

where the unitary group $U(H)$ has its strong operator topology and the automorphism group $\mathrm{Aut}\,\mathcal{K}(H)$ is taken with its norm topology. It is rather easy to see that if H is infinite dimensional then $U(H)$ is contractible. (A much harder theorem of Kuiper states that it is also contractible under the norm topology, but this is not needed here.)

Let X be a locally compact Hausdorff and paracompact space. Locally trivial bundles over X with fibers $\mathcal{K}(H)$ and structure group $\mathrm{Aut}(\mathcal{K}(H))$ are classified by their classifying class in the Čech cohomology group $H^1(X, \mathrm{Aut}(\mathcal{K}(H))$. Now, since the middle term in (8) is contractible, it stands to reason to expect that

$$H^1(X, \mathrm{Aut}\,\mathcal{K}(H)) \simeq H^2(X, U(1)) \simeq H^3(X, \mathbb{Z}).$$

The first isomorphism is actually not so obvious because (8) is an exact sequence of nonabelian groups, but it is true (cf. [107] for a proof). The resulting class

$$\delta(E) \in H^3(X, \mathbb{Z})$$

is a complete isomorphism invariant of such bundles.

Continuous fields of C^*-algebras are often constructed by crossed product algebras. Given a crossed product algebra $A \rtimes_\alpha \mathbb{R}$, by rescaling the action α_t to α_{st}, we obtain a one-parameter family of crossed product algebras $A \rtimes_{\alpha^s} \mathbb{R}$ over $[0, \infty)$ whose fiber at 0 is $A \otimes C_0(\mathbb{R})$. For example, the translation action of $\mathbb{R}$ on itself will give a field of C^*-algebras whose fiber at 0 is $C_0(\mathbb{R}) \otimes C_0(\mathbb{R}) \simeq C_0(\mathbb{R}^2)$ and at $t > 0$ is $C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R} \simeq \mathcal{K}(L^2(\mathbb{R}))$.

Example 2.17. Let H be the discrete *Heisenberg group* of upper triangular matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

with integer entries. It is a solvable, and hence amenable, group which means that the reduced and full group C^*-algebras of H coincide. The group C^*-algebra $C^*(H)$ can also be defined as the C^*-algebra generated by three unitaries U, V, W subject to relations

$$UV = WVU, \quad UW = WU, \quad VW = WV. \tag{9}$$

They correspond to elements

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that W is central and in fact it generates the center $Z(A) \simeq C(S^1)$. For each $e^{2\pi i\theta} \in S^1$, we have a surjective C^*-morphism

$$f_\theta : C^*(H) \longrightarrow A_\theta$$

by sending $U \mapsto U$, $V \mapsto V$ and $W \mapsto e^{2\pi i\theta}1$. This is clear from (9). Thus, roughly speaking, the Heisenberg group C^*-algebra $C^*(H)$ can be viewed as a 'noncommutative bundle' over the circle with noncommutative tori A_θ as fibers. Notice, however, that this bundle is not a locally trivial bundle of algebras as different fibers are not isomorphic to each other. One can show in fact $C^*(H)$ is the C^*-algebra of continuous sections of a field of C^*-algebras over the circle whose fiber at $e^{2\pi i\theta}$ is the noncommutative torus A_θ (cf. [7, 57] and references therein).

2.5. *Noncommutative tori*

These algebras can be defined in a variety of ways, e.g. as the C^*-algebra of the Kronecker foliation of the two-torus by lines of constant slope $dy = \theta dx$, as the crossed product algebra $C(S^1) \rtimes \mathbb{Z}$ associated to the automorphism of the circle through rotation by an angle $2\pi\theta$, as strict deformation quantization, or by generators and relations as we do here.

Let $\theta \in \mathbb{R}$ and $\lambda = e^{2\pi i\theta}$. The *noncommutative torus* A_θ is the *universal* unital C^*-algebra generated by unitaries U and V subject to the relation

$$UV = \lambda VU. \tag{10}$$

The universality property here means that given any unital C^*-algebra B with two unitaries u and v satisfying $uv = \lambda vu$, there exists a unique unital C^* morphism $A_\theta \to B$ sending $U \to u$ and $V \to v$.

Unlike the purely algebraic case where any set of generators and relations automatically defines a universal algebra, this is not the case for universal C^*-algebras. Care must be applied in defining a norm satisfying the C^*-identity, and in general the universal problem does not have a solution (cf. [7] for more on this). For the noncommutative torus we proceed as follows. Consider the unitary operators $U, V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined by

$$(Uf)(n) = e^{2\pi i\theta n} f(n), \qquad (Vf)(n) = f(n-1).$$

They satisfy $UV = \lambda VU$. Let A_θ be the unital C^*-subalgebra of $\mathcal{L}(\ell^2(\mathbb{Z}))$ generated by U and V. To check the universality condition, it suffices to check that the operators $U^m V^n$, m, $n \in \mathbb{Z}$, are linearly independent, which we leave as an exercise.

Using the Fourier isomorphism $\ell^2(\mathbb{Z}) \simeq L^2(S^1)$, we see that A_θ can also be described as the C^*-subalgebra of $\mathcal{L}(L^2(S^1))$ generated by the operators $(Uf)(z) = f(\lambda z)$ and $(Vf)(z) = zf(z)$ for all $f \in L^2(S^1)$. This presentation also makes it clear that we have an isomorphism

$$A_\theta \simeq C(S^1) \rtimes \mathbb{Z},$$

where $\mathbb{Z}$ acts by rotation though the angle $2\pi\theta$.

Since the Haar measure on S^1 is invariant under rotations, the formula

$$\tau\Big(\sum f_i V^i\Big) = \int_{S^1} f_0$$

defines a faithful positive normalized trace

$$\tau : A_\theta \to \mathbb{C}.$$

This means that τ is a trace and

$$\tau(aa^*) > 0, \qquad \tau(1) = 1,$$

for all $a \neq 0$. On the dense subalgebra of A_θ generated by U and V we have

$$\tau\left(\sum a_{mn} U^m V^n\right) = a_{00}. \tag{11}$$

Exercise 2.6. *Show that if θ is irrational then there is a unique trace, given by* (11), *on the subalgebra of A_θ generated by U and V. For rational values of θ show that there are uncountably many traces.*

The map $U \mapsto V$ and $V \mapsto U$ defines an isomorphism $A_\theta \simeq A_{1-\theta}$. This shows that we can restrict to $\theta \in [0, \frac{1}{2})$. It is known that in this range A_{θ_1} is isomorphic to A_{θ_2} if and only if $\theta_1 = \theta_2$. Notice that A_θ is commutative if and only if θ is an integer, and simple Fourier theory shows that A_0 is isomorphic to the algebra $C(\mathbb{T}^2)$ of continuous functions on the 2-torus. For irrational θ, A_θ is known to be a simple C^*-algebra, i.e. it has no proper closed two-sided ideal. In particular it has no finite dimensional representations [7, 49, 57].

For $\theta = \frac{p}{q}$ a rational number, A_θ has a finite dimensional representation. To see this consider the unitary $q \times q$ matrices

$$u = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & \cdots & 0 & \lambda^{q-1} \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

They satisfy the relations

$$uv = \lambda vu, \qquad u^q = v^q = 1. \tag{12}$$

It follows that there is a unique C^* map from $A_\theta \to M_q(\mathbb{C})$ sending $U \to u$ and $V \to v$. This, of course, implies that A_θ is not simple for θ a rational number. In fact it is known that in this case A_θ is Morita equivalent to the commutative algebra $C(\mathbb{T}^2)$, as we shall see shortly.

Exercise 2.7. *Show that the C^*-algebra generated by relations* (12) *is isomorphic to $M_q(\mathbb{C})$.* (*hint: it suffices to show that the matrices $u^i v^j$, $1 \leq i$, $j \leq q$, are linearly independent.*)

Assuming $\theta = \frac{p}{q}$ is rational, we have $U^q V = V U^q$ and $V^q U = U V^q$, which show that U^q and V^q are in the center of A_θ. In fact it can be shown that they generate the center and we have

$$Z(A_\theta) \simeq C(\mathbb{T}^2).$$

One shows that there is a (flat) vector bundle E of rank q over $\mathbb{T}^2$ such that $A_{\frac{p}{q}}$ is isomorphic to the algebra of continuous sections of the endomorphism bundle of E,

$$A_{\frac{p}{q}} \simeq \Gamma(\mathbb{T}^2, \operatorname{End}(E)).$$

Exercise 2.8. *Show that if θ is irrational then $Z(A_\theta) = \mathbb{C}1$.*

There is a dense $*$-subalgebra $\mathcal{A}_\theta \subset A_\theta$ that deserves to be called the algebra of "*smooth functions*" on the noncommutative torus (see Section 3.1 for more on the meaning of the word smooth in noncommutative geometry). By definition $\mathcal{A}_\theta$ consists of elements of the form

$$\sum_{(m,n)\in\mathbb{Z}^2} a_{mn} U^m V^n,$$

where $(a_{mn}) \in \mathcal{S}(\mathbb{Z}^2)$ is a rapid decay Schwartz class function, i.e. for all $k \geq 1$

$$\operatorname*{Sup}_{m,n} |a_{mn}|(1 + |m| + |n|)^k < \infty.$$

Exercise 2.9. *Show that $\mathcal{A}_0 \simeq C^\infty(\mathbb{T}^2)$.*

A *derivation* or *infinitesimal automorphism* of an algebra $\mathcal{A}$ is a linear map $\delta : \mathcal{A} \to \mathcal{A}$ satisfying $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. For the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on a manifold M one checks that there is a one to one correspondence between derivations on $\mathcal{A}$ and vector fields on M. For this reason derivations are usually considered as noncommutative analogues of vector fields. The fundamental derivations of the noncommutative torus, δ_1, $\delta_2 : \mathcal{A}_\theta \to \mathcal{A}_\theta$ are the derivations uniquely defined by their values on generators of the algebra

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \text{and} \quad \delta_2(U) = 0, \quad \delta_2(V) = V,$$

or equivalently

$$\delta_1\Bigg(\sum_{(m,n)\in\mathbb{Z}^2} a_{mn} U^m V^n\Bigg) = \sum_{(m,n)\in\mathbb{Z}^2} m a_{mn} U^m V^n,$$

and similarly for δ_2.

The invariance property of the Haar measure for the torus has a noncommutative counterpart. It is easy to see that the canonical trace $\tau : \mathcal{A}_\theta \to \mathbb{C}$ is invariant under δ_1 and δ_2, i.e. $\tau(\delta_i(a)) = 0$ for $i = 1, 2$ and all $a \in \mathcal{A}_\theta$.

Exercise 2.10. *Find all derivations of $\mathcal{A}_\theta$.*

Remark 1. Although they are deformations of Hopf algebras (in fact deformations of a group), noncommutative tori are not Hopf algebras. They are however in some sense Hopf algebroids. Since they are groupoid algebras, this is not surprising.

The definition of noncommutative tori can be extended to higher dimensions. Let Θ be a real skew-symmetric $n \times n$ matrix and let $\lambda_{jk} = e^{2\pi i \Theta_{jk}}$. The *noncommutative torus* A_Θ is the universal unital C^*-algebra generated by unitaries $U_1, U_2, \ldots, U_n$ with

$$U_k U_j = \lambda_{jk} U_j U_k$$

for all $1 \leq j, k \leq n$. Alternatively, A_Θ is the universal C^*-algebra generated by unitaries U_l, $l \in \mathbb{Z}^n$ with

$$U_l U_m = e^{\pi i \langle l, \Theta m \rangle} U_{l+m},$$

for all $l, m \in \mathbb{Z}^n$. From this it follows hat for any $B \in GL(n, \mathbb{Z})$ we have $A_{B\Theta B^t} \simeq A_\Theta$. The definitions of the invariant trace, smooth subalgebras, and fundamental derivations are easily extended to higher dimensions. The isomorphism types and Morita equivalence classes of higher dimensional noncommutative tori, as well as finite projective modules over them has been extensively studied in recent years.

3. Beyond C^*-algebras

To plunge deeper into noncommutative geometry one must employ various classes of noncommutative algebras that are not C^*-algebras. They include dense subalgebras of C^*-algebras which are *stable under holomorphic functional calculus*, e.g. algebras of smooth functions on noncommutative tori. Another class consists of *almost commutative algebras*. They have a *Poisson algebra* as their *semiclassical limit* and include algebras of differential operators and enveloping algebras. They appear in questions of deformation quantization and its applications.

3.1. *Algebras stable under holomorphic functional calculus*

We are going to describe a situation where many features of the embedding $C^\infty(M) \subset C(M)$ of the algebra of smooth functions on a closed manifold into the algebra of continuous functions is captured and extended to the noncommutative world.

Let A be a unital Banach algebra and let f be a holomorphic function defined on a neighborhood of $\mathrm{sp}\,(a)$, the spectrum of $a \in A$. Let

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z1 - a)^{-1} dz, \tag{13}$$

where the contour γ goes around the spectrum of a only once (counter clockwise). The integral is independent of the choice of the contour and it can be shown that for a fixed a, the map $f \mapsto f(a)$ is a unital algebra map from the algebra of holomorphic functions on a neighborhood of $\mathrm{sp}\,(a)$ to A. It is called the *holomorphic functional calculus.* If f happens to be holomorphic in a disc containing $\mathrm{sp}(a)$ with power series expansion $f(z) = \sum c_i z^i$, then one shows, using the Cauchy integral formula, that $f(a) = \sum c_i a^i$. If A is a C^*-algebra and a is a normal element then, thanks to the Gelfand-Naimark theorem, we have the much more powerful *continuous functional calculus* from $C(\mathrm{sp}\,(a)) \to A$. It extends the holomorphic functional calculus (see below).

Definition 3.1. *Let $B \subset A$ be a unital subalgebra of a unital Banach algebra A. We say B is stable under holomorphic functional calculus if for all $a \in B$ and all holomorphic functions on $sp\,(a)$, we have $f(a) \in B$.*

Example 3.1. (1) The algebra $C^\infty(M)$ of smooth functions on a closed smooth manifold M is stable under holomorphic functional calculus in $C(M)$. The same can be said about the algebra $C^k(M)$ of k-times differentiable functions. The algebra $\mathbb{C}[X]$ of polynomial functions is not stable under holomorphic functional calculus in $C[0, 1]$.
(2) The smooth noncommutative torus $\mathcal{A}_\theta \subset A_\theta$ is stable under holomorphic functional calculus.

Let $\mathrm{sp}_B\,(a)$ denote the spectrum of $a \in B$ with respect to the subalgebra B. Clearly $\mathrm{sp}_A\,(a) \subset \mathrm{sp}_B\,(a)$ but the reverse inclusion holds if and only if invertibility in A implies invertibility in B. A good example to keep in mind is $\mathbb{C}[x] \subset C[0, 1]$. It is easy to see that if B is stable under holomorphic functional calculus, then we have the *spectral permanence property*

$$\mathrm{sp}_A\,(a) \subset \mathrm{sp}_B\,(a).$$

Conversely, under some conditions on the subalgebra B the above spectral permanence property implies that B is stable under holomorphic functional calculus. In fact, in this case for all $z \in \mathrm{sp}_A\,(a)$, $(z1 - a)^{-1} \in B$ and if there is a suitable topology in B, stronger than the topology induced from A, in which B is complete, one can then show that the integral (13) converges in B. We give two instances where this technique works.

Let (H, F) be a Fredholm module over a Banach algebra A and assume that the action of A on H is continuous.

Proposition 3.1. ([24]) *For each $p \in [1, \infty)$, the subalgebra*

$$\mathcal{A} = \{a \in A;\ [F,\, a] \in \mathcal{L}^p(H)\}$$

is stable under holomorphic functional calculus.

Exercise 3.1. (*Smooth compact operators*) *Let $\mathcal{K}^\infty \subset \mathcal{K}$ be the algebra of infinite matrices (a_{ij}) with rapid decay coefficients. Show that $\mathcal{K}^\infty$ is stable under holomorphic functional calculus in the algebra of compact operators $\mathcal{K}$.*

Another source of examples are *smooth vectors* of Lie group actions. Let G be a Lie group acting continuously on a Banach algebra A. An element $a \in A$ is called *smooth* if the map $G \to A$ sending $g \mapsto g(a)$ is smooth. It can be shown that smooth vectors form a dense subalgebra of A which is stable under holomorphic functional calculus.

Example 3.2. The formulas

$$U \mapsto \lambda_1 U, \qquad V \mapsto \lambda_2 V,$$

where $(\lambda_1, \lambda_2) \in \mathbb{T}^2$, define an action of the two-torus $\mathbb{T}^2$ on the noncommutative torus A_θ. Its set of smooth vectors can be shown to coincide with the smooth noncommutative torus $\mathcal{A}_\theta$ [17].

For applications to K-theory and density theorems, the following result is crucial [115].

Proposition 3.2. *If B is a dense subalgebra of a Banach algebra A which is stable under holomorphic functional calculus then so is $M_n(B)$ in $M_n(A)$ for all $n \geq 0$.*

Now let $e \in A$ be an idempotent in A. For any $\epsilon > 0$ there is an idempotent $e' \in B$ such that $\|e - e'\| < \epsilon$. In fact, since B is dense in A we can first approximate it by an element $g \in B$. Since $\mathrm{sp}\,(e) \subset \{0, 1\}$, $\mathrm{sp}\,(g)$ is concentrated around 0 and 1. Let f be a holomorphic function which is identically equal to 1 around 0 and 1. Then

$$e' = f(g) = \frac{1}{2\pi i} \int_\gamma f(z)(z1 - e)^{-1} dz,$$

is an idempotent in B which is close to e. In particular $[e] = [f(g)]$ in $K_0(A)$ (see Section 5.1). Thanks to the above Proposition, we can repeat this argument for $M_n(B) \subset M_n(A)$ for all n. It follows that if B is dense is A and is stable under holomorphic functional calculus, the natural embedding $B \to A$ induces an isomorphism $K_0(B) \simeq K_0(A)$ in K-theory (cf. also the article of J. B. Bost [8] where a more general density theorem along these lines is proved).

Example 3.3. (Toeplitz algebras) The original Toeplitz algebra $\mathcal{T}$ is defined as the universal unital C^*-algebra generated by an *isometry*, i.e. an element S with

$$S^* S = I.$$

It can be concretely realized as the C^*-subalgebra of $\mathcal{L}(\ell^2(\mathbb{N}))$ generated by the unilateral forward shift operator $S(e_i) = e_{i+1}$, $i = 0, 1, \dots$. Since the algebra $C(S^1)$ of continuous functions on the circle is the universal algebra defined by a unitary u, the map $S \mapsto u$ defines a C^*-algebra surjection

$$\sigma : \mathcal{T} \longrightarrow C(S^1),$$

called the *symbol map*. It is an example of the symbol map for pseudodifferential operators of order zero over a closed manifold (see below).

The rank one projection $I - SS^*$ is in the kernel of σ. Since the closed ideal generated by $I - SS^*$ is the ideal $\mathcal{K}$ of compact operators, we have $\mathcal{K} \subset \operatorname{Ker} \sigma$. With some more work one shows that in fact $\mathcal{K} = \operatorname{Ker} \sigma$ and therefore we have a short exact sequence of C^*-algebras, called the *Toeplitz extension* (due to Coburn, cf. [53, 57])

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0. \tag{14}$$

There is an alternative description of the Toeplitz algebra and extension (14) that makes its relation with pseudodifferential operators and index theory more transparent. Let $H = L^2(S^1)^+$ denote the Hilbert space of square integrable functions on the circle whose negative Fourier coefficients vanish and let $P : L^2(S^1) \to L^2(S^1)^+$ denote the canonical projection. Any continuous function $f \in C(S^1)$ defines a *Toeplitz operator*

$$T_f : L^2(S^1)^+ \to L^2(S^1)^+, \qquad T_f(g) = P(gf).$$

It can be shown that the C^*-algebra generated by the set of Toeplitz operators $\{T_f;\ f \in C(S^1)\}$ is isomorphic to the Toeplitz algebra $\mathcal{T}$. The relation

$$T_f T_g - T_{fg} \in \mathcal{K}(H)$$

shows that any Toeplitz operator T can be written as

$$T = T_f + K,$$

where K is a compact operator. In fact this decomposition is unique and gives another definition of the symbol map σ by $\sigma(T_f + K) = f$. It is also clear from extension (14) that a Toeplitz operator T is Fredholm if and only if its symbol $\sigma(T)$ is an invertible function on S^1.

The algebra generated by Toeplitz operators T_f for $f \in C^\infty(S^1)$ is called the *smooth Toeplitz algebra* $\mathcal{T}^\infty \subset \mathcal{T}$. Similar to (14) we have an extension

$$0 \longrightarrow \mathcal{K}^\infty \longrightarrow \mathcal{T}^\infty \xrightarrow{\sigma} C^\infty(S^1) \longrightarrow 0. \tag{15}$$

Exercise 3.2. *Show that $\mathcal{T}^\infty$ is stable under holomorphic functional calculus in $\mathcal{T}$.*

Exercise 3.3. *Show that*

$$\varphi(A, B) = Tr([A, B])$$

defines a cyclic 1*-cocycle on $\mathcal{T}^\infty$. If f is a smooth non-vanishing function on the circle, show that*

$$index(T_f) = \varphi(T_f, T_{f^{-1}}).$$

The Toeplitz extension (14) has a grand generalization. On any closed smooth manifold M, a (scalar) pseudodifferential operator D of order zero defines a bounded linear map $D : L^2(M) \to L^2(M)$ and its principal symbol $\sigma(D)$ is a continuous function on $S^*(M)$, the unit cosphere bundle of M (cf. [89]). Let $\Psi^0(M) \subset \mathcal{L}(L^2(M))$ denote the C^*-algebra generated by all pseudodifferential operators of order zero on M. We then have a short exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K}(L^2(M)) \longrightarrow \Psi^0(M) \stackrel{\sigma}{\longrightarrow} C(S^*M) \longrightarrow 0.$$

For $M = S^1$, the cosphere bundle splits as the disjoint union of two copies of S^1 and the above sequence is the direct sum of two identical copies, each of which is isomorphic to the Toeplitz extension (14).

3.2. *Almost commutative and Poisson algebras*

Let A be a unital complex algebra. We say A is a *filtered algebra* if it has an increasing filtration $F^i(A) \subset F^{i+1}(A)$, $i = 0, 1, 2, \ldots$, with $F^0(A) = \mathbb{C}1$, $F^i(A)F^j(A) \subset F^{i+j}(A)$ for all i, j and $\cup_i F^i(A) = A$. Let $F^{-1}(A) = 0$. The *associated graded algebra* of a filtered algebra is the graded algebra

$$\mathrm{Gr}\,(A) = \bigoplus_{i \geq 0} \frac{F^i(A)}{F^{i-1}(A)}.$$

Definition 3.2. *An almost commutative algebra is a filtered algebra whose associated graded algebra $Gr(A)$ is commutative.*

Being almost commutative is equivalent to the commutator condition

$$[F^i(A),\ F^j(A)] \subset F^{i+j-1}(A),$$

for all i, j. As we shall see Weyl algebras and more generally algebras of differential operators on a smooth manifold, and universal enveloping algebras are examples of almost commutative algebras.

Let A be an almost commutative algebra. The original Lie bracket $[x, y] = xy - yx$ on A induces a Lie bracket $\{\ \}$ on $Gr(A)$ via the formula

$$\{x + F^i,\ y + F^j\} := [x,\ y] + F^{i+j-2}.$$

Notice that by the almost commutativity assumption, $[x,\ y]$ is in $F^{i+j-1}(A)$. The induced Lie bracket on $\mathrm{Gr}\,(A)$ is compatible with its multiplication in the sense that for all $a \in \mathrm{Gr}\,(A)$, the map $b \mapsto \{a,\ b\}$ is a derivation. The algebra $\mathrm{Gr}\,(A)$ is called the *semiclassical limit* of the almost commutative algebra A. It is an example of a Poisson algebra as we recall next. The quotient map

$$\sigma : A \to \mathrm{Gr}\,(A)$$

is called the *principal symbol map.*

Any splitting $q : \mathrm{Gr}\,(A) \to A$ of this map can be regarded as a "naive quantization map". Linear splittings always exist but they are hardly interesting. One usually demands more. For example one wants q to be a Lie algebra map in the sense that

$$q\{a,\ b\} = [q(a),\ q(b)] \tag{16}$$

for all a, b in $\mathrm{Gr}\,(A)$. This is one form of *Dirac's quantization rule* going back to [50]. *No-go theorems*, e.g. Groenvald–van Hove's (cf. [66] for a discussion and precise statements), state that, under reasonable non-degeneracy conditions, this is almost never possible. The remedy is to have (16) satisfied only in an asymptotic sense. As we shall discuss later in this section, this can be done in different ways either in the context of formal deformation quantization [4, 84] or through strict C^*-algebraic deformation quantization [112].

The notion of a Poisson algebra captures the structure of semiclassical limits.

Definition 3.3. *Let P be a commutative algebra. A Poisson structure on P is a Lie algebra bracket $(a, b) \mapsto \{a, b\}$ on A such that for any $a \in A$, the map $b \mapsto \{a, b\} : A \to A$ is a derivation of A. That is, for all b, c in A we have*

$$\{a, bc\} = \{a, b\}c + b\{a, c\}.$$

In geometric examples (see below) the vector field defined by this derivation is called the *Hamiltonian vector field* of the *Hamiltonian function a.*

Definition 3.4. *A Poisson algebra is a pair $(P, \{,\})$ where P is a commutative algebra and $\{\ ,\ \}$ is a Poisson structure on P.*

We saw that the semiclassical limit $P = \mathrm{Gr}\,(A)$ of any almost commutative algebra A is a Poisson algebra. Conversely, given a Poisson algebra P one may ask if it is the semiclassical limit of an almost commutative algebra. This is one form of the problem of quantization of Poisson algebras, the answer to which for general Poisson algebras is negative. We give a few concrete examples of Poisson algebras.

Example 3.4. A *Poisson manifold* is a manifold M whose algebra of smooth functions $A = C^\infty(M)$ is a Poisson algebra (we should also assume that the bracket $\{\ ,\ \}$ is continuous in the Fréchet topology of A). It is not difficult to see that all Poisson structures on A are of the form

$$\{f,\, g\} := \langle df \wedge dg,\ \pi\rangle,$$

where $\pi \in C^\infty(\bigwedge^2(TM))$ is a smooth 2-vector field on M. This bracket clearly satisfies the Leibniz rule in each variable and one checks that it satisfies the Jacobi identity if and only if $[\pi, \pi] = 0$, where the *Schouten bracket* is defined in local coordinates by

$$[\pi,\pi] = \sum_{l=1}^{n} \left(\pi_{lj} \frac{\partial \pi_{ik}}{\partial x_l} + \pi_{li} \frac{\partial \pi_{kj}}{\partial x_l} + \pi_{lk} \frac{\partial \pi_{ji}}{\partial x_1} \right) = 0.$$

The Poisson bracket in local coordinates is given by

$$\{f,\, g\} = \sum_{ij} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Symplectic manifolds are the simplest examples of Poisson manifolds. They correspond to non-degenerate Poisson structures. Given a symplectic form ω, the associated Poisson bracket is given by

$$\{f,\, g\} = \omega(X_f, X_g),$$

where the vector field X_f is the symplectic dual of df.

Let $C^\infty_{\mathrm{poly}}(T^*M)$ be the algebra of smooth functions on T^*M which are polynomial in the cotangent direction. It is a Poisson algebra under the natural symplectic structure of T^*M. This Poisson algebra is the semiclassical limit of the algebra of differential operators on M, as we will see in the next example.

Example 3.5. (Differential operators on commutative algebras). Let A be a commutative unital algebra. Let $\mathcal{D}^0(A)$ denote the set of *differential operators of order zero* on A, i.e. A-linear maps from $A \to A$, and for $n \geq 1$, let $\mathcal{D}^n(A)$ be the set of all operators D in $\mathrm{End}_{\mathbb{C}}(A)$ such that for any $a \in A$, $[D,\ a] \in \mathcal{D}^{n-1}(A)$. The set

$$\mathcal{D}(A) = \bigcup_{n \geq 0} \mathcal{D}^n(A)$$

is a subalgebra of $\mathrm{End}_{\mathbb{C}}(A)$, called the *algebra of differential operators* on A. It is an almost commutative algebra under the filtration given by $\mathcal{D}^n$, $n \geq 0$. Elements of $\mathcal{D}^n(A)$ are called differential operators of order n. For example, a linear map $D : A \to A$ is a differential operator of order one if and only if it is of the form $D = \delta + a$, where δ is a derivation on A and $a \in A$.

For general A, the semiclassical limit $\mathrm{Gr}\,(\mathcal{D}(A))$ and its Poisson structure are not easily identified except for coordinate rings of smooth affine varieties or algebras of smooth functions on a manifold. In this case a differential operator D of order n is locally given by

$$D = \sum_{|I| \leq k} a_I(x) \partial^I,$$

where $I = (i_1, \ldots, i_n)$ is a multi-index and $\partial^I = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_n}$ is a mixed partial derivative. This expression depends on the local coordinates but its leading terms of total degree n have an invariant meaning provided we replace $\partial_i \mapsto \xi_i \in T^*M$. For $\xi \in T_x^*M$, let

$$\sigma_p(D)(x, \xi) := \sum_{|I| = k} a_I(x) \xi^I.$$

Then the function $\sigma_p(D) : T^*M \to \mathbb{C}$, called the *principal symbol* of D, is invariantly defined and belongs to $C^\infty_{\mathrm{poly}}(T^*M)$. The algebra $C^\infty_{\mathrm{poly}}(T^*M)$ inherits a canonical Poisson structure as a subalgebra of the Poisson algebra $C^\infty(T^*M)$ and we have the following

Proposition 3.3. *The principal symbol map induces an isomorphism of Poisson algebras*

$$\sigma_p : Gr\, \mathcal{D}(C^\infty(M)) \overset{\simeq}{\to} C^\infty_{poly}(T^*M).$$

See [16] for a proof of this or, even better, try to prove it yourself by proving it for Weyl algebras first.

Example 3.6. (The Weyl algebra) Let $A_1 := \mathcal{D}\,\mathbb{C}[X]$ be the *Weyl algebra* of differential operators on the line. Alternatively, A_1 can be described as the unital complex algebra defined by generators x and p with

$$px - xp = 1.$$

The map $x \mapsto x$, $p \mapsto \frac{d}{dx}$ defines the isomorphism. Physicists prefer to write the defining relation as the *canonical commutation relation* $pq - qp = \frac{h}{2\pi i}1$, where h is Planck's constant and p and q represent momentum and position operators. This is not without merit because we can then let $h \to 0$ and obtain the commutative algebra of polynomials in p and q as the semiclassical limit. Also, i is necessary if we want to consider p and q as selfadjoint operators (why?).

Any element of A_1 has a unique expression as a differential operator with polynomial coefficients $\sum a_i(x)\frac{d^i}{dx^i}$ where the standard filtration is by degree of the differential operator. We have an algebra isomorphism $Gr(A_1) \simeq \mathbb{C}[x, y]$ under which the (principal) symbol map is given by

$$\sigma\left(\sum_{i=0}^{n} a_i(x)\frac{d^i}{dx^i}\right) = a_n(x)y^n.$$

The induced Poisson bracket on $\mathbb{C}[x, y]$ is the classical Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

In general, the Weyl algebra A_n is the algebra of differential operators on $\mathbb{C}[x_1, \ldots, x_n]$. Alternatively it can be defined as the universal algebra defined by $2n$ generators $x_1, \ldots x_n, p_1, \ldots, p_n$ with

$$[p_i, x_i] = \delta_{ij}, \quad \text{and} \quad [p_i, p_j] = [x_i, x_j] = 0$$

for all i, j. Notice that $A_n \simeq A_1 \otimes \cdots \otimes A_1$ (n factors).

A lot is known about Weyl algebras and a lot remains to be known, including the Dixmier conjecture about the automorphisms of A_n and its relation with the Jacobian conjecture recently studied by Kontsevich in [5]. The Hochschild and cyclic cohomology of A_n is computed in [60].

Exercise 3.4. *Show that A_1 is a simple algebra, i.e. it has no non-trivial two-sided ideals; that any derivation of A_1 is inner; and that $[A_1, A_1] = A_1$, i.e. there are no non-trivial traces on A_1. Prove all of this for A_n. Any derivation of A_1 is inner; is it true that any automorphism of A_1 is inner?*

Exercise 3.5. *Let $A = \mathbb{C}[x]/(x^2)$ be the algebra of dual numbers. Describe its algebra of differential operators.*

Example 3.7. (Universal enveloping algebras) Let $U(\mathfrak{g})$ denote the *enveloping algebra* of a Lie algebra $\mathfrak{g}$. By definition, $U(\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by $x \otimes y - y \otimes x - [x, y]$

for all $x, y \in \mathfrak{g}$. For $p \geq 0$, let $F^p(U(\mathfrak{g}))$ be the subspace generated by tensors of degree at most p. This turns $U(\mathfrak{g})$ into a filtered algebra and the Poincaré–Birkhoff–Witt theorem asserts that its associated graded algebra is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$. The algebra isomorphism is induced by the *symmetrization map* $s : S(\mathfrak{g}) \to \mathrm{Gr}\,(U(\mathfrak{g}))$, defined by

$$s(X_1 X_2 \cdots X_p) = \frac{1}{p!} \sum_{\sigma \in S_p} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(p)}.$$

Note that $S(\mathfrak{g})$ is the algebra of polynomial functions on the dual space $\mathfrak{g}^*$ which is a Poisson manifold under the bracket

$$\{f\,,\, g\}(X) = [Df(X), \;\; Dg(X)]$$

for all $f, g \in C^\infty(\mathfrak{g}^*)$ and $X \in \mathfrak{g}^*$. Here we have used the canonical isomorphism $\mathfrak{g} \simeq \mathfrak{g}^{**}$, to regard the differential $Df(X) \in \mathfrak{g}^{**}$ as an element of $\mathfrak{g}$. The induced Poisson structure on polynomial functions coincides with the Poisson structure in $\mathrm{Gr}\,(U(\mathfrak{g}))$.

Example 3.8. (Algebra of formal pseudodifferential operators on the circle) This algebra is obtained by formally inverting the differentiation operator $\partial := \frac{d}{dx}$ and then completing the resulting algebra. A formal pseudodifferential operator on the circle is an expression of the form $\sum_{-\infty}^{n} a_i(x)\partial^i$, where each $a_i(x)$ is a Laurent polynomial. The multiplication in uniquely defined by the rules $\partial x - x\partial = 1$ and $\partial\partial^{-1} = \partial^{-1}\partial = 1$. We denote the resulting algebra by Ψ_1. The *Adler–Manin trace* on Ψ_1 [95], also called the *noncommutative residue*, is defined by

$$\mathrm{Tr}\left(\sum_{-\infty}^{n} a_i(x)\partial^i\right) = \mathrm{Res}\,(a_{-1}(x);\;0) = \frac{1}{2\pi i}\int_{S^1} a_{-1}(x).$$

This is a trace on Ψ_1. In fact one can show that $\Psi_1/[\Psi_1,\;\Psi_1]$ is one-dimensional which means that any trace on Ψ_1 is a multiple of Tr. Notice that for the Weyl algebra A_1 we have $[A_1,\, A_1] = A_1$.

Another interesting difference between Ψ_1 and A_1 is that Ψ_1 admits non-inner derivations (see exercise below). The algebra Ψ_1 has a nice generalization to algebras of pseudodifferential operators in higher dimensions. The appropriate extension of the above trace is the *noncommutative residue* of Guillemin and Wodzicki (cf. [121]. See also [24] for relations with the Dixmier trace and its place in noncommutative Riemannian geometry).

Exercise 3.6. *Unlike the algebra of differential operators,* Ψ_1 *admits non-inner derivations. Clearly* $\log \partial \notin \Psi_1$, *but show that for any* $a \in \Psi_1$, *we have* $[\log \partial,\ a] \in \Psi_1$ *and therefore the map*

$$a \mapsto \delta(a) := [\log \partial,\ a]$$

defines a non-inner derivation of Ψ_1 [87]. *The corresponding Lie algebra 2-cocycle*

$$\varphi(a, b) = \mathrm{Tr}(a[\log \partial,\ b])$$

is the Radul cocycle.

3.3. *Deformation theory*

In this section we shall freely use results about Hochschild cohomology from Section 6.5. In the last section we saw one way to formalize the idea of quantization through the notion of an almost commutative algebra and its semiclassical limit which is a Poisson algebra. A closely related notion is *formal deformation quantization*, or *star products* [4, 84]. Let $(A, \{\ ,\ \})$ be a Poisson algebra and let $A[[h]]$ be the algebra of formal power series over A. A formal deformation of A is an associative $\mathbb{C}[[h]]$-linear multiplication

$$*_h : A[[h]] \otimes A[[h]] \to A[[h]]$$

such that $*_0$ is the original multiplication and for all a, b in A,

$$\frac{a *_h b - b *_h a}{h} \to \{a, b\}$$

as $h \to 0$. Writing

$$a *_h b = B_0(a, b) + hB_1(a, b) + h^2 B_2(a, b) + \cdots ,$$

where $B_i : A \otimes A \to A$ are Hochschild 2-cochains on A with values in A, we see that the initial conditions on $*_h$ are equivalent to

$$B_0(a, b) = ab, \quad \text{and} \quad B_1(a, b) - B_1(b, a) = \{a,\ b\}.$$

The associativity condition on $*_h$ is equivalent to an infinite system of equations involving the cochains B_i. They are given by

$$B_0 \circ B_n + B_1 \circ B_{n-1} + \cdots + B_n \circ B_0 = 0, \quad \text{for all}\, n \geq 0,$$

or equivalently

$$\sum_{i=1}^{n-1} B_i \circ B_{n-i} = \delta B_n. \tag{17}$$

Here, the *Gerstenhaber* $\circ$ *product* of 2-cochains $f,\, g : A \otimes A \to A$ is defined as the 3-cochain

$$f \circ g(a, b, c) = f(g(a, b), c) - f(a, g(b, c)).$$

Notice that a 2-cochain f defines an associative product if and only if $f \circ f = 0$. Also notice that the Hochschild coboundary δf can be written as $\delta f = -m \circ f - f \circ m$, where $m : A \otimes A \to A$ is the multiplication of A. These observations lead to the associativity equations (17).

To solve these equations starting with $B_0 = m$, by antisymmetrizing we can always assume that B_1 is antisymmetric and hence we can assume $B_1 = \frac{1}{2}\{\, ,\, \}$. Having found $B_0, B_1, \ldots B_{n-1}$, we can find a B_n satisfying (17) if and only if the cocycle $\sum_{i=1}^{n-1} B_i \circ B_{n-i}$ is a coboundary, i.e. its class in $H^3(A, A)$ should vanish. The upshot is that the third Hochschild cohomology $H^3(A, A)$ is the *space of obstructions* for the deformation quantization problem. In particular if $H^3(A, A) = 0$ then any Poisson bracket on A can be deformed. Notice, however, that this is only a sufficient condition and by no means is necessary, as will be shown below.

In the most interesting examples, e.g. for $A = C^\infty(M)$, $H^3(A, A) \neq 0$. To see this consider the differential graded Lie algebra $(C(A,\, A),\, [\, ,\,],\, \delta)$ of continuous Hochschild cochains on A, and the differential graded Lie algebra, with zero differential, $(\bigwedge(TM),\, [\, ,\,],\, 0)$ of polyvector fields on M. The bracket in the first is the Gerstenhaber bracket and in the second is the Schouten bracket of polyvector fields. By a theorem of Connes (see the resolution in Lemma 44 in [22]), we know that the *antisymmetrization map*

$$\alpha : (C^\infty(\bigwedge TM),\, 0) \to (C(A,\, A),\, \delta)$$

sending a polyvector field $X_1 \wedge \cdots \wedge X_k$ to the functional φ defined by

$$\varphi(f^1, \ldots, f^k) = df^1(X_1) df^2(X_2) \cdots df^k(X_k)$$

is a quasi-isomorphism of differential graded algebras. In particular, it induces an isomorphism of graded commutative algebras

$$H^k(A, A) \simeq C^\infty(\bigwedge^k TM).$$

The map α, however, is not a morphism of Lie algebras and that is where the real difficulty of deforming a Poisson structure is hidden. The *formality theorem* of M. Kontsevich [84] states that as a differential graded Lie algebra, $(C(A, A), \delta, [\, ,\,])$ is formal in the sense that it is quasi-isomorphic to its cohomology. Equivalently, it means that one can correct the map α, by adding an infinite number of terms, to a morphism of L_∞ algebras. This shows that the original deformation problem of Poisson structures can be transferred to $(C^\infty(\bigwedge TM), 0)$ where it is unobstructed since the differential in the latter DGL is zero. We give a couple of simple examples where deformations can be explicitly constructed.

Example 3.9. The simplest non-trivial Poisson manifold is the dual $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$. Let $U_h(\mathfrak{g}) = T(\mathfrak{g})/I$, where the ideal I is generated by

$$x \otimes y - y \otimes x - h[x, y], \; x, y \in \mathfrak{g}.$$

By the Poincaré–Birkhoff–Witt theorem, the antisymmetrization map $\alpha_h : S(\mathfrak{g}) \to U_h(\mathfrak{g})$ is a linear isomorphism. We can define a $*$-product on $S(\mathfrak{g})$ by

$$f *_h g = \alpha_h^{-1}(\alpha_h(f)\alpha_h(g)) = \sum_{n=0}^{\infty} h^n B_n(f, g).$$

With some work one can show that B_n are bidifferential operators and hence the formula extends to a $*$-product on $C^\infty(\mathfrak{g}^*)$.

Example 3.10. (Moyal–Weyl quantization) Consider the algebra generated by x and y with relation $xy - yx = \frac{h}{i}1$. Iterated application of the Leibniz rule gives the formula for the product

$$f *_h g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-ih}{2}\right)^n B_n(f, g),$$

where $B_0(f, g) = fg$, $B_1(f, g) = \{f, \; g\}$ is the standard Poison bracket, and for $n \geq 2$,

$$B_n(f, \; g) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\partial_x^k \partial_y^{n-k} f)(\partial_x^{n-k} \partial_y^k g).$$

Notice that this formula makes sense for $f, g \in C^\infty(\mathbb{R}^2)$ and defines a deformation of this algebra with its standard Poisson structure. This can be extended to arbitrary constant Poisson structures

$$\{f, \; g\} = \sum \pi^{ij} \partial_i f \, \partial_j g.$$

The Weyl–Moyal $*$-product is then given by

$$f *_h g = \exp\left(-i\frac{h}{2} \sum \pi^{ij} \partial_i \wedge \partial_j\right) (f, \; g).$$

Remark 2. As we mentioned before, deformation quantization has its origins in Dirac's quantization rule in quantum mechanics [50]. The original idea was to replace the classical observables (functions) by quantum observables (operators) in such a way that Poisson brackets of functions correspond to commutators of operators. It was, however, soon realized that a rigid interpretation of Dirac's rule is impossible and it must be understood only in an asymptotic sense. In fact there are some well known 'no go-theorems' that state that under reasonable non-degeneracy assumptions this rigid notion of quantization is not possible (cf. [66] and references therein). One approach, as in [4], formulates quantization of a classical system as formal deformation quantization of a Poisson structure manifold. We should mention that the algebraic underpinnings of deformation theory of (associative and Lie) algebras and the relevance of Hochschild cohomology goes back to Gerstenhaber's papers [63].

No discussion of deformation quantizations is complete without discussing Rieffel's deformation quantization [112]. Roughly speaking, one demands that formal power series of formal deformation theory should actually be convergent. More precisely, let $(M, \{\ ,\ \})$ be a Poisson manifold. A *strict deformation quantization* of the Poisson algebra $\mathcal{A} = C^\infty(M)$ is a family of pre C^*-algebra structures $(*_h,\ \|\ \|_h)$ on $\mathcal{A}$ for $h \geq 0$ such that the family forms a continuous field of C^*-algebras on $[0,\ \infty)$ (in particular for any $f \in \mathcal{A}$, $\ h \mapsto \|f\|_h$ is continuous) and for all $f, g \in \mathcal{A}$,

$$\left\| \frac{f *_h g - g *_h f}{ih} \right\|_h \to \{f,\ g\}$$

as $h \to 0$. We therefore have a family of C^*-algebras A_h obtained by completing $\mathcal{A}$ with respect to the norm $\|\ \|_h$.

Example 3.11. (Noncommutative tori) In [110] it is shown that the family of noncommutative tori A_θ form a strict deformation quantization of the Poisson algebra of smooth functions on the 2-torus. This in fact appears as a special case of a more general result. Let α be a smooth action of $\mathbb{R}^n$ on $\mathcal{A} = C^\infty(M)$. Let X_i denote the infinitesimal generators for this action. Each skew-symmetric $n \times n$ matrix J defines a Poisson bracket on $\mathcal{A}$ by

$$\{f,\ g\} = \sum J_{ij} X_i(f) X_j(g).$$

For each $h \in \mathbb{R}$, define a new product $*_h$ on $\mathcal{A}$ by

$$f *_h g = \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{hJu}(f)\alpha_v(g) e^{2\pi i u\cdot v} du dv.$$

The $*$ structure is by conjugation and is undeformed (see [110] for the definition of $\|f\|_h$). For $\mathcal{A} = C^\infty(\mathbb{T}^2)$ with the natural $\mathbb{R}^2$ action one obtains A_θ.

Remark 3. Does any Poisson manifold admit a strict deformation quantization? This question is still open (even for symplectic manifolds). In [111], Rieffel shows that the canonical symplectic structure on the 2-sphere admits no $SO(3)$-invariant strict deformation quantization. An intriguing idea promoted by Connes and Marcolli in [34] is that existence of a strict deformation quantization of a Poisson manifold should be regarded as an *integrability condition* for the formal deformation quantization. There is a clear analogy with the case of formal and convergent power series solutions of differential equations around singular points. They ask for a possible 'theory of ambiguity', i.e. a cohomology theory that could capture the difference between the two cases.

4. Sources of Noncommutative Spaces

At present we can identify at least four methods by which noncommutative spaces are constructed:

(i) noncommutative quotients;
(ii) algebraic and C^*-algebraic deformations;
(iii) Hopf algebras and quantum groups;
(iv) cohomological constructions.

It should be stressed that these are by no means mutually exclusive; there are intimate relations between these sources and sometimes a noncommutative space can be described by several methods, as is the case with noncommutative tori. The majority of examples, by far, fall into the first category. We won't discuss the last idea, advanced by Connes and Landi [32] and Connes and Dubois-Violette [26]. Very briefly the idea is that if one writes the conditions for the Chern character of an idempotent in cyclic homology to be trivial on the level of chains, then one obtains interesting examples of algebras such as noncommutative spheres and spherical manifolds, Grassmannians, and Yang–Mills algebras.

4.1. *Noncommutative quotients*

The quotient space $X/\sim$ of a Hausdorff space may easily fail to be Hausdorff and may even be an indiscrete topological space with only two open sets. This happens, for example, when at least one equivalence class is dense in X. Similarly the quotient of a smooth manifold may become singular and fail to be smooth.

The method of *noncommutative quotients* as advanced by Connes in [24] allows one to replace "bad quotients" by nice noncommutative spaces, represented by noncommutative algebras. In general these noncommutative algebras are defined as *groupoid algebras.* In some cases, like quotients by group actions, the noncommutative quotient can be defined as a crossed product algebra too, but in general the use of *groupoids* seem to be unavoidable.

An equivalence relation is usually obtained from a much richer structure by forgetting part of this structure. For example, it may arise from an action of a group G on X where $x \sim y$ if and only if $gx = y$ for some g in G (*orbit equivalence*). Note that there may be, in general, many g with this property. That is x may be identifiable with y in more than one way. Of course when we form the equivalence relation this extra information is lost. The key idea in dealing with bad quotients in Connes' theory is to keep track of this extra information, by first forming a groupoid.

Now Connes' dictum in forming noncommutative quotients can be summarized as follows:

$$\textbf{quotient data} \rightsquigarrow \textbf{groupoid} \rightsquigarrow \textbf{groupoid algebra},$$

where the noncommutative quotient is defined to be the groupoid algebra itself.

Definition 4.1. *A groupoid is a small category in which every morphism is an isomorphism.*

The set of objects of a groupoid $\mathcal{G}$ shall be denoted by $\mathcal{G}^{(0)}$. Every morphism has a *source*, *target* and an *inverse*. They define maps

$$s : \mathcal{G} \longrightarrow \mathcal{G}^{(0)}, \quad t : \mathcal{G} \longrightarrow \mathcal{G}^{(0)}, \quad i : \mathcal{G} \longrightarrow \mathcal{G}.$$

Composition $\gamma_1 \circ \gamma_2$ of morphisms γ_1 and γ_2 is only defined if $s(\gamma_1) = t(\gamma_2)$. Composition defines a map

$$\circ : \mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2);\ s(\gamma_1) = t(\gamma_2)\} \longrightarrow \mathcal{G},$$

which is associative in an obvious sense.

Examples 4.1. (i) Every group G defines a groupoid $\mathcal{G}$ with one object $*$ and

$$\mathrm{Hom}_{\mathcal{G}}(*, *) = G.$$

The composition of morphisms is simply by group multiplication.
(ii) Let X be a G-set. We define a groupoid $\mathcal{G} = X \rtimes G$, called the *transformation groupoid* of the action, as follows. Let obj $\mathcal{G} = X$, and

$$\mathrm{Hom}_{\mathcal{G}}\,(x, y) = \{g \in G;\ gx = y\}.$$

Composition of morphisms is defined via group multiplication. It is easily checked that $\mathcal{G}$ is a groupoid. Its set of morphisms can be identified as

$$\mathcal{G} \simeq X \times G,$$

where the composition of morphisms is given by

$$(gx, h) \circ (x, g) = (x, hg).$$

(iii) Let $\sim$ denote an equivalence relation on a set X. We define a groupoid $\mathcal{G}$, called the *graph of* $\sim$, as follows. Let obj $\mathcal{G} = X$, and let $\mathrm{Hom}_{\mathcal{G}}\,(x, y)$ be a one element set if $x \sim y$, and be the empty set otherwise.

Note that the set of morphisms of $\mathcal{G}$ is identified with the graph of the relation $\sim$ in the usual sense:

$$\mathcal{G} = \{(x, y);\ x \sim y\} \subset X \times X.$$

The *groupoid algebra* of a groupoid $\mathcal{G}$ is the algebra of functions

$$\mathbb{C}\mathcal{G} = \{f : \mathcal{G} \to \mathbb{C};\ f \text{ has finite support}\},$$

with finite support on $\mathcal{G}$. Under the *convolution product*

$$(fg)(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2),$$

and the involution

$$f^*(\gamma) = \overline{f(\gamma^{-1})}$$

it is a $*$-algebra.

Examples 4.2. (i) Let $\mathcal{G}$ be the groupoid defined by a group G. Then clearly the groupoid algebra $\mathbb{C}\mathcal{G}$ is isomorphic to the group algebra $\mathbb{C}G$.
(ii) If $\mathcal{G} = X \rtimes G$ is a transformation groupoid then we have an algebra isomorphism

$$\mathbb{C}\mathcal{G} \simeq C(X) \rtimes G.$$

(iii) Let $\mathcal{G}$ be the *groupoid of pairs* on a set of n elements, i.e.

$$\mathcal{G} = \{(i, j);\ i, j = 1, \ldots, n\}$$

with composition given by

$$(l,k)\circ(j,i)=(l,i)\quad \text{if } k=j.$$

(Composition is not defined otherwise.) We have

$$\mathbb{C}\mathcal{G}\simeq M_n(\mathbb{C}).$$

Indeed, it is easily checked that the map

$$(i,j)\mapsto E_{i,j},$$

where $E_{i,j}$ denote the matrix units, is an algebra isomorphism. This is an extremely important example. In fact, as Connes points out in [24], the matrices in Heisenberg's matrix quantum mechanics [69] were arrived at by a similar procedure.

Exercise 4.1. *Show that the groupoid algebra of a finite groupoid (finite set of objects and finite set of morphisms) can be decomposed as a direct sum of tensor products of group algebras and matrix algebras.*

As the above exercise shows, one cannot get very far with just discrete groupoids and soon one needs to work with topological and smooth groupoids associated to, say, continuous actions of topological groups and to foliations.

A (locally compact) *topological groupoid* is a groupoid such that its set of morphisms $\mathcal{G}$ and set of objects $\mathcal{G}^{(0)}$ are (locally compact) topological spaces, and its composition, source, target and inverse maps are continuous. An *étale groupoid* is a locally compact groupoid such that the fibers of its target map $\mathcal{G}^x=t^{-1}(x)$, $x\in\mathcal{G}^{(0)}$, are discrete.

A *smooth groupoid*, also known as *Lie groupoid*, is a groupoid such that $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are smooth manifolds, the inclusion $\mathcal{G}^{(0)}\to\mathcal{G}$ as well as the maps s,t,i and the composition map $\circ$ are smooth, and s and t are submersions. This last condition will guarantee that the domain of the composition map $\mathcal{G}^{(2)}=\{(\gamma_1,\gamma_2);s(\gamma_1)=t(\gamma_2)\}$ is a smooth manifold.

To define the convolution algebra of a topological groupoid and its C^*-completions, we need an analogue of Haar measure for groupoids. A *Haar measure* on a locally compact groupoid $\mathcal{G}$ is a family of measures μ^x on each t-fiber $\mathcal{G}^x$. The family is supposed to be continuous and left invariant in an obvious sense (cf. [108], unlike locally compact topological groups, groupoids need not have an invariant Haar measure). Given a Haar measure, we can then define, for functions with compact support $f,g\in C_c(\mathcal{G})$

their convolution product

$$(f * g)(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) = \int_{\mathcal{G}^{t(\gamma)}} f(\gamma_1) g(\gamma_1^{-1}\gamma) d\mu^{t(\gamma)}. \tag{18}$$

This turns $C_c(\mathcal{G})$ into a $*$-algebra. The involution is defined by $f^*(\gamma) = \overline{f(\gamma^{-1})})$. For each fiber $\mathcal{G}^x$, we have an $*$-representation π_x of $C_c(\mathcal{G})$ on the Hilbert space $L^2(\mathcal{G}^x, \mu^x)$ defined by

$$(\pi_x f)(\xi)(\gamma) = \int_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) \xi(\gamma_2) = \int_{\mathcal{G}^{t(\gamma)}} f(\gamma_1) \xi(\gamma_1^{-1}\gamma) d\mu^{t(\gamma)}.$$

We can then define a pre C^*-norm on $C_c(\mathcal{G})$ by

$$\|f\| := \sup\{\|\pi_x(f)\|;\, x \in \mathcal{G}^0\}.$$

The completion of $C_c(\mathcal{G})$ under this norm is the *reduced C^*-algebra* of the groupoid $\mathcal{G}$ and will be denoted by $C_r^*(\mathcal{G})$.

There are two special cases that are particularly important and convenient to work with: étale and smooth groupoids. Notice that for an étale groupoid each fiber is a discrete set and with counting measure on each fiber we obtain a Haar measure. The convolution product is then given by

$$(f * g)(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) = \sum_{\mathcal{G}^{t(\gamma)}} f(\gamma_1) g(\gamma_1^{-1}\gamma).$$

Notice that for each γ this is a finite sum since the support of f is compact and hence contains only finitely many points of each fiber.

A second interesting case where one can do away with Haar measures is smooth groupoids. Let $C_c^\infty(\mathcal{G}, \Omega^{\frac{1}{2}})$ be the space of sections, with compact support, of the line bundle of half-densities on a smooth groupoid $\mathcal{G}$. Since the product of two half-densities is a 1-density which has a well defined integral, the integral (18) for $f, g \in C_c^\infty(\mathcal{G}, \Omega^{\frac{1}{2}})$ is well defined and we obtain the smooth convolution algebra $C_c^\infty(\mathcal{G})$.

Examples 4.3. 1. Let Γ be a discrete group acting by homeomorphisms on a locally compact space X. Then the transformation groupoid $X \rtimes \Gamma$ is an étale groupoid and the groupoid algebra recovers the crossed product algebra:

$$C_c(\mathcal{G}) \simeq C_c(X) \rtimes \Gamma \quad \text{and} \quad C_r^*(\mathcal{G}) \simeq C_0(X) \rtimes_r \Gamma.$$

For $X = S^1$ and $\Gamma = \mathbb{Z}$ acting through rotation by an angle θ, we recover the noncommutative torus as a groupoid algebra, which is one among many incarnations of A_θ.

2. Let X be a locally compact space with a Borel probability measure μ and $\mathcal{G}$ be the groupoid of pairs on X. Then for $f, g \in C_c(X \times X)$ the convolution product (18) reduces to

$$(f * g)(x, z) = \int_X f(x, y)g(y, z)d\mu(y),$$

which is reminiscent of matrix multiplication or products of integral operators. In fact the map $T : C_c(X \times X) \to \mathcal{K}(L^2(X, \mu))$ sending f to the integral operator

$$(Tf)(g)(x) = \int_X f(x, y)g(y)d\mu(y)$$

is clearly an algebra map and can be shown to be 1-1 and onto.

On the other extreme, if $\mathcal{G}$ is the groupoid of the discrete equivalence relation on a locally compact space X, also known as the *groupoid of pairs*, then clearly $C_c(\mathcal{G}) \simeq C_c(X)$.

Example 4.1. (Non-Hausdorff manifolds) Let

$$X = S^1 \times 0 \cup S^1 \times 1$$

be the disjoint union of two copies of the circle. We identify $(x, 0) \sim (x, 1)$ for all $x \neq 1$ in S^1. The quotient space $X/\sim$ is a non-Hausdorff manifold. The groupoid of the equivalence relation $\sim$

$$\mathcal{G} = \{(x, y) \subset X \times X;\, x \sim y\}$$

is a smooth étale groupoid. Its smooth groupoid algebra is given by

$$C^\infty(\mathcal{G}) = \{f \in C^\infty(S^1,\, M_2(\mathbb{C}));\; f(1) \text{ is diagonal}\}.$$

There are many interesting examples of noncommutative quotients that we did not discuss here but are of much interest in noncommutative geometry. They include: foliation algebras, the space of Penrose tilings, the adèle space and the space of $\mathbb{Q}$-lattices in number theory. They can all be defined as groupoid algebras and variations thereof. We refer to Connes–Marcolli's article in this volume as well as [33, 34, 36, 24] for a proper introduction.

Exercise 4.2. *Show that the Hecke algebras $\mathcal{H}(\Gamma, \Gamma_0)$ defined in Example* 2.7 *are groupoid algebras.*

The following result of M. Rieffel [109] clarifies the relation between the classical quotients and noncommutative quotients for group actions:

Theorem 4.1. *Assume G acts freely and properly on a locally compact Hausdorff space X. Then we have a strong Morita equivalence between the C^*-algebras $C_0(X/G)$ and $C_0(X) \rtimes_r G$.*

4.2. *Hopf algebras and quantum groups*

Many examples of noncommutative spaces are Hopf algebras or quantum groups. They are either the algebra of coordinates of a quantum group, or, dually, the convolution algebra or the enveloping algebra of a quantum group. In this section we shall make no distinction between Hopf algebras and quantum groups and use these words interchangeably. The theory of Hopf algebras (as well as Hopf spaces) was born in the paper of H. Hopf in his celebrated computation of the rational cohomology of compact connected Lie groups [73]. This line of investigation eventually led to the Cartier–Milnor–Moore theorem [99] characterizing connected cocommutative Hopf algebras as enveloping algebras of Lie algebras.

A purely algebraic theory, with motivations independent from algebraic topology, was created by Sweedler in the 1960's. This line of investigation took a big leap forward with the work of Drinfeld and Jimbo resulting in quantizing all classical Lie groups and Lie algebras [54].

In a different direction, immediately after Pontryagin's duality theorem for locally compact abelian groups, attempts were made to extend it to noncommutative groups. The Tannaka-Krein duality theorem was an important first step. Note that the dual, in any sense of the word, of a noncommutative group is necessarily not a group and one is naturally interested in extending the category of groups to a larger category which is closed under duality and hopefully is even equivalent to its second dual. Hopf von Neumann algebras of G.I. Kac and Vainerman achieve this in the measure theoretic world of von Neumann algebras [55]. The theory of *locally compact quantum groups* of Kustermans and Vaes [88] which was developed much later achieves this goal in the category of C^*-algebras. An important step towards this program was the theory of compact quantum groups of S. L. Woronowicz (cf. [122] for a survey). We refer to [77, 83, 94, 96, 97, 118] for the general theory of Hopf algebras and quantum groups.

The first serious interaction between Hopf algebras and noncommutative geometry started in earnest in the paper of Connes and Moscovici on transverse index theory [38] (cf. also [39, 40, 41] for further developments). In this paper a noncommutative and non-cocommutative Hopf algebra appears as the quantum symmetries of the noncommutative space of codimension one foliations. The same Hopf algebra was later shown to act on the noncommutative space of modular Hecke algebras [42].

To understand the definition of a Hopf algebra, let us see what kind of extra structure exists on the algebra of functions on a group. For simplicity, let G be a finite group, though this is by no means necessary, and let $H = C(G)$ be the algebra of functions from $G \to \mathbb{C}$. The multiplication $m : G \times G \to G$, inversion $i : G \to G$, and unit element $e \in G$, once dualized, define unital algebra maps

$$\Delta : H \to H \otimes H, \quad S : H \to H, \quad \epsilon : H \to \mathbb{C},$$

by the formulas

$$\Delta f(x, y) = f(xy), \quad Sf(x) = f(x^{-1}), \quad \varepsilon(f) = f(e),$$

where we have identified $C(G \times G)$ with $C(G) \otimes C(G)$. Let also

$$m : H \otimes H \to H, \qquad \eta : \mathbb{C} \to H$$

denote the multiplication and unit maps of the algebra H. The associativity, inverse, and unit axioms for groups are dualized and in fact are easily seen to be equivalent to the following *coassociativity, antipode, and counit axioms* for H:

$$\begin{aligned}(\Delta \otimes I)\Delta &= (I \otimes \Delta)\Delta,\\ (\varepsilon \otimes I)\Delta &= (I \otimes \varepsilon)\Delta = I,\\ m(S \otimes I) &= m(I \otimes S) = \eta\varepsilon.\end{aligned}$$

Definition 4.2. *A unital algebra* (H, m, η) *endowed with unital algebra homomorphisms* $\Delta : H \to H \otimes H$, $\varepsilon : H \to \mathbb{C}$ *and a linear map* $S : H \to H$ *satisfying the above equations is called a Hopf algebra.*

It can be shown that the antipode S is unique and is an anti-algebra map. It is also an anti-coalgebra map.

Example 4.2. Commutative or cocommutative Hopf algebras are closely related to groups and Lie algebras. We give a few examples to indicate this connection.

(i) Let Γ be a discrete group (it need not be finite) and $H = \mathbb{C}\Gamma$ the group algebra of Γ. Let

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \varepsilon(g) = 1,$$

for all $g \in \Gamma$ and linearly extend them to H. Then it is easy to check that $(H, \Delta, \varepsilon, S)$ is a cocommutative Hopf algebra. It is commutative if and only if Γ is commutative.

(ii) Let $\mathfrak{g}$ be a Lie algebra and $H = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Using the universal property of $U(\mathfrak{g})$ one checks that there are uniquely defined algebra homomorphisms $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $\varepsilon : U(\mathfrak{g}) \to \mathbb{C}$ and an anti-algebra map $S : U(\mathfrak{g}) \to U(\mathfrak{g})$, determined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad \text{and} \quad S(X) = -X,$$

for all $X \in \mathfrak{g}$. One then checks easily that $(U(\mathfrak{g}), \Delta, \varepsilon, S)$ is a cocommutative Hopf algebra. It is commutative if and only if $\mathfrak{g}$ is an abelian Lie algebra, in which case $U(\mathfrak{g}) = S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$.

(iii) (Compact groups) Let G be a compact topological group. A continuous function $f : G \to \mathbb{C}$ is called *representable* if the set of left translations of f by all elements of G forms a finite dimensional subspace of $C(G)$. It is easy to see that the set of representable functions, $H = \mathrm{Rep}\,(G)$, is a subalgebra of $C(G)$. Let $m : G \times G \to G$ denote the multiplication of G and $m^* : C(G \times G) \to C(G)$, $m^* f(x, y) = f(xy)$, denote its dual map. One checks that if f is representable, then

$$m^* f \in \mathrm{Rep}\,(G) \otimes \mathrm{Rep}\,(G) \subset C(G \times G).$$

Let e denote the identity of G. The formulas

$$\Delta f = m^* f, \quad \varepsilon f = f(e) \quad \text{and} \quad (Sf)(g) = f(g^{-1}),$$

define a Hopf algebra structure on *Rep*(G). Alternatively, one can describe $\mathrm{Rep}\,(G)$ as the linear span of *matrix coefficients* of all finite dimensional complex representations of G. By the *Peter-Weyl Theorem,* $\mathrm{Rep}(G)$ is a dense subalgebra of $C(G)$. This algebra is finitely generated (as an algebra) if and only if G is a Lie group.

(iv) (Affine group schemes) The coordinate ring $H = \mathbb{C}[G]$ of an affine algebraic group G is a commutative Hopf algebra. The maps Δ, ε, and S are the duals of the multiplication, unit, and inversion maps of G, respectively. More generally, an *affine group scheme* over a commutative ring k is a commutative Hopf algebra over k. Given such a Hopf algebra H, it is easy to see that for any commutative k-algebra A, the set $\mathrm{Hom}_{Alg}\,(H, A)$ is a group under convolution product and $A \mapsto \mathrm{Hom}_{Alg}\,(H, A)$ is a functor from the category $ComAlg_k$ of commutative k-algebras to the category of groups. Conversely, any *representable functor* $ComAlg_k \to Groups$ is represented by a, unique up to isomorphism, commutative k-Hopf algebra. Thus the category of affine group schemes is equivalent to the category of representable functors $ComAlg_k \to Groups$.

Example 4.3. (Compact quantum groups) A prototypical example is Woronowicz's $SU_q(2)$, for $0 < q \leq 1$. As a C^*-algebra it is the unital C^*-algebra generated by α and β subject to the relations

$$\beta\beta^* = \beta^*\beta, \;\; \alpha\beta = q\beta\alpha, \;\; \alpha\beta^* = q\beta^*\alpha, \;\; \alpha\alpha^* + q^2\beta^*\beta = \alpha^*\alpha + \beta^*\beta = I.$$

Notice that these relations amount to saying that

$$U = \begin{pmatrix} \alpha & q\beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

is unitary, i.e. $UU^* = U^*U = I$. Its coproduct and antipode are defined by

$$\Delta \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},$$

$$S(\alpha) = \alpha^*, \;\; S(\beta) = -q^{-1}\beta^*, \;\; S(\beta^*) = -q\beta, \;\; S(\alpha^*) = \alpha.$$

Notice that the coproduct is only defined on the algebra $\mathcal{O}(SU_q(2))$ of matrix elements on the quantum group, and its extension to $C(SU_q(2))$ lands in the completed tensor product

$$\Delta : C(SU_q(2)) \longrightarrow C(SU_q(2))\hat{\otimes}C(SU_q(2)).$$

At $q = 1$ we obtain the algebra of continuous functions on $SU(2)$. We refer to [88] for a survey of compact and locally compact quantum groups.

Example 4.4. (Connes–Moscovici Hopf algebras) A very important example for noncommutative geometry and its applications to transverse geometry and number theory is the family of *Connes-Moscovici Hopf algebras* $\mathcal{H}_n$ for $n \geq 0$ [39, 40, 41]. They are deformations of the group $G = \text{Diff}\,(\mathbb{R}^n)$ of diffeomorphisms of $\mathbb{R}^n$ and can also be thought of as deformations of the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$. These algebras appeared for the first time as quantum symmetries of transverse frame bundles of codimension n foliations. We briefly treat the case $n = 1$ here. The main features of $\mathcal{H}_1$ stem from the fact that the group $G = \text{Diff}\,(\mathbb{R}^n)$ has a *factorization* of the form

$$G = G_1G_2,$$

where G_1 is the subgroup of diffeomorphisms φ that satisfy

$$\varphi(0) = 0, \quad \varphi'(0) = 1,$$

and G_2 is the $ax + b$-group of affine diffeomorphisms. We introduce two Hopf algebras corresponding to G_1 and G_2 respectively. Let F denote the Hopf algebra of polynomial functions on the pro-unipotent group G_1. It can also be defined as the *continuous dual* of the enveloping algebra of the Lie algebra of G_1. It is a commutative Hopf algebra generated by the Connes-Moscovici coordinate functions δ_n, $n = 1, 2, \ldots$, defined by

$$\delta_n(\varphi) = \frac{d^n}{dt^n}(\log(\varphi'(t)))|_{t=0}.$$

The second Hopf algebra, U, is the universal enveloping algebra of the Lie algebra $\mathfrak{g}_2$ of the $ax+b$-group. It has generators X and Y and one relation $[X, Y] = X$.

The factorization $G = G_1G_2$ defines a *matched pair* of Hopf algebras consisting of F and U. More precisely, The Hopf algebra F has a right U-module algebra structure defined by

$$\delta_n(X) = -\delta_{n+1} \quad \text{and} \quad \delta_n(Y) = -n\delta_n.$$

The Hopf algebra U, on the other hand, has a left F-comodule coalgebra structure via

$$X \mapsto 1 \otimes X + \delta_1 \otimes X \quad \text{and} \quad Y \mapsto 1 \otimes Y.$$

One can check that they are a matched pair of Hopf algebras in the sense of G.I. Kac and Majid [94] and the resulting bicrossed product Hopf algebra

$$\mathcal{H}_1 = F \bowtie U$$

is the Connes–Moscovici Hopf algebra $\mathcal{H}_1$. (See [39] for a slightly different approach and fine points of the proof.)

Thus $\mathcal{H}_1$ is the universal Hopf algebra generated by $\{X, Y, \delta_n;\ n = 1, 2, \ldots\}$ with relations

$$[Y, X] = X, \ \ [Y, \delta_n] = n\delta_n, \ \ [X, \delta_n] = \delta_{n+1}, \ \ [\delta_k, \delta_l] = 0,$$

$$\Delta Y = Y \otimes 1 + 1 \otimes Y, \quad \Delta\delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1,$$

$$\Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,$$

$$S(Y) = -Y, \ \ S(X) = -X + \delta_1 Y, \ \ S(\delta_1) = -\delta_1,$$

for $n, k, l = 1, 2, \ldots$.

Another recent point of interaction between Hopf algebras and noncommutative geometry is the work of Connes and Kreimer in renormalization schemes of quantum field theory. We refer to [29, 30, 31, 34, 36, 35] and references therein for this fascinating new subject.

5. Topological K-theory

The topological K-theory of spaces and its main theorem, *the Bott periodicity theorem*, can be extended to noncommutative Banach algebras. Of all topological invariants of spaces, K-theory has the distinct feature that it is the easiest to extend to noncommutative spaces. Moreover, on a large class of C^*-algebras the theory can be characterized by a few simple axioms. In the next section we take up the question of Chern character in noncommutative geometry. It is to address this and similar questions that cyclic cohomology and Connes' Chern character map enter the game.

K-theory was first introduced by Grothendieck in 1958 in his extension of the Riemann–Roch theorem to algebraic varieties. The isomorphism classes of bounded complexes of coherent sheaves on a variety X form an abelian monoid and the group that they generated was called $K_0(X)$. Soon after, Atiyah and Hirzebruch realized that in a similar fashion complex vector bundles over a compact space X define a group $K^0(X)$ and, moreover, using standard methods of algebraic topology, one obtains a *generalized cohomology theory* for spaces in this way. Bott's periodicity theorem for homotopy groups of stable unitary groups immediately implies that the new functor is 2-periodic. By the mid-1970s it was clear to operator algebraists that topological K-theory and Bott periodicity theorem can be extended to all Banach algebras. Our references for this section include [6, 57, 24].

5.1. *The K_0 functor*

Since the definition of $K_0(A)$ depends only on the underlying ring structure of A and makes sense for any ring, we shall define $K_0(A)$ for any ring A. Let A be a unital noncommutative ring. A right A-module P is called *projective* if it is a direct summand of a free module, i.e. there exists a right A-module Q such that

$$P \oplus Q \simeq A^I.$$

Equivalently, P is projective if and only if any short exact sequence of right A-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

splits. Let $\mathcal{P}(A)$ denote the set of isomorphism classes of finitely generated projective right A-modules. Under the operation of direct sum, $\mathcal{P}(A)$ is an abelian monoid. The group $K_0(A)$ is, by definition, the universal group generated by the monoid $\mathcal{P}(A)$. Thus elements of $K_0(A)$ can be written as $[P]-[Q]$ for P, $Q \in \mathcal{P}(A)$, with $[P]-[Q]=[P']-[Q']$ if and only if there is an $R \in \mathcal{P}(A)$ such that $P \oplus Q' \oplus R \simeq P' \oplus Q \oplus R$.

A unital ring homomorphism $f : A \to B$ defines a map (*base change*) $f_* : \mathcal{P}(A) \to \mathcal{P}(B)$ by

$$f_*(P) = P \otimes_A B$$

where the left A-module structure on B is induced by f. This map is clearly additive and hence induces an additive map

$$f_* : K_0(A) \to K_0(B).$$

This shows that $A \to K_0(A)$ is a functor.

We need to define K_0 of non-unital rings. Let A^+ be the *unitization* of a non-unital ring A. By definition, $A^+ = A \oplus \mathbb{Z}$ with multiplication $(a, m)(b, n) = (ab + na + mb, mn)$ and unit element $(0, 1)$. A non-unital ring map $f : A \to B$ clearly induces a unital ring map $f^+ : A^+ \to B^+$ by $f^+(a, n) = (f(a), n)$. The canonical morphism $A^+ \to \mathbb{Z}$, sending $(a, n) \to n$, is unital and we define $K_0(A)$ as the kernel of the induced map $K_0(A^+) \to K_0(\mathbb{Z})$. If A is already unital then the surjection $A^+ \to \mathbb{Z}$ splits and one shows that the two definitions coincide.

The first important result about K_0 is its *half-exactness* (cf., for example, [6] for a proof): for any exact sequence of rings

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0,$$

the induced sequence

$$K_0(I) \longrightarrow K_0(A) \longrightarrow K_0(A/I) \tag{19}$$

is *exact in the middle.* Simple examples show that exactness at the left and right ends can fail and in fact the extent to which they fail is measured by higher K-groups as we define them in the next section.

Remark 4. When A is commutative, the tensor product $P \otimes_A Q$ of A-modules is well defined and is an A-module again. It is finite and projective if P and Q are finite and projective. This operation turns $K_0(A)$ into a commutative ring. In general, for noncommutative rings no such multiplicative structure exists on $K_0(A)$.

There is an alternative description of $K_0(A)$ in terms of idempotents in matrix algebras over A that is often convenient. An idempotent $e \in M_n(A)$ defines a right A-module map

$$e : A^n \longrightarrow A^n$$

by left multiplication by e. Let $P_e = eA^n$ be the image of e. The relation

$$A^n = eA^n \oplus (1 - e)A^n$$

shows that P_e is a finite projective right A-module. Different idempotents can define isomorphic modules. This happens, for example, if e and f are *equivalent idempotents* (sometimes called *similar*) in the sense that

$$e = ufu^{-1}$$

for some invertible $u \in GL(n, A)$. Let $M(A) = \cup M_n(A)$ be the direct limit of matrix algebras $M_n(A)$ under the embeddings $M_n(A) \to M_{n+1}(A)$ defined by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Similarly let $GL(A)$ be the direct limit of the groups $GL(n, A)$. It acts on $M(A)$ by conjugation.

Definition 5.1. *Two idempotents $e \in M_k(A)$ and $f \in M_l(A)$ are called stably equivalent if their images in $M(A)$ are equivalent under the action of $GL(A)$.*

The following is easy to prove and answers our original question:

Lemma 5.1. *The projective modules P_e and P_f are isomorphic if and only if the idempotents e and f are stably equivalent.*

Let $\mathrm{Idem}(M(A))/GL(A)$ denote the set of stable equivalence classes of idempotents over A. This is an abelian monoid under the operation

$$(e,\, f) \mapsto e \oplus f := \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

It is clear that any finite projective module is of the type P_e for some idempotent e. In fact writing $P \oplus Q \simeq A^n$, one can let e be the idempotent corresponding to the projection map $(p, q) \mapsto (p, 0)$. These observations prove the following lemma:

Lemma 5.2. *For any unital ring A, the map $e \mapsto P_e$ defines an isomorphism of monoids*

$$Idem\,(M(A))/GL(A) \simeq \mathcal{P}(A).$$

Given an idempotent $e = (e_{ij}) \in M_n(A)$, its image under a homomorphism $f : A \to B$ is the idempotent $f_*(e) = (f(e_{ij}))$. This is our formula for $f_* : K_0(A) \to K_0(B)$ in the idempotent picture of K-theory.

For a Banach algebra A, $K_0(A)$ can be described in terms of connected components of the space of idempotents of $M(A)$ under its inductive limit topology (a subset $V \subset M(A)$ is open in the inductive limit topology if and only if $V \cap M_n(A)$ is open for all n). It is based on the following important observation: Let e and f be idempotents in a unital Banach algebra A and assume $\|e - f\| < 1/\|2e - 1\|$. Then $e \sim f$. In fact with

$$v = (2e - 1)(2f - 1) + 1$$

and $u = \frac{1}{2}v$, we have $ueu^{-1} = f$. To see that u is invertible note that $\|u - 1\| < 1$. One consequence of this fact is that if e and f are in the same path component of the space of idempotents in A, then they are equivalent. As a result we have, for any Banach algebra A, an isomorphism of monoids

$$\mathcal{P}(A) \simeq \pi_0(\mathrm{Idem}\,(M(A))),$$

where π_0 is the functor of path components.

For C^*-algebras, instead of idempotents it suffices to consider only the *projections*. A projection is a self-adjoint idempotent ($p^2 = p = p^*$). The reason is that every idempotent in a C^*-algebra is similar to a projection [6]: let e be an idempotent and set $z = 1 + (e - e^*)(e^* - e)$. Then z is invertible and positive and one shows that $p = ee^*z^{-1}$ is a projection and is similar to e.

Exercise 5.1. *Show that the set of projections of a C^*-algebra is homotopy equivalent* (*in fact a retraction*) *of the set of idempotents.*

Let $\mathrm{Proj}(M(A))$ denote the space of projections in $M(A)$. We have established isomorphisms of monoids

$$\mathcal{P}(A) \simeq \pi_0(\mathrm{Idem}\,(M(A))) \simeq \pi_0(\mathrm{Proj}\,(M(A))).$$

From the above homotopic interpretation of K_0 for Banach algebras, its *homotopy invariance* and *continuity* easily follows. Let $f, g : A \to B$ be continuous homomorphisms between Banach algebras. They are called *homotopic* if there exists a continuous homomorphism

$$F : A \to C([0,1],\, B)$$

such that $f = e_0 F$ and $g = e_1 F$, where e_0, $e_1 : C([0,1], B) \to B$ are the evaluations at 0 and 1 maps. Now by our definition of K_0 via π_0, it is clear that $e_{0*} = e_{1*} : K_0(C([0,\,1],\,B)) \to K_0(B)$ and hence

$$f_* = g_* : K_0(A) \to K_0(B),$$

which shows that K_0 is homotopy invariant.

In a similar way one can also show that K_0 preserves direct limits of Banach algebras: if $A = \varinjlim\,(A_i,\, f_{ij})$ is an inductive limit of Banach algebras then $K_0(A) = \varinjlim\,(K_0(A_i),\, f_{ij*})$. This property is referred to as *continuity* of K_0.

In addition to its homotopy invariance and continuity, we collect a couple of other properties of K_0 which hold for all rings:

- *Morita Invariance*: if A and B are Morita equivalent unital rings then $K_0(A) \simeq K_0(B)$. This is clear since Morita equivalent rings, by definition, have equivalent categories of modules and the equivalence can be shown to preserve the categories of finite projective modules. Therefore $\mathcal{P}(A) \simeq \mathcal{P}(B)$.

- *Additivity*: $K_0(A \oplus B) \simeq K_0(A) \oplus K_0(B)$ for unital rings A and B. This is a consequence of $\mathcal{P}(A \oplus B) \simeq \mathcal{P}(A) \oplus \mathcal{P}(B)$, which is easy to prove.

Example 5.1. (Commutative algebras) For $A = C_0(X)$ we have

$$K_0(C_0(X)) \simeq K^0(X), \tag{20}$$

where K^0 is the topological K-theory of spaces. The reason for this is the *Swan theorem* [117] (cf. also Serre [116] for the corresponding result in the context of affine varieties), according to which for any compact Hausdorff space X the category of finite projective $C(X)$-modules is equivalent to the category of complex vector bundles on X. The equivalence is via the *global section functor*. Given a vector bundle $p : E \to X$, let

$$P = \Gamma(E) = \{s : X \to E;\ ps = \mathrm{id}_X\}$$

be the set of all continuous global sections of E. It is clear that under fiberwise scalar multiplication and addition, P is a $C(X)$-module. Using the local triviality of E and a partition of unity one shows that there is a vector bundle F on X such that $E \oplus F$ is a trivial bundle, or, equivalently,

$$P \oplus Q \simeq A^n,$$

where Q is the module of global sections of F. This shows that P is finite and projective. It is not difficult to show that all finite projective modules are obtained in this way and Γ is an equivalence of categories (see exercise below). Now the rest of the proof of (20) is clear since $K^0(X)$ is, by definition, the universal group defined by the monoid of complex vector bundles on X.

Exercise 5.2. *Given a finite projective $C(X)$-module P, let Q be a $C(X)$-module such that $P \oplus Q \simeq A^n$, for some integer n. Let $e : A^n \to A^n$ be the right A-linear projection map $(p, q) \mapsto (p, 0)$. It is obviously an idempotent in $M_n(C(X))$. Define the vector bundle E to be the image of e :*

$$E = \{(x, v);\ e(x)v = v, \textit{for all } x \in X,\ v \in \mathbb{C}^n\} \subset X \times \mathbb{C}^n.$$

Now it is easily seen that $\Gamma(E) \simeq P$. With some more work it is shown that the functor Γ is full and faithful and hence defines an equivalence of categories [117].

Motivated by the Serre–Swan theorem, one usually thinks of finite projective modules over noncommutative algebras as *noncommutative vector bundles.*

Example 5.2. Here is a nice example of a projection in $M_2(C_0(\mathbb{R}^2)^+)$. Let

$$p = \frac{1}{1+|z|^2} \begin{pmatrix} |z|^2 & z \\ \bar{z} & 1 \end{pmatrix}.$$

It does not define an element of $K_0(C_0(\mathbb{R}^2))$ since $p(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$. The difference

$$\beta = p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is however the generator of $K_0(C_0(\mathbb{R}^2)) \simeq \mathbb{Z}$. This is a consequence of the Bott periodicity theorem that we recall later in this section. β is called the *Bott generator* and p the *Bott projection*. Now we have $C(S^2) = C_0(\mathbb{R}^2)^+$. Let $[1]$ denote the class of the trivial line bundle on S^2. It follows that $[1]$ and β form a basis for $K_0(C(S^2)) \simeq K^0(S^2) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Examples 5.1. (i) $K_0(\mathbb{C}) \simeq \mathbb{Z}$. In fact any finite projective module over $\mathbb{C}$ is simply a finite dimensional complex vector space whose isomorphism class is determined by its dimension. This shows that $\mathcal{P}(\mathbb{C}) \simeq \mathbb{N}$, from which our claim follows.
(ii) By Morita invariance, we then have $K_0(M_n(\mathbb{C})) \simeq \mathbb{Z}$.
(iii) The algebra of compact operators $\mathcal{K} = \mathcal{K}(H)$ on a separable Hilbert space is the direct limit of matrix algebras. Using the continuity of K_0 we conclude that $K_0(\mathcal{K}) \simeq \mathbb{Z}$.
(iv) On the other hand, $K_0(\mathcal{L}) = 0$ where $\mathcal{L} = \mathcal{L}(H)$ is the algebra of bounded operators on an infinite dimensional Hilbert space. To prove this let $e \in M_n(\mathcal{L}) = \mathcal{L}(H^n)$ be an idempotent. The idempotents $e \oplus I$ and $0 \oplus I$ are equivalent in $M_{2n}(\mathcal{L}) = \mathcal{L}(H^{2n})$ since both have infinite dimensional range. This shows that $[e] = 0$. Notice that for commutative algebras $A = C(X)$, $K_0(A)$ always contains a copy of $\mathbb{Z}$.

Consider the exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L}(H) \longrightarrow \mathcal{C} \longrightarrow 0,$$

where $\mathcal{C} := \mathcal{L}(H)/\mathcal{K}$ is called the *Calkin algebra*. From the above example we see that the corresponding K_0 sequence (19) fails to be exact on the left.

Example 5.3. (The trace map) Let A be a unital algebra, V be a vector space and $\tau : A \to V$ be a trace on A. Then τ induces an additive map

$$\tau : K_0(A) \to V$$

as follows. Given an idempotent $e = (e_{ij}) \in M_k(A)$, let

$$\tau([e]) = \sum_{i=1}^{k} \tau(e_{ii}). \tag{21}$$

Using the trace property of τ, one has $\tau(ueu^{-1}) = \tau(e)$. The additivity property $\tau(e \oplus f) = \tau(e) + \tau(f)$ is clear. This shows that (21) is a well defined map on K-theory.

Alternatively, given a finite projective A-module P, let

$$\tau([P]) = \mathrm{Tr}\,(\mathrm{id}_P),$$

where

$$\mathrm{Tr} : \mathrm{End}_A\,(P) \simeq P^* \otimes_A P \to V,$$

the *Hattori–Stallings trace*, is the natural extension of τ defined by

$$\mathrm{Tr}\,(f \otimes \xi) = \tau(f(\xi))$$

for all $f \in P^* = \mathrm{Hom}_A(P, A)$ and $\xi \in P$. This is the simplest example of a pairing between cyclic cohomology and K-theory (Connes' Chern character), to be defined in full generality later in these notes. Notice that if $\tau([e]) \neq 0$ then we can conclude that $[e] \neq 0$ in $K_0(A)$. This is often very useful in applications.

Example 5.4. (Hopf line bundle on quantum spheres) Let $0 < q \leq 1$ be a real number. The algebra $C(S_q^2)$ of functions on the *standard Podleś quantum sphere* S_q^2 is, by definition, the unital C^*-algebra generated by elements a and b with relations

$$aa^* + q^{-4}b^2 = 1,\ a^*a + b^2 = 1,\ ab = q^{-2}ba,\ a^*b = q^2ba^*.$$

The quantum analogue of the Dirac (or Hopf) monopole line bundle over S^2 is given by the following projection in $M_2(C(S_q^2))$ [14]:

$$\mathbf{e}_q = \frac{1}{2}\begin{bmatrix} 1 + q^{-2}b & qa \\ q^{-1}a^* & 1 - b \end{bmatrix}.$$

It can be directly checked that $\mathbf{e}_q^2 = \mathbf{e}_q = \mathbf{e}_q^*$. For $q = 1$, $C(S_1^2) = C(S^2)$ and the corresponding projection defines the Hopf line bundle on S^2. We refer to the article of Landi and van Suijlekom in this volume [91] for a survey of noncommutative bundles and instantons in general.

Example 5.5. ($K_0(A_\theta)$) We shall see later in this section, using the Pimsner–Voiculescu exact sequence, that

$$K_0(A_\theta) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

One generator is the class of the trivial idempotent [1]. Notice that $[1] \neq 0$ because $\tau(1) = 1 \neq 0$, where $\tau : A_\theta \to \mathbb{C}$ is the canonical trace. When

θ is irrational a second generator is given by the *Powers-Rieffel projection* $p \in A_\theta$. The projection p is of the form

$$p = U_2^* \, g(U_1)^* + f(U_1) + g(U_1)\, U_2, \tag{22}$$

where $f, g \in C^\infty(S^1)$. By $g(U_1)$ we mean of course $\sum \hat{g}_n U_1^n$ where $\hat{g}_n$ are the Fourier coefficients of g. To fulfill the projection condition $p^2 = p = p^*$, f and g must satisfy certain relations (cf. [24]) one of which implies that $\int_0^1 f(t)dt = \theta$. There are many such solutions but their corresponding projections are all homotopic and hence define the same class in $K_0(A_\theta)$. Now in (22), the only contribution to the trace $\tau(p)$ comes from the constant term of the middle term and hence

$$\tau(p) = \int_0^1 f(t)dt = \theta.$$

It follows that the range of the trace map $\tau : K_0(A_\theta) \to \mathbb{C}$ is in fact the subgroup $\mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}$.

Example 5.6. (Relation with Fredholm operators) The space of Fredholm operators, under the norm topology, is a *classifying space* for the K-theory of spaces. Let $[X, \mathcal{F}]$ denote the set of homotopy classes of continuous maps from a compact space X to the space of Fredholm operators $\mathcal{F}$ on an infinite dimensional Hilbert space. Such continuous maps should be thought of as families of Fredholm operators parameterized by X. By a theorem of Atiyah and Jänich (cf. [3] for a proof and a generalization) there exists a well defined *index map* index : $[X, \mathcal{F}] \to K^0(X)$ which induces an isomorphism

$$\text{index} : [X, \mathcal{F}] \simeq K^0(X). \tag{23}$$

Its definition is as follows. Given a Fredholm family $T : X \to \mathcal{F}$, if $\dim \operatorname{Ker}(T_x)$ and $\dim \operatorname{Coker}(T_x)$ are locally constant functions of x, then the family of finite dimensional subspaces $\operatorname{Ker}(T_x)$ and $\operatorname{Coker}(T_x)$, $x \in X$, define vector bundles denoted $\operatorname{Ker}(T)$ and $\operatorname{Coker}(T)$ on X, and their difference

$$\text{index}\,(T) := \operatorname{Ker}(T) - \operatorname{Coker}(T)$$

is the K-theory class associated to T. For a general family the dimensions of the subspaces $\operatorname{Ker}(T_x)$ and $\operatorname{Coker}(T_x)$ may be discontinuous, but one shows that it can always be continuously deformed to a family where these dimensions are continuous.

The isomorphism (23) is fundamental. For example, the index of a family of elliptic operators which are fiberwise elliptic, by this result, is an element of the K-theory of the base manifold (and not an integer). In noncommutative geometry, for example in transverse index theory on foliated manifolds, the parameterizing space X is highly singular and is replaced by a noncommutative algebra A. The above analytic index map (23), with values in $K_0(A)$, still can be defined and its identification is one of the major problems of noncommutative index theory [24].

5.2. *The higher K-functors*

Starting with K_1, algebraic and topological K-theory begin to differ from each other. In this section we shall first briefly indicate the definition of algebraic K_1 of rings and then define, for Banach algebras, a sequence of functors K_n for $n \geq 1$.

For a unital ring A, let $GL(A)$ be the *direct limit* of groups $GL(n, A)$ of invertible $n \times n$ matrices over A where the direct system $GL(n, A) \hookrightarrow GL(n+1, A)$ is defined by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. The algebraic K_1 of A is defined as the abelianization of $GL(A)$:

$$K_1^{\text{alg}}(A) := GL(A)/[GL(A),\, GL(A)],$$

where $[\ ,\ ]$ denotes the commutator subgroup.

Applied to $A = C(X)$, this definition does not reproduce the topological $K^1(X)$. For example for $A = \mathbb{C} = C(\text{pt})$ we have $K_1^{\text{alg}}(\mathbb{C}) \simeq \mathbb{C}^\times$ where the isomorphism is induced by the determinant map

$$\det :\, GL(\mathbb{C}) \to \mathbb{C}^\times,$$

while $K^1(\text{pt}) = 0$. It turns out that, to obtain the right result, one should divide $GL(A)$ by a bigger subgroup, i.e. by the *closure* of its commutator subgroup. This works for all Banach algebras and will give the right definition of topological K_1. A better approach however is to define the higher K groups in terms of K_0 and the *suspension functor.*

The *suspension* of a Banach algebra A is the Banach algebra

$$SA = C_0(\mathbb{R},\ A)$$

of continuous functions from $\mathbb{R}$ to A vanishing at infinity. Notice that for $A = C(X)$, SA is isomorphic to the algebra of continuous functions on $X \times [0,1]$ vanishing on $X \times \{0,1\}$. It follows that $SA^+ \simeq C(\Sigma X)$, where ΣX is the *suspension* of X obtained by collapsing $X \times \{0,1\}$ to a point in $X \times [0,1]$.

Definition 5.2. *The higher topological K groups of a Banach algebra A are the K_0 groups of the iterated suspensions of A :*

$$K_n(A) = K_0(S^n A), \qquad n \geq 1.$$

This is a bit too abstract. It is better to think of higher K groups of A as higher homotopy groups of $GL(A)$. To do this we need the following Lemma. Let $GL^\circ(n, A)$ denote the connected component of the identity in $GL(n, A)$.

Lemma 5.3. (i) *Let $f : A \to B$ be a surjective unital homomorphism of unital Banach algebras. Then $f : GL^\circ(1, A) \to GL^\circ(1, B)$ is surjective.*
(ii) *For any $u \in GL(n, A)$,* $\mathrm{diag}\,(u, u^{-1}) \in GL^\circ(2n,\, A)$.

To prove the first statement notice that the group generated by the exponentials e^y, $y \in B$, coincides with $GL^\circ(1, B)$. Now since f is surjective we have $e^y = e^{f(x)} = f(e^x)$ which implies that any product of exponentials is in the image of f. To prove the second statement we can use the path

$$z_t = \mathrm{diag}\,(u,\, u^{-1}) u_t \,\mathrm{diag}\,(u,\, u^{-1})\, u_t^{-1},$$

where

$$u_t = \begin{pmatrix} \cos \dfrac{\pi}{2} t & -\sin \dfrac{\pi}{2} t \\ \sin \dfrac{\pi}{2} t & \cos \dfrac{\pi}{2} t \end{pmatrix},$$

connecting $\mathrm{diag}(u,\, u^{-1})$ to $\mathrm{diag}(1,\, 1)$.

We can now show that

$$K_1(A) \simeq \pi_0(GL(A)), \tag{24}$$

the group of connected components of $GL(A)$. To see this let $u \in GL(n, A)$. Then, by the above lemma, there is a path α_t in $GL(2n,\, A)$ connecting diag $(u,\, u^{-1}) \in GL^0(2n,\, A)$ to I_{2n}. Let $p_n = \mathrm{diag}(I_n,\, 0)$. Then $e_t = \alpha_t p_n \alpha_t^{-1}$ is an idempotent in $(SA)^+$ and the map $[u] \mapsto [e_t] - [p_n]$ implements the isomorphism in (24).

Now since $\pi_n(GL(A)) \simeq \pi_{n-1}(GL(SA))$, using (24) we obtain

$$K_n(A) \simeq \pi_{n-1}(GL(A)). \tag{25}$$

Example 5.7. Let $A = \mathbb{C}$. Then by (25), we have

$$K_n(\mathbb{C}) \simeq \pi_{n-1}(GL(\mathbb{C})) \simeq \pi_{n-1}(U(\mathbb{C})).$$

By the Bott periodicity theorem, the homotopy groups of the stable unitary groups $U(\mathbb{C})$ are periodic, i.e. for all n we have

$$\pi_n(U(\mathbb{C})) \simeq \pi_{n+2}(U(\mathbb{C})),$$

and hence

$$K_{n+2}(\mathbb{C}) \simeq K_n(\mathbb{C}).$$

This is the simplest instance of the general Bott periodicity theorem for the K-theory of Banach algebras to be discussed in the next section.

Example 5.8. For any Banach algebra A, we have a surjection

$$K_1^{alg}(A) \twoheadrightarrow K_1(A).$$

Using (24), this follows from

$$GL^\circ(A) = \overline{[GL(A), GL(A)]},$$

which we leave as an exercise. For $A = \mathbb{C}$, we have $K_1^{\mathrm{alg}}(\mathbb{C}) \simeq \mathbb{C}^\times$ with the isomorphism given by the determinant map $\det : GL(\mathbb{C}) \to \mathbb{C}^\times$, while $K_1(\mathbb{C}) = 0$ (see the next example).

Example 5.9. (i) Since $GL(n, \mathbb{C})$ is connected for all n, we have $K_1(\mathbb{C}) = 0$. Similarly, using polar decomposition, one shows that for any von Neumann algebra A, $GL(n, A)$ is connected for all n and hence $K_1(A) = 0$.
(ii) By Morita invariance we have $K_1(M_n(\mathbb{C})) = 0$.
(iii) Since the algebra $\mathcal{K}$ of compact operators is the direct limit of finite matrices, by continuity we have $K_1(\mathcal{K}) = 0$.

Exercise 5.3. *Starting from the definitions, show that for $i = 0, 1$*

$$K_i(C(S^1)) \simeq \mathbb{Z}.$$

Under these isomorphisms a projection $e : S^1 \to M_n(\mathbb{C})$ is sent to $tr(e) \in \mathbb{Z}$ and an invertible $u : S^1 \to GL(n, \mathbb{C})$ is sent to the winding number of $det\,(u(z))$.

The suspension functor is *exact* in the sense that for any exact sequence of Banach algebras $0 \to I \to A \to A/I \to 0$ the sequence

$$0 \longrightarrow SI \longrightarrow SA \longrightarrow S(A/I) \longrightarrow 0$$

is exact too. Coupled with the half exactness of K_0, we conclude that the sequences

$$K_n(I) \longrightarrow K_n(A) \longrightarrow K_n(A/I)$$

are exact in the middle for all $n \geq 0$.

One can splice these half exact sequences into a long exact sequence

$$\cdots \to K_n(I) \to K_n(A) \to K_n(A/I) \to \cdots \to K_0(I) \to K_0(A) \to K_0(A/I). \tag{26}$$

To do this it suffices to show that there exists a *connecting homomorphism*

$$\partial : K_1(A) \longrightarrow K_0(I)$$

which renders the sequence

$$K_1(I) \to K_1(A) \to K_1(A/I) \xrightarrow{\partial} K_0(I) \to K_0(A) \to K_0(A/I) \tag{27}$$

exact. It is sometimes called the (*generalized*) *index map* since for the Calkin extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L}(H) \longrightarrow \mathcal{C} \longrightarrow 0 \tag{28}$$

it coincides with the index of Fredholm operators

$$\partial = \text{index} : K_1(\mathcal{C}) \longrightarrow K_0(\mathcal{K}) \simeq \mathbb{Z}.$$

Let $u \in GL_n(A/I)$ and let $w \in GL_{2n}(A)$ be a lift of $\operatorname{diag}(u, u^{-1})$. Define

$$\partial([u]) = [wp_nw^{-1}] - [p_n] \in K_0(I),$$

where the projection $p_n = \operatorname{diag}(I_n, 0)$. It can be shown that this map is well defined and (27) is exact.

Example 5.10. Let A be a unital C^*-algebra. Using polar decomposition one shows that any invertible in $GL(n, A)$ is homotopic to a unitary. Given such a unitary u, we can find a partial isometry v in $M_{2n}(A)$ lifting $\operatorname{diag}(u, 1)$. Now the unitary

$$w = \begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{pmatrix}$$

lifts $\operatorname{diag}((u, 1), (u^{-1}, 1))$ and hence

$$\partial([u]) = [wp_{2n}w^{-1}] - [p_{2n}] = [1 - v^*v] - [1 - vv^*].$$

For the Calkin extension (28) this maps sends an invertible in the Calkin algebra $\mathcal{C}$, i.e. a Fredholm operator, to its Fredholm index in $K_0(\mathcal{K}) \simeq \mathbb{Z}$.

Remark 5. A more conceptual way to get the long exact sequence (26) would be to derive it from the homotopy exact sequence of a fibration.

5.3. *Bott periodicity theorem*

Homotopy invariance, Morita invariance, additivity and the exact sequence (26) are essential features of topological K-theory. The deepest result of K-theory, however, at least in the commutative case, is the Bott periodicity theorem. It states that there is a natural isomorphism between K_0 and K_2. The isomorphism is given by the *Bott map*

$$\beta : K_0(A) \to K_1(SA).$$

Since $K_1(SA) \simeq \pi_1(GL(A))$ is the homotopy group of the stable general linear group of A, β should somehow turn an idempotent in $M(A)$ into a loop of invertibles in $GL(A)$. We assume A is unital (the general case easily follows). Given an idempotent $e \in M_n(A)$, define a map $u_e : S^1 \to GL(n, A)$ by $u_e(z) = ze + (1 - z)e$. It defines a loop in $GL(A)$ based at 1, whose homotopy class is an element of $\pi_1(GL(A)) \simeq K_1(SA)$. Now the *Bott map* $\beta : K_0(A) \longrightarrow K_1(SA)$ is defined by

$$\beta([e] - [f]) = u_e u_f^{-1}.$$

Notice that, since u_e is a group homomorphism the additivity of β follows.

Theorem 5.1. (Bott periodicity theorem) *For a complex Banach algebra A the Bott map*

$$\beta : K_0(A) \to K_2(A)$$

is a natural isomorphism.

It follows that for all $n \geq 0$,

$$K_n(A) \simeq K_{n+2}(A)$$

and the long exact sequence (26) reduces to a periodic 6-term exact sequence

$$\begin{array}{ccccc}
K_0(I) & \xrightarrow{i_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/I) \\
\partial \uparrow & & & & \downarrow \partial \\
K_1(A/I) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[i_*]{} & K_1(I)
\end{array}$$

Example 5.11. For $A = \mathbb{C}$ we already knew that $K_0(\mathbb{C}) \simeq \mathbb{Z}$ and $K_1(\mathbb{C}) \simeq 0$. Using Bott periodicity we obtain $K_{2n}(\mathbb{C}) \simeq \mathbb{Z}$, and $K_{2n+1}(\mathbb{C}) \simeq 0$. This is a non-trivial result and in fact of the same magnitude of difficulty as Bott periodicity. Since the spheres S^n are iterated suspensions of a point, we obtain $K^0(S^{2n}) \simeq \mathbb{Z} \oplus \mathbb{Z}$, $K^1(S^{2n}) \simeq 0$, and $K^0(S^{2n+1}) \simeq K^1(S^{2n+1}) \simeq \mathbb{Z}$.

Example 5.12. (The Toeplitz extension) The Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0$$

was introduced in Section 3.1. Now the index map $\partial : K_1(C(S^1)) \to K_0(\mathcal{K})$ in the 6-term exact sequence is an isomorphism (this is, more or less, equivalent to the Gohberg-Krein index theorem for Toeplitz operators). Since $K_0(\mathcal{K}) = \mathbb{Z}$, $K_0(\mathcal{K}) = 0$, and $K_i(C(S^1)) = \mathbb{Z}$ for $i = 0, 1$, from the above 6-term exact sequence we deduce that $K_0(\mathcal{T}) \simeq K_1(\mathcal{T}) = \mathbb{Z}$.

5.4. *Further results*

In the commutative case the properties of homotopy invariance, long exact sequence, and Bott periodicity, suffice to give a good understanding of the K-theory of spaces such as CW complexes. Noncommutative spaces are of course much richer and more complicated. Here we give two further results that have no counterpart in the commutative case. Proofs of both can be found in [6] or in the original articles cited below.

Theorem 5.2. (Connes' Thom isomorphism [20]) *If $\alpha : \mathbb{R} \to Aut(A)$ is a continuous one-parameter group of automorphisms of a C^*-algebra A, then*

$$K_i(A \rtimes_\alpha \mathbb{R}) \simeq K_{1-i}(A), \quad i = 0, 1.$$

In particular this result shows that the K-theory of $A \rtimes_\alpha \mathbb{R}$ is independent of the action α.

The dimension shift is reminiscent of the dimension shift in the classical Thom isomorphism theorem relating the K-theory with compact support of the total space of a vector bundle with the K-theory of its base. Note that if the action is trivial then the theorem reduces to the Bott periodicity theorem. In fact in this case we have

$$A \rtimes_\alpha \mathbb{R} \simeq A \otimes C_0(\mathbb{R}) \simeq SA.$$

The second result we would like to highlight in this section is the 6-term exact sequence of Pimsner and Voiculescu:

Theorem 5.3. (Pimsner–Voiculescu exact sequence [105]) *For any automorphism $\alpha \in Aut(A)$ of a C^*-algebra A there is a 6-term exact sequence*

$$\begin{tikzcd}
K_0(A) \arrow[r, "1-\alpha_*"] & K_0(A) \arrow[r, "i_*"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l, "i_*"'] & K_1(A) \arrow[l, "1-\alpha_*"']
\end{tikzcd}$$

Example 5.13. (K-theory of noncommutative tori) A beautiful application of this result is to the K-theory of the noncommutative torus. We have $A_\theta = C(S^1) \rtimes_\alpha \mathbb{Z}$, where the automorphism α is through rotation by $2\pi\theta$. But α is homotopic to the identity through rotations by $2\pi\theta t$, $t \in [0, 1]$. By homotopy invariance of K_i, we obtain $\alpha_* = 1$. Using $K_i(C(S^1)) \simeq \mathbb{Z}$ for $i = 0, 1$ and a simple diagram chase we conclude from the Pimsner-Voiculescu exact sequence that

$$K_i(A_\theta) \simeq \mathbb{Z} \oplus \mathbb{Z} \quad \text{for} \;\; i = 0, 1.$$

Thus it seems that K-theory by itself cannot distinguish the isomorphism class of A_θ for different θ. There is however an extra piece of structure in $K_0(A)$, for A a C^*-algebra, that can be used in this regard. Notice that $K_0(A)$ is an *ordered group* with its positive cone defined by projections in $M(A)$. Equipped with this extra structure one can then show that $A_{\theta_1} \simeq A_{\theta_2}$ iff $\theta_1 = \theta_2$ or $\theta_1 = 1 - \theta_2$ (cf. [6] and references therein).

5.5. *Twisted K-theory*

Twisted K-theory has been around for quite some time (cf. Donovon-Karoubi [52], Rosenberg [113]). The recent surge of interest in the subject has to do with both mathematics and high energy physics. In mathematics, a recent result of Freed, Hopkins and Teleman shows that the twisted equivariant K-theory of a compact Lie group is isomorphic to the Verlinde algebra (fusion algebra) of the group [58, 59]. The latter algebra is the algebra of projective representations of the loop group of the group at a fixed *level.* In some semiclassical limits of string theory over a background spacetime X, the strengths of B-fields are elements of $H^3(X, \mathbb{Z})$. When this B-field is non-trivial the topological charges of D-branes are interpreted as elements of twisted K-theory with respect to the twisting defined by B (cf. [10, 11] for a mathematical perspective). A recent comprehensive study of twisted K-theory can be found in Atiyah and Segal's article [3].

The twisting coefficients (local systems) of twisted K-theory are cohomology classes in $H^3(X, \mathbb{Z})$. There are at least two approaches to the subject. One can either extend the definition of K-theory through Fredholm operators and the relation (23) to include twistings as in [3], or one can define the twisted K-theory as the K-theory of a noncommutative algebra as is done in [113]. We shall briefly describe this latter definition.

Let X be a locally compact, Hausdorff, and second countable space. We recall the classification of locally trivial bundles of algebras with fibers isomorphic to the algebra $\mathcal{K} = \mathcal{K}(H)$ of compact operators on an infinite dimensional Hilbert space, and with structure group $\mathrm{Aut}\,(\mathcal{K})$. As we saw in Section 2.4 there is a one to one correspondence between isomorphism classes of such bundles and $H^3(X, \mathbb{Z})$. Given such a bundle of algebras $\mathcal{A}$, its Dixmier-Douady invariant

$$\delta(\mathcal{A}) \in H^3(X, \mathbb{Z})$$

is a complete isomorphism invariant of such bundles.

Now given a pair (X, δ) as above, the *twisted K-theory* of X can be defined as the K-theory of the C^*-algebra $A = \Gamma(X, \mathcal{A})$ of continuous sections of $\mathcal{A}$ vanishing at infinity:

$$K^i_\delta(X) := K_i(A).$$

There is also an equivariant version of twisted K-theory, denoted by $K^\delta_G(X)$, that is specially important in view of the recent work [58]. The coefficients for this theory are elements of the equivariant cohomology $H^3_G(X, \mathbb{Z})$, where G is a compact Lie group acting on a space X. We refer to [3] for its definition. For simplicity, let G be a compact, connected, simply connected, and simple Lie group. Then the central extensions of the loop group LG of G are characterized by a positive integer k, called the level [106]. For each positive integer k, the positive energy representations of this central extension, up to equivalence, constitute a finite set and we denote by $V_k(G)$ the free abelian group generated by this set. There is a commutative ring structure on this set, corresponding to tensor product of representations.

Now let G act on itself by conjugation ($G = X$). Then the equivariant cohomology $H^3(G, \mathbb{Z})$ is a free group of rank one whose elements we shall denote by integers. The theorem of Freed-Hopkins-Teleman states that, at each level k, the fusion ring $V_k(G)$ is isomorphic to the twisted equivariant K-theory of G:

$$K^n_G(G) \simeq V_k(G),$$

where the integer n can be explicitly defined in terms of $k \geq 0$ and G [58, 59].

Example 5.14. Let $G = SU(2) \simeq S^3$. Then $H^3(G, \mathbb{Z}) \simeq \mathbb{Z}$. For each integer n representing a class in $H^3(G, \mathbb{Z})$ there is a bundle $\mathcal{A}_n$ of algebras of compact operators $\mathcal{K}$ over S^3 obtained by gluing the trivial bundles $S^3_+ \times \mathcal{K}$ and $S^3_- \times \mathcal{K}$ on the upper and lower hemispheres respectively. The gluing is defined by a map $S^2 \to \mathrm{Aut}\,(\mathcal{K})$ of degree n. Let A_n denote the C^*-algebra of continuous sections of $\mathcal{A}_n$. We have then, by definition,

$$K^{i,n}(S^3) = K_i(A_n).$$

The representation ring of $G = SU(2)$ is a polynomial algebra whose generator is the fundamental 2-dimensional representation of G. The twisted equivariant K-theory of $SU(2)$ can be explicitly computed and shown to be a quotient of the representation ring (cf. [58, 59]).

5.6. *K-homology*

In [2] and a little later and independently in [13], Atiyah and Brown, Douglas and Fillmore proposed theories dual to topological K-theory, using techniques of functional analysis and operator algebras. The cycles for Atiyah's theory are *abstract elliptic operators* (H, F) over $C(X)$ where $H = H^+ \oplus H^-$ is a $\mathbb{Z}_2$-graded Hilbert space, $\pi : C(X) \to \mathcal{L}(H)$ is an even representation of $C(X)$, and $F : H \to H$ is an even bounded operator with $F^2 - I \in \mathcal{K}(H)$. This data must satisfy the condition

$$[F, \pi(a)] \in \mathcal{K}(H).$$

We see that an abstract elliptic operator is the same as a Fredholm module over $C(X)$ as in definition (4) except that now instead of $F^2 = I$ we have the above condition. The two definitions are however essentially equivalent. In particular the formula

$$\langle (H, F), [e] \rangle = \operatorname{index} F_e^+,$$

where the Fredholm operator F_e^+ is defined in (6) defines a pairing between the K-theory of X and abstract elliptic operators on X.

Let $\mathrm{Ext}(A)$ denote the set of isomorphism classes of extensions of a C^*-algebra A of the form

$$0 \longrightarrow \mathcal{K} \longrightarrow E \longrightarrow A \longrightarrow 0.$$

There is a natural operation of addition of extensions which turn $\mathrm{Ext}(A)$ into an abelian monoid. It can be shown that if A is a nuclear C^*-algebra, for example if $A = C(X)$ is commutative, then $\mathrm{Ext}(A)$ is actually a group. There is a pairing

$$K_1(A) \times \mathrm{Ext}(A) \to \mathbb{Z}$$

which is defined as follows. Let δ be the connecting homomorphism in the 6-term exact sequence of an extension $\mathcal{E}$ representing an element of $\mathrm{Ext}\,(A)$, and let $[u] \in K_1\,(A)$. Then

$$\langle [u]\,, [\mathcal{E}] \rangle := \delta([u]) \in K_0\,(\mathcal{K}) \simeq \mathbb{Z}.$$

Example 5.15. A simple example of a non-trivial extension is the Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

from Section 3.1. It can be shown that the class of this extension generates the K-homology group $\mathrm{Ext}(C(S^1))$ [13]. A more elaborate example is the pseudodifferential extension of the algebra of functions on the cosphere bundle:

$$0 \longrightarrow \mathcal{K}(L^2(M)) \longrightarrow \Psi^0(M) \xrightarrow{\sigma} C(S^*M) \longrightarrow 0.$$

briefly discussed in Section 3.1. For a comprehensive introduction to K-homology see Higson and Roe's [71].

For noncommutative spaces, the above approach to K-homology works best when the corresponding C^*-algebra is nuclear. For arbitrary C^*-algebras, Kasparov's KK-theory provides a unified approach to both K-theory and K-homology in a bivariant theory (cf. [78] and [6]).

6. Cyclic Cohomology

Let A be an algebra and $e, f \in M_n(A)$ be two idempotents. How can we show that $[e] \neq [f]$ in $K_0(A)$? Here is a simple device that is often helpful in this regard. As we saw in Example 5.3, any trace $\tau : A \longrightarrow \mathbb{C}$ induces an additive map

$$\tau : K_0(A) \longrightarrow \mathbb{C},$$

via the formula

$$\tau([e]) := \sum_{i=1}^{n} e_{ii}.$$

Now if $\tau([e]) \neq \tau([f])$, then, of course, $[e] \neq [f]$.

Exercise 6.1. *Let $A = C(X)$, where X is compact and connected, and let $\tau(f) = f(x)$, for some fixed $x \in X$. Show that if E is a vector bundle on X and $e \in M_n(C(X))$ an idempotent defining E, then $\tau([e]) = \dim(E_x)$ where E_x is the fibre of E over x. Thus τ simply measures the rank of the vector bundle.*

The topological information hidden in an idempotent is much more subtle than just its "rank" and in fact traces can only capture zero-dimensional information. To know more about idempotents and K-theory we need higher dimensional analogues of traces. They are called *cyclic cocycles* and their study is the subject of *cyclic cohomology.* As we shall see in this section, cyclic cohomology is the right noncommutative analogue of de Rham homology of currents on smooth manifolds.

Cyclic cohomology was first discovered by Alain Connes [19, 21, 22]. Let us first recall a remarkable subcomplex of the Hochschild complex called the *Connes complex* that was introduced by him for the definition of cyclic cohomology. For an algebra $\mathcal{A}$ let

$$C^n(\mathcal{A}) = \mathrm{Hom}\,(\mathcal{A}^{\otimes(n+1)}, \mathbb{C}), \quad n = 0, 1, \ldots,$$

denote the space of $(n+1)$-linear functionals on $\mathcal{A}$ with values in $\mathbb{C}$. The Hochschild differential $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ is defined by

$$\begin{aligned}(b\varphi)(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1})\\ + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \ldots, a_n).\end{aligned}$$

One checks that $b^2 = 0$. The cohomology of the complex $(C(\mathcal{A}), b)$ is, by definition, the Hochschild cohomology of $\mathcal{A}$ (with coefficients in the A-bimodule A^*) and will be denoted by $HH^n(\mathcal{A})$. An n-cochain $\varphi \in C^n(\mathcal{A})$ is called *cyclic* if

$$\varphi(a_n, a_0, \ldots, a_{n-1}) = (-1)^n \varphi(a_0, a_1, \ldots, a_n)$$

for all $a_0, \ldots, a_n$ in $\mathcal{A}$. Though it is not obvious at all, one can check that cyclic cochains form a subcomplex

$$(C_\lambda(\mathcal{A}), b) \subset (C(\mathcal{A}), b)$$

of the Hochschild complex.

We shall refer to $(C_\lambda(\mathcal{A}), b)$ as the *Connes complex* of $\mathcal{A}$. Its cohomology, by definition, is the cyclic cohomology of $\mathcal{A}$ and will be denoted by $HC^n(\mathcal{A})$. We start our introduction to cyclic cohomology by some concrete examples of cyclic cocycles à la Connes.

6.1. *Cyclic cocycles*

We give an example of a cyclic cocycle. Let M be a closed (i.e. compact without boundary), smooth, oriented, n-manifold. For $f^0, \ldots, f^n \in \mathcal{A} = C^\infty(M)$, let

$$\varphi(f^0, \ldots, f^n) = \int_M f^0 df^1 \ldots df^n.$$

The $(n+1)$-linear cochain

$$\varphi : \mathcal{A} \times \cdots \times \mathcal{A} \to \mathbb{C}$$

has three properties: it is continuous with respect to the natural Fréchet space topology on $\mathcal{A}$; it is a Hochschild cocycle; and it is cyclic. For the cocycle condition, notice that

$$
\begin{aligned}
(b\varphi)(f^0, \ldots, f^{n+1}) &= \sum_{i=0}^{n} (-1)^i \int_M f^0 df^1 \cdots d(f^i f^{i+1}) \cdots df^{n+1} \\
&\quad + (-1)^{n+1} \int_M f^{n+1} f^0 df^1 \cdots df^n \\
&= 0,
\end{aligned}
$$

where we only used the Leibniz rule for the de Rham differential d and the graded commutativity of the algebra $(\Omega M, d)$ of differential forms on M. The cyclic property of φ

$$\varphi(f^n, f^0, \ldots, f^{n-1}) = (-1)^n \varphi(f^0, \ldots, f^n)$$

is more interesting. In fact since

$$\int_M (f^n df^0 \cdots df^{n-1} - (-1)^n f^0 df^1 \cdots df^n) = \int_M d(f^n f^0 df^1 \cdots df^{n-1}),$$

we see that the cyclic property of φ follows from Stokes' formula

$$\int_M d\omega = 0,$$

which is valid for any $(n-1)$-form ω on a closed manifold M.

A remarkable property of cyclic cocycles is that, unlike de Rham cocycles which make sense only over commutative algebras, they can be defined over any noncommutative algebra and the resulting cohomology theory is the right generalization of de Rham homology of currents on a smooth manifold. Before developing cyclic cohomology any further we give one more example. In the above situation it is clear that if $V \subset M$ is a closed p-dimensional oriented submanifold then the formula

$$\varphi(f^0, \ldots, f^p) = \int_V f^0 df^1 \cdots df^p$$

defines a cyclic p-cocycle on $\mathcal{A}$. We can replace V by *closed currents* on M and obtain more cyclic cocycles.

Recall that a p-dimensional *current* C on M is a continuous linear functional $C : \Omega^p M \to \mathbb{C}$ on the space of p-forms on M. We write $\langle C, \omega \rangle$ instead of $C(\omega)$. For example a zero dimensional current on M is just a distribution on M. The differential of a current is defined by $\langle dC, \omega \rangle = \langle C, d\omega \rangle$ and in this way one obtains the complex of currents on M whose homology is the *de Rham homology* of M.

Exercise 6.2. *Let C be a p-dimensional current on M. Show that the $(p+1)$-linear functional*

$$\varphi_C(f^0, \ldots, f^p) = \langle C\,,\, f^0 df^1 \cdots df^p \rangle$$

is a Hochschild cocycle on $\mathcal{A}$. Show that if C is closed then φ_C is a cyclic p-cocycle on $\mathcal{A}$.

Let $\mathcal{A}$ be an algebra. Define the operators

$$b' : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A}), \quad \text{and} \quad \lambda : C^n(\mathcal{A}) \to C^n(\mathcal{A}),$$

by

$$\begin{aligned}(b'\varphi)(a_0, \ldots, a_{n+1}) &= \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}),\\ (\lambda\varphi)(a_0, \ldots, a_n) &= (-1)^n \varphi(a_n, a_0, \ldots, a_{n-1}),\end{aligned}$$

By a direct computation one checks that

$$(1-\lambda)b = b'(1-\lambda), \qquad b'^2 = 0. \tag{29}$$

Notice that a cochain $\varphi \in C^n$ is cyclic if and only if $(1-\lambda)\varphi = 0$. Using (29) we obtain

Lemma 6.1. *The space of cyclic cochains is invariant under b, i.e. for all n,*

$$b\, C_\lambda^n(\mathcal{A}) \subset C_\lambda^{n+1}(\mathcal{A}).$$

We therefore have a subcomplex of the Hochschild complex, called the *Connes complex* of $\mathcal{A}$:

$$C_\lambda^0(\mathcal{A}) \xrightarrow{b} C_\lambda^1(\mathcal{A}) \xrightarrow{b} C_\lambda^2(\mathcal{A}) \xrightarrow{b} \cdots \tag{30}$$

The cohomology of this complex is called the *cyclic cohomology* of $\mathcal{A}$ and will be denoted by $HC^n(\mathcal{A})$, $n = 0, 1, 2, \ldots$. A cocycle for cyclic cohomology is called a *cyclic cocycle*. It satisfies the two conditions:

$$(1-\lambda)\varphi = 0, \quad \text{and} \quad b\varphi = 0.$$

Examples 6.1. 1. Clearly $HC^0(A) = HH^0(A)$ is the space of traces on A. In particular if A is commutative then $HC^0(\mathcal{A}) \simeq A^*$ is the linear dual of A.

2. Let $\mathcal{A} = C^\infty(M, M_n(\mathbb{C}))$ be the space of smooth matrix valued functions on a closed smooth oriented manifold M. For any closed de Rham p-current on M

$$\varphi_C(f^0, \ldots, f^p) = \langle C\,,\, Tr(f^0 df^1 \cdots df^p)\rangle$$

is a cyclic p-cocycle on $\mathcal{A}$.

3. Let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation and $\tau : \mathcal{A} \to \mathbb{C}$ an invariant trace, i.e. $\tau(\delta(a)) = 0$ for all $a \in \mathcal{A}$. Then one checks that

$$\varphi(a_0, a_1) = \tau(a_0 \delta(a_1)) \tag{31}$$

is a cyclic 1-cocycle on $\mathcal{A}$. This example can be generalized. Let δ_1 and δ_2 be a pair of *commuting* derivations which leave a trace τ invariant. Then

$$\varphi(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))) \tag{32}$$

is a cyclic 2-cocycle on $\mathcal{A}$.

Here is a concrete example with $\mathcal{A} = \mathcal{A}_\theta$ a smooth noncommutative torus. Let $\delta_1,\, \delta_2 : \mathcal{A}_\theta \to \mathcal{A}_\theta$ be the unique derivations defined by

$$\delta_1(U) = U, \quad \delta_1(V) = 0; \quad \delta_2(U) = 0, \quad \delta_2(V) = V.$$

They commute with each other and preserve the standard trace τ on $\mathcal{A}_\theta$. The resulting cyclic 1-cocycles $\varphi_1(a_0, a_1) = \tau(a_0\delta_1(a_1))$ and $\varphi_1'(a_0, a_1) = \tau(a_0\delta_2(a_1))$ form a basis for the periodic cyclic cohomology $HP^1(\mathcal{A}_\theta)$. Similarly, the corresponding cocycle (32) together with τ form a basis for $HP^0(\mathcal{A}_\theta)$.

Consider the short exact sequence of complexes

$$0 \to C_\lambda \to C \to C/C_\lambda \to 0.$$

Its associated long exact sequence is

$$\cdots \longrightarrow HC^n(\mathcal{A}) \longrightarrow HH^n(\mathcal{A}) \longrightarrow H^n(C/C_\lambda) \longrightarrow HC^{n+1}(\mathcal{A}) \longrightarrow \cdots \tag{33}$$

We need to identify the cohomology groups $H^n(C/C_\lambda)$. To this end, consider the short exact sequence

$$0 \longrightarrow C/C_\lambda \xrightarrow{1-\lambda} (C, b') \xrightarrow{N} C_\lambda \longrightarrow 0, \tag{34}$$

where the operator N is defined by

$$N = 1 + \lambda + \lambda^2 + \cdots + \lambda^n : C^n \longrightarrow C^n.$$

The relations

$$N(1-\lambda) = (1-\lambda)N = 0, \quad \text{and} \quad bN = Nb'$$

can be verified and they show that $1-\lambda$ and N are morphisms of complexes in (34).

Exercise 6.3. *Show that* (34) *is exact (the interesting part is to show that* $Ker N \subset Im(1-\lambda)$*).*

Now, assuming A is unital, the middle complex (C, b') in (34) can be shown to be exact with contracting homotopy $s : C^n \to C^{n-1}$ given by

$$(s\varphi)(a_0, \ldots, a_{n-1}) = (-1)^n \varphi(a_0, \ldots, a_{n-1}, 1).$$

It follows that $H^n(C/C_\lambda) \simeq HC^{n-1}(\mathcal{A})$. Using this in (33), we obtain *Connes' long exact* sequence relating Hochschild and cyclic cohomology:

$$\cdots \longrightarrow HC^n(\mathcal{A}) \xrightarrow{I} HH^n(\mathcal{A}) \xrightarrow{B} HC^{n-1}(\mathcal{A}) \xrightarrow{S} HC^{n+1}(\mathcal{A}) \longrightarrow \cdots \tag{35}$$

The operators B and S can be made more explicit by finding the connecting homomorphisms in the above long exact sequences. Remarkably, there is a formula for *Connes' boundary operator* B on the level of cochains given by

$$B = Ns(1-\lambda) = NB_0,$$

where $B_0 : C^n \to C^{n-1}$ is defined by

$$B_0\varphi(a_0, \ldots, a_{n-1}) = \varphi(1, a_0, \ldots, a_{n-1}) - (-1)^n \varphi(a_0, \ldots, a_{n-1}, 1).$$

The operator $S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A})$ is called the *periodicity operator* and is in fact related to Bott periodicity. The *periodic cyclic cohomology* of $\mathcal{A}$ is defined as the direct limit under the operator S of cyclic cohomology groups:

$$HP^i(\mathcal{A}) = \varinjlim HC^{2n+i}(\mathcal{A}), \qquad i = 0, 1.$$

A typical application of (35) is to extract information about cyclic cohomology from Hochschild cohomology. For example, assume $f : \mathcal{A} \to \mathcal{B}$ is an algebra homomorphism such that $f^* : HH^n(\mathcal{B}) \to HH^n(\mathcal{A})$ is an isomorphism for all $n \geq 0$. Then, using the five lemma, we conclude that $f^* : HC^n(\mathcal{B}) \to HC^n(\mathcal{A})$ is an isomorphism for all n. In particular from Morita invariance of Hochschild cohomology one obtains the Morita invariance of cyclic cohomology.

6.2. *Connes' spectral sequence*

The cyclic complex (30) and the long exact sequence (35), as useful as they are, are not powerful enough for computations. A much deeper relation between Hochschild and cyclic cohomology groups is encoded in Connes' spectral sequence that we recall now. This spectral sequence resembles in many ways the Hodge to de Rham spectral sequence for complex manifolds. About this connection we shall say nothing in these notes but see [74] where a conjecture of Kontsevich and Soibelman about the degeneration of this spectral sequence is proved.

Let $\mathcal{A}$ be a unital algebra. Connes' (b, B)-bicomplex of $\mathcal{A}$ is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
C^2(\mathcal{A}) & \xrightarrow{B} & C^1(\mathcal{A}) & \xrightarrow{B} & C^0(\mathcal{A}) \\
{\scriptstyle b}\uparrow & & {\scriptstyle b}\uparrow & & \\
C^1(\mathcal{A}) & \xrightarrow{B} & C^0(\mathcal{A}) & & \\
{\scriptstyle b}\uparrow & & & & \\
C^0(\mathcal{A}) & & & &
\end{array}$$

Of the three relations

$$b^2 = 0, \quad bB + Bb = 0, \quad B^2 = 0,$$

only the middle relation is not obvious. But this follows from the relations $b's + sb' = 1$, $(1-\lambda)b = b'(1-\lambda)$ and $Nb' = bN$, already used in this section.

Theorem 6.1. (Connes [22]) *The map $\varphi \mapsto (0, \ldots, 0, \varphi)$ is a quasi-isomorphism of complexes*

$$(C_\lambda(\mathcal{A}),\, b) \to (Tot\mathcal{B}(\mathcal{A}),\, b + B)$$

This is a consequence of the vanishing of the E^2 term of the second spectral sequence (filtration by columns) of $\mathcal{B}(A)$. To prove this consider the short exact sequence of b-complexes

$$0 \longrightarrow \operatorname{Im} B \longrightarrow \operatorname{Ker} B \longrightarrow \operatorname{Ker} B/\operatorname{Im} B \longrightarrow 0$$

By a hard lemma of Connes ([22], Lemma 41), the induced map

$$H_b(\operatorname{Im} B) \longrightarrow H_b(\operatorname{Ker} B)$$

is an isomorphism. It follows that $H_b(\operatorname{Ker} B/\operatorname{Im} B)$ vanish. To take care of the first column one appeals to the fact that

$$\operatorname{Im} B \simeq \operatorname{Ker}(1-\lambda)$$

is the space of cyclic cochains.

6.3. *Topological algebras*

There is no difficulty in defining *continuous* analogues of Hochschild and cyclic cohomology groups for Banach algebras. One simply replaces bimodules by Banach bimodules (where the left and right module actions are bounded operators) and cochains by continuous cochains. Since the multiplication of a Banach algebra is a bounded map, all operators including the Hochschild boundary and the cyclic operator extend to this continuous setting. The resulting Hochschild and cyclic theory for Banach and C^*-algebras, however, is hardly useful and tends to vanish in many interesting examples. This is hardly surprising since the definition of any Hochschild and cyclic cocycle of dimension bigger than zero involves differentiating the elements of the algebra in one way or another. This is in sharp contrast with topological K-theory where the right setting is the setting of Banach or C^*-algebras.

Exercise 6.4. *Let X be a compact Hausdorff space. Show that any derivation $\delta : C(X) \longrightarrow C(X)$ is identically zero. (Hint: first show that if $f = g^2$ and $g(x) = 0$ for some $x \in X$, then $\delta(f)(x) = 0$.)*

Remark 6. By results of Connes and Haagerup (cf. [24] and references therein), we know that a C^*-algebra is *amenable* if and only if it is *nuclear.* Amenability refers to the property that for all $n \geq 1$,

$$H^n_{cont}(A, M^*) = 0,$$

for any Banach dual bimodule M^*. In particular, by using Connes' long exact sequence, we find that, for any nuclear C^*-algebra A,

$$HC^{2n}_{cont}(A) = A^*, \quad \text{and} \quad HC^{2n+1}_{cont}(A) = 0,$$

for all $n \geq 0$.

The right class of topological algebras for cyclic cohomology turns out to be the class of *locally convex algebras* [22]. An algebra $\mathcal{A}$ equipped with a locally convex topology is called a locally convex algebra if its multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is jointly continuous. Basic examples of locally convex algebras include the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on a closed manifold and the smooth noncommutative tori $\mathcal{A}_\theta$ and their higher dimensional analogues. The topology of $C^\infty(M)$ is defined by the sequence of seminorms

$$\|f\|_n = \sup |\partial^\alpha f|; \quad |\alpha| \leq n,$$

where the supremum is over a fixed, finite, coordinate cover for M (see the exercise below for the topology of $\mathcal{A}_\theta$).

Given locally convex topological vector spaces V_1 and V_2, their *projective tensor product* is a locally convex space $V_1 \hat{\otimes} V_2$ together with a universal jointly continuous bilinear map $V_1 \otimes V_2 \to V_1 \hat{\otimes} V_2$[65]. It follows from the universal property that for any locally convex space W, we have a natural isomorphism between continuous bilinear maps $V_1 \times V_2 \to W$ and continuous linear maps $V_1 \hat{\otimes} V_2 \to W$. One of the nice properties of the projective tensor product is that for smooth compact manifolds M and N, the natural map

$$C^\infty(M) \hat{\otimes} C^\infty(N) \to C^\infty(M \times N)$$

is an isomorphism.

A *topological left $\mathcal{A}$-module* is a locally convex topological vector space $\mathcal{M}$ endowed with a continuous left $\mathcal{A}$-module action $\mathcal{A} \times \mathcal{M} \to \mathcal{M}$. A *topological free* left $\mathcal{A}$-module is a module of the type $\mathcal{M} = \mathcal{A} \hat{\otimes} V$ where V is a locally convex space. A *projective* module is a module which is a direct summand in a free module.

Given a locally convex algebra $\mathcal{A}$, let

$$C^n_{\text{cont}}(\mathcal{A}) = \text{Hom}_{\text{cont}}(\mathcal{A}^{\hat{\otimes} n}, \mathbb{C})$$

be the space of continuous $(n+1)$-linear functionals on A and let $C^n_{\text{cont},\,\lambda}(\mathcal{A})$ denote the space of continuous cyclic cochains on $\mathcal{A}$. All the algebraic definitions and results of Sections 6.1 and 6.2 extend to this topological setting. In particular one defines topological Hochschild and cyclic cohomology groups of a locally convex algebra. The right class of topological projective and free resolutions are those resolutions that admit a continuous linear splitting. This extra condition is needed when one wants to prove comparison theorems for resolutions. We won't go into details here since this is very well explained in Connes' original article [22].

Exercise 6.5. *The sequence of norms*

$$p_k(a) = Sup\{(1 + |n| + |m|)^k |a_{mn}|\}$$

defines a locally convex topology on the smooth noncommutative torus $\mathcal{A}_\theta$. Show that the multiplication of $\mathcal{A}_\theta$ is continuous in this topology.

6.4. *The deformation complex*

What we called the Hochschild cohomology of $\mathcal{A}$ and denoted by $HH^n(\mathcal{A})$ is in fact the Hochschild cohomology of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-bimodule $\mathcal{A}^*$. In general, given an $\mathcal{A}$-bimodule $\mathcal{M}$, the Hochschild complex of $\mathcal{A}$ with coefficients in the bimodule $\mathcal{M}$ is the complex

$$C^0(\mathcal{A},\, \mathcal{M}) \xrightarrow{\delta} C^1(\mathcal{A},\, \mathcal{M}) \xrightarrow{\delta} C^2(\mathcal{A},\, \mathcal{M}) \longrightarrow \cdots$$

where $C^0(\mathcal{A},\, \mathcal{M}) = \mathcal{M}$ and $C^n(\mathcal{A},\, \mathcal{M}) = \mathrm{Hom}_{\mathbb{C}}\,(A^{\otimes n},\, \mathcal{M})$ is the space of n-linear functionals on $\mathcal{A}$ with values in $\mathcal{M}$. The differential δ is given by

$$\begin{aligned}(\delta\varphi)(a_1, \ldots, a_{n+1}) = a_1\varphi(a_2, \ldots, a_{n+1})& \\ + \sum_{i=1}^{n}(-1)^{i+1}\varphi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1})& \\ + (-1)^{n+1}\varphi(a_1, \ldots, a_n)a_{n+1}.&\end{aligned}$$

Two special cases are particularly important. For $\mathcal{M} = \mathcal{A}^*$, the linear dual of $\mathcal{A}$ with the bimodule action

$$(afb)(c) = f(bca)$$

for all a, b, c in $\mathcal{A}$ and $f \in \mathcal{A}^*$, we obtain the Hochschild groups $H^n(\mathcal{A}, \mathcal{A}^*) = HH^n(A)$. This is important in cyclic cohomology since as we saw it enters into a long exact sequence with cyclic groups. The second important case is when $\mathcal{M} = \mathcal{A}$ with bimodule structure given by left and right multiplication. The resulting complex $(C(\mathcal{A},\, \mathcal{A}),\, \delta)$ is called the *deformation complex* of $\mathcal{A}$. It is the complex that underlies the deformation theory of associative algebras as studied by Gerstenhaber [62].

There is a much deeper structure hidden in the deformation complex $(C(\mathcal{A},\, \mathcal{A}),\, \delta)$ than meets the eye and we will only barely scratch the surface. The first piece of structure is the cup product. The *cup product* $\cup : C^p \times C^q \to C^{p+q}$ is defined by

$$(f \cup g)(a^1, \ldots, a^{p+q}) = f(a^1, \ldots, a^p)g(a^{p+1}, \ldots, a^{p+q}).$$

Notice that $\cup$ is associative and one checks that this product is compatible with the differential δ and hence induces an associative graded product on $H(\mathcal{A},\, \mathcal{A}) = \oplus H^n(\mathcal{A},\, \mathcal{A})$. What is not so obvious however is that this product is graded commutative for any $\mathcal{A}$ [62].

The second piece of structure on $(C(\mathcal{A},\, \mathcal{A}),\, \delta)$ is a graded Lie bracket. It is based on the Gerstenhaber circle product $\circ : C^p \times C^q \longrightarrow C^{p+q-1}$

defined by

$$(f \circ g)(a_1, \dots, a_{p+q-1}) = \sum_{i=1}^{p-1} (-1)^{|g|(|f|+i-1)} f(a^1, \dots, g(a^i, \dots, a^{i+p}), \dots, a^{p+q-1}).$$

Notice that $\circ$ is not an associative product. Nevertheless one can show that [62] the corresponding graded bracket $[\,,\,] : C^p \times C^q \to C^{p+q-1}$

$$[f, g] = f \circ g - (-1)^{(p-1)(q-1)} g \circ f$$

defines a graded Lie algebra structure on deformation cohomology $H(\mathcal{A}, \mathcal{A})$. Notice that the Lie algebra grading is now shifted by one.

What is most interesting is that the cup product and the Lie algebra structure are compatible in the sense that $[\,,\,]$ is a graded derivation for the cup product; or in short $(H(\mathcal{A}, \mathcal{A}), \cup, [\,,\,])$ is a graded Poisson algebra.

The *fine structure* of the Hochschild cochain complex $(C(\mathcal{A}, \mathcal{A}), \delta)$, e.g. the existence of higher order products and homotopies between them is the subject of many studies in recent years [84, 85, 86]. While it is relatively easy to write down these higher order products in the form of a brace algebra structure on the Hochschild complex, relating them to known geometric structures such as moduli of curves is quite hard.

Remark 7. The graded Poisson algebra structure on deformation cohomology $H(\mathcal{A}, \mathcal{A})$ poses a natural question: is $H(\mathcal{A}, \mathcal{A})$ the semiclassical limit of a quantum cohomology theory for algebras?

Example 6.1. Let $\mathcal{A} = C^\infty(M)$, where M is a compact n-dimensional manifold. In [22] Connes gives a projective resolution of the topological left $\mathcal{A}\hat{\otimes}\mathcal{A}$-module $\mathcal{A}$,

$$\mathcal{A} \leftarrow \mathcal{M}_0 \leftarrow \mathcal{M}_1 \cdots \leftarrow \mathcal{M}_n \leftarrow 0 \tag{36}$$

where $\mathcal{M}_i$ is the space of smooth sections of the vector bundle $p_1^*(\bigwedge^i TM)$ and $p_1 : M \times M \to M$ is the projection on the second factor. After applying the $\mathrm{Hom}_{\mathcal{A}\hat{\otimes}\mathcal{A}}(-, \mathcal{A})$ functor to (36), one obtains a complex with zero differentials, which shows that

$$H^p(\mathcal{A}, \mathcal{A}) \simeq C^\infty(\bigwedge^p TM), \quad p = 0, 1, \dots.$$

The latter is the space of polyvector fields on M.

6.5. *Cyclic homology*

Cyclic cohomology is a contravariant functor on the category of algebras. There is a dual covariant theory called *cyclic homology* that we introduce now. The relation between the two is similar to the relation between currents and differential forms on manifolds.

For each $n \geq 0$, let $C_n(\mathcal{A}) = \mathcal{A}^{\otimes(n+1)}$. Define the operators

$$\begin{aligned} b : C_n(\mathcal{A}) &\longrightarrow C_{n-1}(\mathcal{A}) \\ b' : C_n(\mathcal{A}) &\longrightarrow C_{n-1}(\mathcal{A}) \\ \lambda : C_n(\mathcal{A}) &\longrightarrow C_n(\mathcal{A}) \\ s : C_n(\mathcal{A}) &\longrightarrow C_{n+1}(\mathcal{A}) \\ N : C_n(\mathcal{A}) &\longrightarrow C_n(\mathcal{A}) \\ B : C_n(\mathcal{A}) &\longrightarrow C_{n+1}(\mathcal{A}) \end{aligned}$$

by

$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad + (-1)^n (a_n a_0 \otimes a_1 \cdots \otimes a_{n-1}) \\ b'(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ \lambda(a_0 \otimes \cdots \otimes a_n) &= (-1)^n (a_n \otimes a_0 \cdots \otimes a_{n-1}) \\ s(a_0 \otimes \cdots \otimes a_n) &= (-1)^n (a_0 \otimes \cdots \otimes a_n \otimes 1) \\ N &= 1 + \lambda + \lambda^2 + \cdots + \lambda^n \\ B &= (1-\lambda) s N. \end{aligned}$$

They satisfy the relations

$$b^2 = 0, \qquad b'^2 = 0, \qquad (1-\lambda) b' = b(1-\lambda),$$
$$b' N = N b, \qquad B^2 = 0, \qquad bB + Bb = 0.$$

The complex $(C_\bullet(\mathcal{A}), b)$ is the Hochschild complex of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-bimodule $\mathcal{A}$. The complex

$$C_n^\lambda(\mathcal{A}) := C_n(\mathcal{A}) / \mathrm{Im}(1-\lambda)$$

is called the *Connes complex* of $\mathcal{A}$ for cyclic homology. Its homology, denoted by $HC_n(\mathcal{A})$, $n = 0, 1, \ldots$, is called the *cyclic homology* of $\mathcal{A}$. It is clear that the space of cyclic cochains is the linear dual of the space of cyclic chains

$$C^n_\lambda(\mathcal{A}) \simeq \mathrm{Hom}\,(C^\lambda_n(\mathcal{A}),\, \mathbb{C})$$

and

$$HC^n(\mathcal{A}) \simeq HC_n(\mathcal{A})^*.$$

Similar to cyclic cohomology, there is a long exact sequence relating Hochschild and cyclic homologies, and also there is a spectral sequence from Hochschild to cyclic homology. In particular cyclic homology can be computed using the following bicomplex.

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd}$$

Example 6.2. (Hochschild–Kostant–Rosenberg and Connes theorems) Let

$$\mathcal{A} \xrightarrow{d} \Omega^1\mathcal{A} \xrightarrow{d} \Omega^2\mathcal{A} \xrightarrow{d} \cdots$$

denote the de Rham complex of a commutative unital algebra $\mathcal{A}$. By definition $d : \mathcal{A} \to \Omega^1\mathcal{A}$ is a *universal derivation* into a symmetric $\mathcal{A}$-bimodule and $\Omega^n\mathcal{A} := \wedge^n_{\mathcal{A}}\Omega^1\mathcal{A}$ is the k-th exterior power of $\Omega^1\mathcal{A}$ over $\mathcal{A}$. One usually defines $\Omega^1\mathcal{A}$, the *module of Kähler differentials*, as I/I^2, where I is the kernel of the multiplication map $A \otimes A \to A$. d is then defined by

$$d(a) = a \otimes 1 - 1 \otimes a \mod(I^2).$$

The universal derivation d has a unique extension to a graded derivation of degree one on $\Omega\mathcal{A}$, denoted by d.

The *antisymmetrization map*

$$\varepsilon_n : \Omega^n\mathcal{A} \longrightarrow \mathcal{A}^{\otimes(n+1)}, \quad n = 0, 1, 2, \ldots,$$

is defined by

$$\varepsilon_n(a_0 da_1 \wedge \cdots \wedge da_n) = \sum_{\sigma \in S_n} sgn(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

where S_n is the symmetric group of order n. We also have a map

$$\mu_n : \mathcal{A}^{\otimes n} \longrightarrow \Omega^n\mathcal{A}, \quad n = 0, 1, \ldots$$

$$\mu_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 da_1 \wedge \cdots \wedge da_n.$$

One checks that the resulting maps

$$(\Omega\mathcal{A}, 0) \to (C(\mathcal{A}), b) \quad \text{and} \quad (C(\mathcal{A}), b) \to (\Omega\mathcal{A}, 0)$$

are morphisms of complexes, i.e.

$$b \circ \varepsilon_n = 0, \quad \mu_n \circ b = 0.$$

Moreover, one can easily check that

$$\mu_n \circ \varepsilon_n = n!\,\mathrm{Id}_n.$$

It follows that, for any commutative algebra $\mathcal{A}$, the antisymmetrization map induces an inclusion

$$\varepsilon_n : \Omega^n\mathcal{A} \hookrightarrow HH_n(\mathcal{A}),$$

for all n.

The Hochschild–Kostant–Rosenberg theorem [72] states that if $\mathcal{A}$ is a *regular algebra*, e.g. the algebra of regular functions on a smooth affine variety, then ε_n defines an algebra isomorphism

$$\varepsilon_n : \Omega^n\mathcal{A} \simeq HH_n(\mathcal{A})$$

between Hochschild homology of $\mathcal{A}$ and the algebra of differential forms on $\mathcal{A}$.

To compute the cyclic homology of $\mathcal{A}$, we first show that under the map μ the operator B corresponds to the de Rham differential d. More precisely, for each integer $n \geq 0$ we have a commutative diagram:

$$\begin{array}{ccc} C_n(\mathcal{A}) & \xrightarrow{\ \mu\ } & \Omega^n\mathcal{A} \\ \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle d} \\ C_{n+1}(\mathcal{A}) & \xrightarrow{\ \mu\ } & \Omega^{n+1}\mathcal{A} \end{array}$$

We have

$$\begin{aligned}
\mu B(f_0 \otimes \cdots \otimes f_n) &= \mu \sum_{i=0}^{n} (-1)^{ni} (1 \otimes f_i \otimes \cdots \otimes f_{i-1} \\
&\quad - (-1)^n f_i \otimes \cdots \otimes f_{i-1} \otimes 1) \\
&= \frac{1}{(n+1)!} \sum_{i=0}^{n} (-1)^{ni} df_i \cdots df_{i-1} \\
&= \frac{1}{(n+1)!} (n+1) df_0 \cdots df_n \\
&= d\mu(f_0 \otimes \cdots \otimes f_n).
\end{aligned}$$

It follows that μ defines a morphism of bicomplexes

$$\mathcal{B}(\mathcal{A}) \longrightarrow \Omega(\mathcal{A}),$$

where $\Omega(\mathcal{A})$ is the bicomplex

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
\Omega^2\mathcal{A} \arrow[d, "0"] & \Omega^1\mathcal{A} \arrow[l, "d"'] \arrow[d, "0"] & \Omega^0\mathcal{A} \arrow[l, "d"'] \\
\Omega^1\mathcal{A} \arrow[d, "0"] & \Omega^0\mathcal{A} \arrow[l, "d"'] & \\
\Omega^0\mathcal{A} & &
\end{tikzcd}$$

Since μ induces isomorphisms on row homologies, it induces isomorphisms on total homologies as well. Thus we have [22, 92]:

$$HC_n(\mathcal{A}) \simeq \Omega^n\mathcal{A}/\operatorname{Im} d \oplus H_{dR}^{n-2}(\mathcal{A}) \oplus \cdots \oplus H_{dR}^{k}(\mathcal{A}),$$

where $k = 0$ if n is even and $k = 1$ if n is odd.

Using the same map μ acting between the corresponding periodic complexes, one concludes that the periodic cyclic homology of $\mathcal{A}$ is given by

$$HP_k(\mathcal{A}) \simeq \bigoplus_i H_{dR}^{2i+k}(\mathcal{A}), \quad k = 0, 1.$$

By a completely similar method one can compute the *continuous cyclic homology* of the algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on a smooth closed manifold M. Here by continuous cyclic homology we mean the homology of the cyclic complex where instead of algebraic tensor products $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$, one uses the topological *projective tensor product* $\mathcal{A}\hat{\otimes} \cdots \hat{\otimes}\mathcal{A}$. The continuous Hochschild homology of $\mathcal{A}$ can be computed using Connes' topological resolution for $\mathcal{A}$ as an $\mathcal{A}$-bimodule as in Example (6.1). The result is

$$HH_n^{cont}(C^\infty(M)) \simeq \Omega^n M$$

with isomorphism induced by the map

$$f_0 \otimes f_1 \otimes \cdots \otimes f_n \mapsto f_0 df_1 \cdots df_n.$$

The rest of the computation of continuous cyclic homology follows the same pattern as in the case of regular algebras above. The end result is [22]:

$$HC_n^{\text{cont}}(C^\infty(M)) \simeq \Omega^n M/\operatorname{Im} d \oplus H_{dR}^{n-2}(M) \oplus \cdots \oplus H_{dR}^k(M),$$

and

$$HP_k^{cont}(C^\infty(M)) \simeq \bigoplus_i H_{dR}^{2i+k}(M), \quad k = 0, 1.$$

The cyclic (co)homology of (topological) algebras is computed in many cases. We refer to [22] for smooth noncommutative tori, to [15] for group algebras, and to [60, 104] for crossed product algebras. In [1] the spectral sequence for group crossed products has been extended to Hopf algebra crossed products. We refer to [46–48] for an alternative approach to cyclic (co)homoloy due to Cuntz and Quillen.

6.6. *Connes–Chern character*

We can now indicate Connes' generalization of the pairing between $K_0(\mathcal{A})$ and traces on $\mathcal{A}$ (Example (5.3)) to a full fledged pairing between K-theory and cyclic cohomology:

$$K_0(\mathcal{A}) \times HC^{2n}(\mathcal{A}) \longrightarrow \mathbb{C}, \tag{37}$$

$$K_1^{\text{alg}}(\mathcal{A}) \times HC^{2n+1}(\mathcal{A}) \longrightarrow \mathbb{C}. \tag{38}$$

These maps are defined for all $n \geq 0$ and are compatible with the S-operation on cyclic cohomology. In the dual setting of cyclic homology, these pairings translate into noncommutative *Connes-Chern characters*

$$\mathrm{Ch}_0^{2n} : K_0(\mathcal{A}) \longrightarrow HC_{2n}(\mathcal{A}),$$
$$\mathrm{Ch}_1^{2n+1} : K_1^{\text{alg}}(\mathcal{A}) \longrightarrow HC_{2n+1}(\mathcal{A}),$$

compatible with the S-operation on cyclic homology. As a consequence of compatibility with the periodicity operator S, we obtain maps

$$\begin{aligned} \mathrm{Ch}_0 &: K_0(\mathcal{A}) \longrightarrow HP_0(\mathcal{A}), \\ \mathrm{Ch}_1 &: K_1^{\mathrm{alg}}(\mathcal{A}) \longrightarrow HP_1(\mathcal{A}). \end{aligned}$$

For $\mathcal{A} = C^\infty(M)$, these maps reduce to the classical Chern character as defined via the connection and curvature formalism of Chern–Weil theory [100].

The definition of these pairings rest on the following three facts [22, 24, 92]:

(1) For any $k \geq 1$, the map $\varphi \mapsto \varphi_k$ from $C^n(\mathcal{A}) \to C^n(M_k(\mathcal{A}))$ defined by

$$\varphi_k(m_0 \otimes a_0, m_1 \otimes a_1, \ldots, m_n \otimes a_n) = \mathrm{Tr}\,(m_0 m_1 \cdots m_n)\varphi(a_0, a_1, \ldots, a_n)$$

commutes with the operators b and λ. It follows that if φ is a cyclic cocycle on $\mathcal{A}$, then φ_k is a cyclic cocycle on $M_k(\mathcal{A})$.

(2) Inner automorphisms act by the identity on Hochschild and hence on cyclic cohomology.

(3) (normalization) The inclusion of *normalized cochains* $C_\lambda^{\mathrm{norm}}(\mathcal{A}) \to C_\lambda(\mathcal{A})$ is a quasi-isomorphism in dimensions $n \geq 1$. A cyclic cochain φ is called *normalized* if $\varphi(a_0, \ldots, a_n) = 0$ if $a_i = 1$ for some i.

Now let $e \in M_k(A)$ be an idempotent and $[\varphi] \in HC^{2n}(\mathcal{A})$. The pairing (37) is defined by the bilinear map

$$\langle [\varphi],\, [e] \rangle = \varphi_k(e, \ldots, e).$$

Let us first check that the value of the pairing depends only on the cyclic cohomology class of φ. It suffices to assume $k = 1$ (why?). Let $\varphi = b\psi$ with $\psi \in C_\lambda^{2n-1}(A)$. Then we have

$$\begin{aligned} \varphi(e, \ldots, e) &= b\psi(e, \ldots, e) \\ &= \psi(ee, e, \ldots, e) - \psi(e, ee, \ldots, e) + \cdots + (-1)^{2n}\psi(ee, e, \ldots, e) \\ &= \psi(e, \ldots, e) \\ &= 0, \end{aligned}$$

where the last relation follows from the cyclic property of ψ. The pairing is clearly invariant under the inclusion $M_k(\mathcal{A}) \to M_{k+1}(\mathcal{A})$.

It remains to show that the value of $\langle [\varphi]\,, [e] \rangle$, for fixed φ, only depends on the class of $[e] \in K_0(\mathcal{A})$. It suffices to check that for $u \in GL_k(\mathcal{A})$, $\langle [\varphi],\, [e] \rangle = \langle [\varphi],\, [ueu^{-1}] \rangle$. But this is exactly fact 2) above.

Exercise 6.6. *Let $e \in \mathcal{A}$ be an idempotent. Show that*

$$Ch_0^{2k}(e) := (-1)^k \frac{(2k)!}{k!} tr\left(\left(e - \frac{1}{2}\right) \otimes e^{\otimes(2k+1)}\right), \quad k = 0, \ldots, n,$$

defines a cycle in the (b, B)-bicomplex of $\mathcal{A}$. This is the formula for the Connes-Chern character in the bicomplex picture of cyclic homology.

Dually, given a cocycle $\varphi = (\varphi_0, \varphi_2, \ldots, \varphi_2 n)$ in the (b, B)-bicomplex, its pairing with an idempotent $e \in M_k(\mathcal{A})$ is given by

$$\langle[\varphi],\ [e]\rangle = \sum_{k=1}^{n} (-1)^k \frac{k!}{(2k)!} \varphi_{2k}\left(e - \frac{1}{2}, e, \ldots, e\right).$$

Given a normalized cyclic cocycle $\varphi \in HC^{2n+1}(\mathcal{A})$ and an invertible $u \in GL(k, A)$, let

$$\langle[\varphi],\ [u]\rangle = \varphi_k(u, u^{-1}, \ldots, u, u^{-1}).$$

It can be shown (cf. [22], Part II) that the above formula defines a pairing between K_1^{alg} and HC^{2n+1}.

Exercise 6.7. *Given an invertible $u \in GL(n, \mathcal{A})$, show that*

$$Ch_1^{2k+1} := (-1)^k k!\, Tr(u^{-1} \otimes u)^{\otimes 2k}$$

defines a cycle in the normalized (b, B)-bicomplex of $\mathcal{A}$. This is the formula for the Connes–Chern character in the (b, B)-bicomplex picture of cyclic homology. Dually, given a normalized (b, B)-cocycle $\varphi = (\varphi_1, \ldots, \varphi_{2n+1})$, the formula

$$\langle[\varphi],\ [u]\rangle = \sum_{k=1}^{n} (-1)^k \varphi_{2k+1}(u, u^{-1}, \ldots, u, u^{-1})$$

defines the pairing between K_1^{alg} and HC^{2n+1}.

It often happens that an element of $K_0(\mathcal{A})$ is represented by a finite projective module and not by an explicit idempotent. It is then important to have a formalism that would give the value of its pairing with cyclic cocycles. This is based on a noncommutative version of Chern–Weil theory developed by Connes in [18, 22] that we sketch next.

Let $\mathcal{E}$ be a finite projective right $\mathcal{A}$-module, $(\Omega,\ d)$ a differential calculus on $\mathcal{A}$ and let $\int : \Omega^{2n} \to \mathbb{C}$ be a closed graded trace representing a cyclic cocycle φ on $\mathcal{A}$. Thanks to its projectivity, $\mathcal{E}$ admits a *connection*, i.e. a degree one map

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega \to \mathcal{E} \otimes_{\mathcal{A}} \Omega$$

which satisfies the graded Leibniz rule

$$\nabla(\xi\omega) = \nabla(\xi)\omega + (-1)^{\deg \xi}\xi d\omega$$

with respect to the right Ω-module structure on $\mathcal{E} \otimes_{\mathcal{A}} \Omega$. The *curvature* of ∇ is the operator ∇^2, which can be easily checked to be Ω-linear,

$$\nabla^2 \in \operatorname{End}_{\Omega}(E \otimes_{\mathcal{A}} \Omega) = \operatorname{End}_{\mathcal{A}}(\mathcal{E}) \otimes \Omega.$$

Now since $\mathcal{E}$ is finite projective over $\mathcal{A}$ it follows that $\mathcal{E} \otimes_{\mathcal{A}} \Omega$ is finite projective over Ω and therefore the trace $\int : \Omega \to \mathbb{C}$ extends to a trace, denoted again by $\int$, on $\operatorname{End}_{\mathcal{A}}(\mathcal{E}) \otimes \Omega$ (cf. formula (21)). The following result of Connes relates the value of the pairing as defined above to its value computed through the Chern-Weil formalism:

$$\langle [\mathcal{E}],\, [\varphi] \rangle = \frac{1}{n!} \int \nabla^n.$$

Example 6.3. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing functions on the real line. The operators $u, v : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined by

$$(uf)(x) = f(x - \theta), \qquad (vf)(x) = e^{2\pi i \theta} f(x)$$

satisfy the relation $uv = e^{2\pi i\theta} vu$ and hence turn $\mathcal{S}(\mathbb{R})$ into a right $\mathcal{A}_\theta$-module via the maps $U \mapsto u$, $V \mapsto v$. We denote this module by $\mathcal{E}_{0,1}$. It is the simplest of a series of modules $\mathcal{E}_{p,q}$ defined by Connes in [18]. It turns out that $\mathcal{E}_{0,1}$ is finite projective, and for the canonical trace τ on $\mathcal{E}_{0,1}$ we have

$$\langle \tau,\, \mathcal{E}_{0,1} \rangle = -\theta.$$

Using the two derivations δ_1, δ_2 as a basis for "invariant vector fields" one can define a differential calculus $\Omega^0 \oplus \Omega^1 \oplus \Omega^2$ on $\mathcal{A}_\theta$ with $\Omega^i = \bigwedge^i \{de_1, de_2\} \otimes \mathcal{A}_\theta$. A connection on $\mathcal{E}$ with respect to this calculus is a pair of operators $\nabla_1,\ \nabla_2 : \mathcal{E} \to \mathcal{E}$ satisfying

$$\nabla_j(\xi a) = (\nabla_j)(\xi a) + \xi \delta_j(a)$$

for all $\xi \in \mathcal{E}_{0,1}$ and $a \in \mathcal{A}_\theta$ and $j = 1, 2$. One can check that the following formula defines a connection on $\mathcal{E}_{0,1}$ [18, 24]:

$$\nabla_1(\xi)(s) = -\frac{s}{\theta}\, \xi(s), \qquad \nabla_2(\xi)(s) = \frac{d\xi}{ds}(s).$$

The curvature of this connection is constant and is given by $\nabla^2 = [\nabla_1, \nabla_2] = \frac{1}{\theta} I \in \operatorname{End}_{\mathcal{A}}(\mathcal{E}_{0,1})$.

Remark 8. Chern–Weil theory is a theory of characteristic classes for smooth principal G-bundles, where G is a Lie group. The above theory for noncommutative vector bundles should be generalized to noncommutative analogues of principal bundles. A good point to start would be the theory of Hopf–Galois extensions.

In the remainder of this section we shall briefly introduce Connes' Chern character in K-homology (cf. [22, 24] for a full account). In fact one of the main reasons for introducing cyclic cohomology was to define a Chern character in K-homology [19]. In the even case, let (H, F) be an even p-summable Fredholm module over an algebra $\mathcal{A}$ as in definition (2.2). For each even integer $n \geq p-1$, define an n-cochain φ_n on $\mathcal{A}$ by

$$\varphi_n(a_0, \ldots, a_n) = \mathrm{Trace}\,(\varepsilon\, a_0[F, a_1] \cdots [F, a_n]).$$

The p-summability condition on (H, F) ensures that the above product of commutators is in fact a trace class operator and φ_n is finite for all $n \geq p-1$ (this is obvious if $n > p-1$; in general one has to manipulate the commutators a bit to prove this).

Exercise 6.8. *Show that φ_n is a cyclic cocycle. Also show that if n is odd then $\varphi_n = 0$.*

Although this definition depends on n, it can be shown that the cyclic cocycles φ_n are related to each other by the periodicity operator S,

$$S\varphi_n = \varphi_{n+2},$$

and therefore define an even periodic cyclic cohomology class. This is Connes' Chern character of a K-homology class in the even case. For applications it is important to have an index formula which computes the value of the pairing $\langle[\varphi_n], [e]\rangle$ as the index of a Fredholm operator similar to Example 2.11. We refer to [22, 24] for this.

Let A be a nuclear C^*-algebra. In the odd case, smooth p-summable elements of K-homology group $K^1(A)$ can be represented either by smooth Brown-Douglas-Fillmore extensions

$$0 \longrightarrow \mathcal{L}^p \longrightarrow \mathcal{E} \longrightarrow \mathcal{A} \longrightarrow 0$$

or by Kasparov modules. We refer to [22] for the definition of the Connes-Chern character in the odd case. We have already met one example of this though in the case of smooth Toeplitz extension

$$0 \longrightarrow \mathcal{K}^\infty \longrightarrow \mathcal{T}^\infty \longrightarrow C^\infty(S^1) \longrightarrow 0.$$

The map $f \to T_f$, sending a function on the circle to the corresponding Toeplitz operator, is a section for the symbol map σ. The extension is p-summable for all $p \geq 1$. Its Connes character is represented by the cyclic 1-cocycle on $C^\infty(S^1)$ defined by

$$\varphi_1(f,\, g) = \mathrm{Tr}\,([T_f,\, T_g]).$$

6.7. *Cyclic modules*

Cyclic cohomology of algebras was first defined by Connes through explicit complexes or bicomplexes [19, 22]. Soon after he introduced the notion of *cyclic module* and defined its cyclic cohomology [21]. Later developments proved that this extension was of great significance. Apart from earlier applications, here we have the very recent work [37] in mind where the abelian category of cyclic modules plays the role of the category of motives in noncommutative geometry. Another recent example is the cyclic cohomology of Hopf algebras [39, 40, 67, 68], which cannot be defined as the cyclic cohomology of an algebra or a coalgebra but only as the cyclic cohomology of a cyclic module naturally attached to the given Hopf algebra.

The original motivation of [21] was to define cyclic cohomology of algebras as a derived functor. Since the category of algebras and algebra homomorphisms is not even an additive category (for the simple reason that the sum of two algebra homomorphisms is not an algebra homomorphism in general), the standard (abelian) homological algebra is not applicable. In Connes' approach, the category Λ_k of cyclic k-modules appears as an "abelianization" of the category of k-algebras. Cyclic cohomology is then shown to be the derived functor of the functor of traces, as we explain in this section.

The *simplicial category* Δ is the category whose objects are totally ordered sets

$$[n] = \{0 < 1 < \cdots < n\},$$

for $n = 0, 1, 2, \ldots$. A morphism $f : [n] \to [m]$ is an order preserving, i.e. monotone non-decreasing, map $f : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$. Of particular interest among the morphisms of Δ are *faces* δ_i and *degeneracies* σ_i,

$$\delta_i : [n-1] \to [n], \quad \sigma_i : [n] \to [n-1], \qquad i = 1, 2, \ldots.$$

By definition δ_i is the unique injective morphism missing i and σ_i is the unique surjective morphism identifying i with $i+1$. It can be checked that they satisfy the following *simplicial identities*:

$$
\begin{aligned}
\delta_j\delta_i &= \delta_{j-1}\delta_i \quad \text{if} \quad i < j,\\
\sigma_i\sigma_i &= \sigma_i\sigma_i \qquad \text{if} \quad i < j,\\
\sigma_i\delta_i &= \begin{cases} \sigma_{j-1}\delta_i & i < j \\ \mathrm{id} & i = j \text{ or } i = j+1 \\ \sigma_j\delta_{i-1} & i > j+1. \end{cases}
\end{aligned}
$$

Every morphism of Δ can be uniquely decomposed as a product of faces followed by a product of degeneracies.

The cyclic category Λ has the same set of objects as Δ and in fact contains Δ as a subcategory. An, unfortunately unintuitive, definition of its morphisms is as follows (see [24] for a more intuitive definition in terms of homotopy classes of maps from $S^1 \to S^1$). Morphisms of Λ are generated by simplicial morphisms δ_i, σ_i as above and $\tau_n : [n] \to [n]$ for $n \geq 0$. They are subject to the above simplicial as well as the following extra relations:

$$
\begin{aligned}
\tau_n\delta_i &= \delta_{i-1}\tau_{n-1} \quad 1 \leq i \leq n\\
\tau_n\delta_0 &= \delta_n\\
\tau_n\sigma_i &= \sigma_{i-1}\tau_{n-1} \quad 1 \leq i \leq n\\
\tau_n\sigma_0 &= \sigma_n\tau_{n+1}^2\\
\tau_n^{n+1} &= \mathrm{id}.
\end{aligned}
$$

A *cyclic object* in a category $\mathcal{C}$ is a functor $\Lambda^{\mathrm{op}} \to \mathcal{C}$. A *cocyclic object* in $\mathcal{C}$ is a functor $\Lambda \to \mathcal{C}$. For any commutative ring k, we denote the category of cyclic k-modules by Λ_k. A morphism of cyclic k-modules is a natural transformation between the corresponding functors. Equivalently, a morphism $f : X \to Y$ consists of a sequence of k-linear maps $f_n : X_n \to Y_n$ compatible with the face, degeneracy, and cyclic operators. It is clear that Λ_k is an abelian category. The kernel and cokernel of a morphism f is defined pointwise: $(\operatorname{Ker} f)_n = \operatorname{Ker} f_n : X_n \to Y_n$ and $(\operatorname{Coker} f)_n = \operatorname{Coker} f_n : X_n \to Y_n$. More generally, if $\mathcal{A}$ is any abelian category then the category $\Lambda\mathcal{A}$ of cyclic objects in $\mathcal{A}$ is itself an abelian category.

Let Alg_k denote the category of unital k-algebras and unital algebra homomorphisms. There is a functor

$$\natural : Alg_k \longrightarrow \Lambda_k,$$

defined as follows. To an algebra A, we associate the cyclic module $A^\natural$ defined by $A_n^\natural = A^{\otimes(n+1)}, n \geq 0$, with face, degeneracy and cyclic operators given by

$$\begin{aligned}
\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \\
\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \\
\sigma_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n, \\
\tau_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n \otimes a_0 \cdots \otimes a_{n-1}.
\end{aligned}$$

A unital algebra map $f : A \to B$ induces a morphism of cyclic modules $f^\natural : A^\natural \to B^\natural$ by $f^\natural(a_0 \otimes \cdots \otimes a_n) = f(a_0) \otimes \cdots \otimes f(a_n)$.

Example 6.4. We have

$$\mathrm{Hom}_{\Lambda_k}(A^\natural,\ k^\natural) \simeq T(A),$$

where $T(A)$ is the space of traces from $A \to k$. Under this isomorphism a trace τ is sent to the cyclic map $(f_n)_{n\geq 0}$, where

$$f_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \tau(a_0 a_1 \cdots a_n), \quad n \geq 0.$$

Now we can state the following fundamental theorem of Connes [21]:

Theorem 6.2. *For any unital k-algebra A, there is a canonical isomorphism*

$$HC^n(A) \simeq Ext^n_{\Lambda_k}(A^\natural, k^\natural), \qquad \textit{for all } n \geq 0.$$

Now the above Example and Theorem, combined together, say that cyclic cohomology is the derived functor of the functor of traces $A \to T(A)$ where the word derived functor is understood to mean as above.

Motivated by the above theorem, one defines the cyclic cohomology and homology of any cyclic module M by

$$HC^n(M) := \mathrm{Ext}^n_{\Lambda_k}(M, k^\natural),$$

and

$$HC_n(M) := \mathrm{Tor}_n^{\Lambda_k}(M, k^\natural),$$

One can use the injective resolution used to prove the above Theorem to show that these Ext and Tor groups can be computed by explicit complexes and bicomplexes, similar to the situation with algebras. For example one has the following first quadrant bicomplex, called the *cyclic bicomplex* of M

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
M_2 & \xleftarrow{1-\lambda} & M_2 & \xleftarrow{N} & M_2 & \xleftarrow{1-\lambda} & \dots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_1 & \xleftarrow{1-\lambda} & M_1 & \xleftarrow{N} & M_1 & \xleftarrow{1-\lambda} & \dots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_0 & \xleftarrow{1-\lambda} & M_0 & \xleftarrow{N} & M_0 & \xleftarrow{1-\lambda} & \dots
\end{array}$$

whose total homology is naturally isomorphic to cyclic homology. Here the operator $\lambda : M_n \to M_n$ is defined by $\lambda = (-1)^n \tau_n$, while

$$b = \sum_{i=0}^{n} (-1)^i \delta_i, \qquad b' = \sum_{i=0}^{n-1} (-1)^i \delta_i,$$

and $N = \sum_{i=0}^{n} \lambda^i$. Using the simplicial and cyclic relations, one can check that $b^2 = b'^2 = 0$, $b(1-\lambda) = (1-\lambda)b'$ and $b'N = Nb'$. These relations amount to saying that the above is a bicomplex.

The (b, B)-bicomplex of a cyclic module is the bicomplex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
M_2 & \xleftarrow{B} & M_1 & \xleftarrow{B} & M_0 \\
\downarrow b & & \downarrow b & & \\
M_1 & \xleftarrow{B} & M_0 & & \\
\downarrow b & & & & \\
M_0 & & & &
\end{array}$$

whose total homology is again isomorphic to the cyclic homology of M (this time we have to assume that k is a field of characteristic 0). Here $B : M_n \to M_{n+1}$ is Connes' boundary operator defined by $B = (1-\lambda)sN$, where $s = (-1)^n \sigma_n$.

A remarkable property of the cyclic category Λ, not shared by the simplicial category, is its *self-duality* in the sense that there is a natural isomorphism of categories $\Lambda \simeq \Lambda^{\mathrm{op}}$. Roughly speaking, Connes' duality functor $\Lambda^{\mathrm{op}} \longrightarrow \Lambda$ acts as the identity on objects of Λ and exchanges face and degeneracy operators while sending the cyclic operator to its inverse. Thus to a cyclic (resp. cocyclic) module one can associate a cocyclic (resp. cyclic) module by applying Connes' duality isomorphism. In the next section we shall see examples of cyclic modules in Hopf cyclic (co)homology that are dual to each other in the above sense.

6.8. *Hopf cyclic cohomology*

In their fundamental work on index theory of transversally elliptic operators [39], Connes and Moscovici developed a new cohomology theory for Hopf algebras based on ideas in cyclic cohomology. This theory can be regarded as the right noncommutative analogue of both group and Lie algebra homology, although this was not the original motivation behind it. Instead, the main reason was to obtain a *noncommutative characteristic map*

$$\chi_\tau : HC^*_{(\delta,\sigma)}(H) \longrightarrow HC^*(A), \tag{39}$$

for an action of a Hopf algebra H on an algebra A endowed with an "invariant trace" $\tau : A \to \mathbb{C}$. Here, the pair (δ, σ) consists of a grouplike element $\sigma \in H$ and a character $\delta : H \to \mathbb{C}$ satisfying certain compatibility conditions to be discussed later in this section. While in this section we confine ourselves to Hopf algebras, we refer to the recent surveys [43] and [82] for later developments in the subject inspired by [39].

The characteristic map (39) is induced, on the level of cochains, by a map

$$\chi_\tau : H^{\otimes n} \longrightarrow C^n(A)$$

defined by

$$\chi_\tau(h_1 \otimes \cdots \otimes h_n)(a^0, \ldots, a^n) = \tau(a_0 h_1(a^1) \cdots h_n(a^n)). \tag{40}$$

Maps like this have quite a history in cyclic cohomology, going back to [18]. Notice, for example, that the fundamental cyclic cocycles (31), (32) on the noncommutative 2-torus are of this form, where H is the enveloping algebra of a two-dimensional abelian Lie algebra acting by a pair of commuting derivations δ_1 and δ_2 on $\mathcal{A}_\theta$.

Exercise 6.9. *Let $\delta_1, \ldots, \delta_p$ be a commuting family of derivations on an algebra $\mathcal{A}$ and let $\tau : \mathcal{A} \to \mathbb{C}$ be an invariant trace, i.e. $\tau(\delta_i(a)) = 0$ for all $a \in \mathcal{A}$ and $i = 1, \ldots, p$. Show that*

$$\varphi(a^0, \ldots, a^p) = \sum_{\sigma \in S_n} (-1)^\sigma \tau(a^0 \delta_{\sigma_1}(a^1) \cdots \delta_{\sigma_p}(a^p))$$

is a cyclic p-cocycle on $\mathcal{A}$.

Applied to higher dimensional noncommutative tori, these cocycles give a basis for its periodic cyclic cohomology.

For applications to transverse geometry and number theory [41–43], it is important to formulate a notion of "invariant trace" under the presence of a modular pair. Let A be an H-module algebra, δ a character of H, and $\sigma \in H$ a grouplike element. A linear map $\tau : A \to \mathbb{C}$ is called δ-*invariant* if for all $h \in H$ and $a \in A$,

$$\tau(h(a)) = \delta(h)\tau(a).$$

τ is called a σ-*trace* if for all a, b in A,

$$\tau(ab) = \tau(b\sigma(a)).$$

For $a, b \in A$, let

$$\langle a,\, b\rangle := \tau(ab).$$

Then the δ-invariance property of τ is equivalent to the *integration by parts formula*:

$$\langle h(a),\, b\rangle = \langle a,\, \widetilde{S}_\delta(h)(b)\rangle, \tag{41}$$

where the δ-*twisted antipode* $\widetilde{S}_\delta : H \to H$ is defined by $\widetilde{S}_\delta = \delta * S$. That is,

$$\widetilde{S}_\delta(h) = \delta(h^{(1)})S(h^{(2)}).$$

Loosely speaking, this amounts to saying that the formal adjoint of the differential operator h is $\widetilde{S}_\delta(h)$. Following [39, 40], we say $(\delta,\, \sigma)$ is a *modular pair* if $\delta(\sigma) = 1$, and a *modular pair in involution* if

$$\widetilde{S}_\delta^2(h) = \sigma h \sigma^{-1},$$

for all h in H.

Examples 6.2. 1. For any commutative or cocommutative Hopf algebra we have $S^2 = 1$. It follows that $(\varepsilon, 1)$ is a modular pair in involution.

2. The original non-trivial example of a modular pair in involution is the pair $(\delta,\, 1)$ for the Connes-Moscovici Hopf algebra $\mathcal{H}_1$. Let δ denote the unique extension of the modular character

$$\delta : \mathfrak{g}_{\text{aff}} \to \mathbb{R}; \quad \delta(X) = 1, \quad \delta(Y) = 0,$$

to a character $\delta : U(\mathfrak{g}_{\text{aff}}) \to \mathbb{C}$. There is a unique extension of δ to a character, denoted by the same symbol $\delta : \mathcal{H}_1 \to \mathbb{C}$. Indeed the relations $[Y, \delta_n] = n\delta_n$ show that we must have $\delta(\delta_n) = 0$, for $n = 1, 2, \ldots$. One can then check that these relations are compatible with the algebra structure of $\mathcal{H}_1$.

Now the algebra $A_\Gamma = C_0^\infty(F^+(M)) \rtimes \Gamma$ from Section 4.2 admits a δ-invariant trace $\tau : A_\Gamma \to \mathbb{C}$ under its canonical $\mathcal{H}_1$ action. It is given by [39]:

$$\tau(fU_\varphi^*) = \int_{F^+(M)} f(y, y_1) \frac{dydy_1}{y_1^2}, \quad \text{if } \varphi = 1,$$

and $\tau(fU_\varphi^*) = 0$, otherwise.

3. Let $H = A(SL_q(2))$ denote the Hopf algebra of functions on quantum $SL(2)$. As an algebra it is generated by $x,\ u,\ v,\ y$, subject to the relations

$$ux = qxu, \;\; vx = qxv, \;\; yu = quy, \;\; yv = qvy,$$

$$uv = vu, \;\; xy - q^{-1}uv = yx - quv = 1.$$

The coproduct, counit and antipode of H are defined by

$$\Delta(x) = x \otimes x + u \otimes v, \quad \Delta(u) = x \otimes u + u \otimes y,$$

$$\Delta(v) = v \otimes x + y \otimes v, \quad \Delta(y) = v \otimes u + y \otimes y,$$

$$\epsilon(x) = \epsilon(y) = 1, \quad \epsilon(u) = \epsilon(v) = 0,$$

$$S(x) = y, \;\; S(y) = x, \;\; S(u) = -qu, \;\; S(v) = -q^{-1}v.$$

Define a character $\delta : H \to \mathbb{C}$ by

$$\delta(x) = q, \;\; \delta(u) = 0, \;\; \delta(v) = 0, \;\; \delta(y) = q^{-1}.$$

One checks that $\widetilde{S}_\delta^2 = \mathrm{id}$. This shows that $(\delta,\, 1)$ is a modular pair in involution for H. This example and its Hopf cyclic cohomology are studied in [80].

More generally, it is shown in [40] that *coribbon Hopf algebras* and compact quantum groups are endowed with canonical modular pairs in involution of the form $(\delta,\, 1)$ and, dually, ribbon Hopf algebras have canonical modular pairs in involution of the type $(1, \sigma)$.

4. It is shown in [67] that modular pairs in involution are in fact one-dimensional examples of *stable anti-Yetter-Drinfeld modules* over Hopf algebras introduced there. These modules are noncommutative coefficient systems for the general Hopf cyclic cohomology theory developed in [68].

Now let $(H,\, \delta,\, \sigma)$ be a Hopf algebra endowed with a modular pair in involution. In [39] Connes and Moscovici attach a cocyclic module $H^\natural_{(\delta,\sigma)}$ to this data as follows. Let

$$H^{\natural,0}_{(\delta,\sigma)} = \mathbb{C}, \quad \text{and} \quad H^{\natural,n}_{(\delta,\sigma)} = H^{\otimes n}, \quad \text{for} \;\; n \geq 1.$$

Its face, degeneracy and cyclic operators δ_i, σ_i, and τ_n are defined by

$$\begin{aligned}
\delta_0(h_1 \otimes \cdots \otimes h_n) &= 1 \otimes h_1 \otimes \cdots \otimes h_n \\
\delta_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_n \ \text{ for } \ 1 \le i \le n \\
\delta_{n+1}(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_n \otimes \sigma \\
\sigma_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \epsilon(h_{i+1}) \otimes \cdots \otimes h_n \ \text{ for } \ 0 \le i \le n \\
\tau_n(h_1 \otimes \cdots \otimes h_n) &= \Delta^{n-1}\widetilde{S}(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma).
\end{aligned}$$

The cyclic cohomology of the cocyclic module $H^\natural_{(\delta,\sigma)}$ is called the Hopf cyclic cohomology of the triple (H, δ, σ) and will be denoted by $HC^n_{(\delta,\sigma)}(H)$.

Examples 6.3. 1. For $H = \mathcal{H}_n$, the Connes–Moscovici Hopf algebra, we have [39]

$$HP^n_{(\delta,1)}(\mathcal{H}_n) \simeq \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i(\mathfrak{a}_n, \mathbb{C})$$

where $\mathfrak{a}_n$ is the Lie algebra of formal vector fields on $\mathbb{R}^n$.

2. For $H = U(\mathfrak{g})$ the enveloping algebra of a Lie algebra $\mathfrak{g}$, we have [39]

$$HP^n_{(\delta,1)}(H) \cong \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H_i(\mathfrak{g}, \mathbb{C}_\delta)$$

3. For $H = \mathbb{C}[G]$ the coordinate ring of a nilpotent affine algebraic group G, we have [39]

$$HP^n_{(\epsilon,1)}(H) \cong \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i(\mathfrak{g}, \mathbb{C}),$$

where $\mathfrak{g} = Lie(G)$.

4. If H admits a normalized left Haar integral, then [45]

$$HP^1_{(\delta,\sigma)}(H) = 0, \qquad HP^0_{(\delta,\sigma)}(H) = \mathbb{C}.$$

Recall that a linear map $\int : H \to \mathbb{C}$ is called a normalized left Haar integral if for all $h \in H$, $\int h = \int (h^{(1)})h^{(2)}$ and $\int 1 = 1$. It is known that a Hopf algebra defined over a field admits a normalized left Haar integral if and only if it is cosemisimple [118]. Compact quantum groups and group algebras are known to admit a normalized Haar integral in the above sense. In the latter case $\int : \mathbb{C}G \to k$ sending $g \mapsto 0$ for all $g \neq e$ and $e \mapsto 1$ is a Haar integral. Note that G need not be finite. In this regard, we should also mention that there are interesting examples of finite-dimensional non-cosemisimple Hopf algebras defined as quantum groups at roots of unity. Nothing is known about the cyclic (co)homology of these Hopf algebras.

5. If $H = U_q(\mathrm{sl}_2)$ is the quantum universal algebra of sl_2, we have [45]

$$HP^0_{(\epsilon,\sigma)}(H) = 0, \quad HP^1_{(\epsilon,\sigma)}(H) = \mathbb{C} \oplus \mathbb{C}.$$

6. Let H be a commutative Hopf algebra. The periodic cyclic cohomology of the cocyclic module $\mathcal{H}^\natural_{(\epsilon,1)}$ can be computed in terms of the Hochschild homology of the coalgebra H with trivial coefficients.

Proposition 6.1. ([80]) *Let H be a commutative Hopf algebra. Its periodic cyclic cohomology in the sense of Connes-Moscovici is given by*

$$HP^n_{(\epsilon,1)}(H) = \bigoplus_{i=n \pmod 2} H^i(H, \mathbb{C}).$$

For example, if $H = \mathbb{C}[G]$ is the algebra of regular functions on an affine algebraic group G, the coalgebra complex of H is isomorphic to the group cohomology complex of G where instead of regular cochains one uses regular functions $G \times G \times \cdots \times G \to k\mathbb{C}$. Denote this cohomology by $H^i(G, \mathbb{C})$. It follows that

$$HP^n_{(\epsilon,1)}(\mathbb{C}[G]) = \bigoplus_{i=n \pmod 2} H^i(G, \mathbb{C}).$$

As is remarked in [80], when the Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ is nilpotent, it follows from Van Est's theorem that $H^i(G, \mathbb{C}) \simeq H^i(\mathfrak{g}, \mathbb{C})$. This gives an alternative proof of Proposition 4 and Remark 5 in [39].

Now given (H, δ, σ), a Hopf algebra endowed with a modular pair in involution as above, let $\mathcal{A}$ be an algebra with an H-action and let $\tau : \mathcal{A} \to \mathbb{C}$ be a δ-invariant σ-trace. Then one can check, using in particular the integration by parts formula (41), that the characteristic map (40) is a morphism of cocyclic modules. It follows that we have a well defined map

$$\chi_\tau : HC^n_{(\delta,\sigma)}(H) \to HC^n(\mathcal{A}).$$

Example 6.5. Let $\mathcal{A}$ be an n-dimensional smooth noncommutative torus with canonical commutating derivations $\delta_1, \ldots, \delta_n$ defined by $\delta_i(U_j) = \delta_{ij} U_i$. They define an action of $H = U(\mathfrak{g})$ on $\mathcal{A}$ where $\mathfrak{g}$ is the abelian n-dimensional Lie algebra. The canonical trace τ is invariant under this action. The characteristic map χ_τ combined with the antisymmetrization map

$$\bigwedge^k \mathfrak{g} \to U(\mathfrak{g})^{\otimes k}$$

defines a map

$$\bigwedge^k \mathfrak{g} \to HC^k(\mathcal{A}).$$

This is the map of Exercise 6.8.

In the rest of this section we recall a dual cyclic theory for Hopf algebras which was defined and studied in [80]. This theory is needed, for example, when one studies coactions of Hopf algebras and quantum groups on noncommutative spaces. Notice that for compact quantum groups coactions are more natural. Also, as we mentioned before, for cosemisimple Hopf algebras, i.e. Hopf algebras endowed with a normalized Haar integral, the Hopf cyclic cohomology is trivial in positive dimensions, but the dual theory is non-trivial. There is a clear analogy with continuous group cohomology here.

Let (δ, σ) be a modular pair on H such that $\widehat{S}^2 = \mathrm{id}_H$, where $\widehat{S}(h) := h^{(2)}\sigma S(h^{(1)})$. We define a cyclic module $\widetilde{H}_\natural^{(\delta,\sigma)}$ by

$$\widetilde{H}_{\natural,0}^{(\delta,\sigma)} = \mathbb{C}, \qquad \widetilde{H}_{\natural,n}^{(\delta,\sigma)} = H^{\otimes n}, \quad n > 0.$$

Its face, degeneracy, and cyclic operators are defined by

$$\begin{aligned}
\delta_0(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= \epsilon(h_1) h_2 \otimes h_3 \otimes \cdots \otimes h_n \\
\delta_i(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= h_1 \otimes h_2 \otimes \cdots \otimes h_i h_{i+1} \otimes \cdots \otimes h_n \\
\delta_n(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= \delta(h_n) h_1 \otimes h_2 \otimes \cdots \otimes h_{n-1} \\
\sigma_0(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= 1 \otimes h_1 \otimes \cdots \otimes h_n \\
\sigma_i(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= h_1 \otimes h_2 \cdots \otimes h_i \otimes 1 \otimes h_{i+1} \cdots \otimes h_n \\
\sigma_n(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= h_1 \otimes h_2 \otimes \cdots \otimes 1, \\
\tau_n(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= \delta(h_n^{(2)}) \sigma S(h_1^{(1)} h_2^{(1)} \cdots h_{n-1}^{(1)} h_n^{(1)}) \\
&\qquad \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}.
\end{aligned}$$

We denote the cyclic homology of this cyclic module by $\widetilde{HC}_\bullet^{(\delta,\sigma)}(H)$.

Remark 9. It is not difficult to check that $(\delta \circ S^{-1}, \sigma^{-1})$, is a modular pair in involution if and only if (δ, σ) is a modular pair with $\widehat{S}^2 = id_H$. In other words (δ, σ) is a modular pair in involution in the sense of Connes and Moscovici [39] if and only if $(\delta \circ S, \sigma^{-1})$ is a modular pair in involution in the sense of [80].

Now let A be an H-comodule algebra. A linear map, $\tau : A \to \mathbb{C}$ is called a δ-trace if

$$\tau(ab) = \tau(b^{(0)} a) \delta(b^{(1)}) \qquad \forall\, a, b \in A.$$

It is called σ-invariant if for all $a \in A$,

$$\tau(a^{(0)}) a^{(1)} = \tau(a) \sigma.$$

Now consider the map $\chi_\tau : A_\natural \to \widetilde{H}_\natural^{(\delta,\sigma)}$ defined by

$$\chi_\tau(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \tau(a_0 a_1^{(0)} \cdots a_n^{(0)}) a_1^{(1)} \otimes a_2^{(1)} \otimes \cdots \otimes a_n^{(1)}.$$

It is proved in [80] that χ_τ is a morphism of cyclic modules. This looks rather uninspiring, but once dualized we obtain a characteristic map for coactions

$$\chi_\tau^* : \widetilde{HC}^n_{(\delta,\sigma)}(H) \longrightarrow HC^n(A),$$

which can be useful as will be shown in Example 6.7 below.

Next we state a theorem which computes the Hopf cyclic homology of cocommutative Hopf algebras in terms of the Hochschild homology of the underlying algebra:

Theorem 6.3. ([80]) *If H is a cocommutative Hopf algebra, then*

$$\widetilde{HC}_n^{(\delta,1)}(H) = \bigoplus_{i\geq 0} H_{n-2i}(H, \mathbb{C}_\delta),$$

where $\mathbb{C}_\delta$ is the one-dimensional module defined by δ.

Example 6.6. One knows that for any Lie algebra $\mathfrak{g}$,

$$H_n(U(\mathfrak{g}), \mathbb{C}_\delta) \simeq H_n^{Lie}(\mathfrak{g}, \mathbb{C}_\delta).$$

So by Theorem 6.3 we have

$$\widetilde{HC}_n^{(\delta,1)}(U(\mathfrak{g})) \simeq \bigoplus_{i\geq 0} H_i^{Lie}(\mathfrak{g}, \mathbb{C}_\delta).$$

Example 6.7. Let $H = \mathbb{C}\Gamma$ be the group algebra of a discrete group Γ. Then from Theorem 6.3 we have

$$\widetilde{HC}_n^{(\epsilon,1)}(\mathbb{C}\Gamma) \simeq \bigoplus\nolimits_{i\geq 0} H_{n-2i}(\Gamma, \mathbb{C}),$$
$$\text{and } \widetilde{HP}_n^{(\epsilon,1)}(\mathbb{C}\Gamma) \simeq \bigoplus\nolimits_{i=n \pmod 2} H_i(\Gamma, \mathbb{C}).$$

Now any Hopf algebra H is a comodule algebra over itself via the coproduct map $H \longrightarrow H \otimes H$. The map $\tau : \mathbb{C}\Gamma \to \mathbb{C}$ defined by

$$\tau(g) = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases}$$

is a δ-invariant σ-trace for $\delta = \epsilon$, $\sigma = 1$. The dual characteristic map

$$\chi_\tau^* : \widetilde{HC}^n_{(\epsilon,1)}(\mathbb{C}\Gamma) \to HC^n(\mathbb{C}\Gamma)$$

combined with the inclusion $H^n(\Gamma, \mathbb{C}) \hookrightarrow \widetilde{HC}^n_{(\epsilon,1)}(\mathbb{C}\Gamma)$ gives us a map

$$H^n(\Gamma, \mathbb{C}) \to HC^n(\mathbb{C}\Gamma)$$

from group cohomology to cyclic cohomology. The image of a normalized group n-cocycle $\varphi(g_1, \ldots, g_n)$ under this map is the cyclic n-cocycle $\hat{\varphi}$ defined by

$$\hat{\varphi}(g_0, \ldots, g_n) = \begin{cases} \varphi(g_1, \ldots, g_n) & g_0 g_1 \cdots g_n = e \\ 0 & g_0 g_1 \cdots g_n \neq e. \end{cases}$$

Thus the characteristic map for Hopf cyclic homology reduces to a well known map in noncommutative geometry [24]. This should be compared with Example 6.5.

It would be very interesting to compute the Hopf cyclic homology $\widetilde{HC}_n$ of quantum groups. We cite one of the very few results known in this direction. Let $H = A(SL_q(2, \mathbb{C}))$ be the Hopf algebra of quantum SL_2. As an algebra it is generated by a, b, c, d, with relations

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd,$$

$$bc = cb, \quad ad - q^{-1}bc = da - qbc = 1.$$

The coproduct, counit and antipode of H are defined by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes c$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d$$

$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0,$$

$$S(a) = d, \quad S(d) = a, \quad S(b) = -qb, \quad S(c) = -q^{-1}c.$$

We define a modular pair (σ, δ) by

$$\delta(a) = q, \quad \delta(b) = 0, \quad \delta(c) = 0, \quad \delta(d) = q^{-1},$$

$\sigma = 1$. Then we have $\widetilde{S}^2_{(1,\delta)} = \mathrm{id}$.

Theorem 6.4. ([80]) *For q not a root of unity, one has*

$$\widetilde{HC}_1(A(SL_q(2, \mathbb{C}))) = \mathbb{C} \oplus \mathbb{C}, \qquad \widetilde{HC}_n(A(SL_q(2, \mathbb{C}))) = 0, \quad n \neq 1,$$

and

$$\widetilde{HP}_0(A(SL_q(2, \mathbb{C}))) = \widetilde{HP}_1(A(SL_q(2, \mathbb{C}))) = 0.$$

In [67, 68], following the lead of [1, 80, 81], Hajac–Khalkhali–Rangipour-Sommerhäuser define a full fledged Hopf cyclic cohomology theory for algebras or coalgebras endowed with actions or coactions of a Hopf algebra. This extends the pioneering work of Connes and Moscovici in two different directions. It allows coefficients for the theory and instead of Hopf algebras one now works with algebras or coalgebras with a Hopf action. It turns out that the periodicity condition $\tau_n^{n+1} = \mathrm{id}$ for the cyclic operator puts very stringent conditions on the type of coefficients that are allowable and the correct class of Hopf modules turned to be the class of *stable anti-Yetter–Drinfeld modules* over a Hopf algebra. It also sheds light on Connes–Moscovici's modular pairs in involution by interpreting them as one-dimensional stable anti-Yetter–Drinfeld modules.

The category of anti-Yetter–Drinfeld modules over a Hopf algebra H is a twisting, or variant rather, of the category of Yetter-Drinfeld H-modules. Notice that this latter category is widely studied primarily because of its connections with quantum group theory and with invariants of knots and low-dimensional topology [123]. Technically it is obtained from the latter by replacing the antipode S by S^{-1} although this connection is hardly illuminating. We refer to [82] and references therein for a survey of anti-Yetter–Drinfeld modules and Hopf cyclic cohomology.

Remark 10. The picture that is emerging is quite intriguing and seems to be at the crossroads of three different areas: von Neumann algebras, quantum groups and low-dimensional topology, and cyclic cohomology. We have Connes–Moscovici's modular pairs in involution which was suggested by type III factors and non-unimodular Lie groups and turned out to be examples of anti-Yetter–Drinfeld modules. As we saw above the latter category is of fundamental importance in Hopf cyclic cohomology. One obviously needs to understand these connections much better.

References

[1] R. Akbarpour and M. Khalkhali, Hopf algebra equivariant cyclic homology and cyclic homology of crossed product algebras, J. Reine Angew. Math., **559**, 137–152 (2003).

[2] M. F. Atiyah, Global theory of elliptic operators. 1970 Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969) pp. 21–30, Univ. of Tokyo Press.

[3] M. F. Atiyah and G. Segal, Twisted K-theory, math.KT/0407054.

[4] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Deformation theory and quantization. I. Deformations of symplectic structures, Ann. Phys. **111**, 61–110 (1978) and II. Physical applications, Ann. Phys. **111**, 111–151 (1978).

[5] A. Belov-Kanel and M. Kontsevich, The Jacobian conjecture is stably equivalent to the Dixmier conjecture, math.RA/0512171.
[6] B. Blackadar, K-theory for operator algebras, Mathematical Sciences Research Institute Publications, **5** Springer-Verlag, New York, 1986.
[7] B. Blackadar, Operator algebras, Theory of C^*-algebras and von Neumann algebras, Encyclopaedia of Mathematical Sciences, 122, Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006.
[8] J. B. Bost, Principe d'Oka, K-théorie et systèmes dynamiques non commutatifs. (French) [The Oka principle, K-theory and noncommutative dynamical systems], Invent. Math. **101**, 261–333 (1990).
[9] J. B. Bost and A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, Selecta Math. (N.S.) **1**, 411–457 (1995).
[10] P. Bouwknegt, A. L. Carey, V. Mathai, M. K. Murray and D. Stevenson, Twisted K-theory and K-theory of bundle gerbes, Commun. Math. Phys. **228**, 17–49 (2002).
[11] P. Bouwknegt and V. Mathai, D-branes, B-fields and twisted K-theory, JHEP 0003 (2000).
[12] O. Bratteli and D. Robinson, Operator algebras and quantum statistical mechanics. 2 Vols. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.
[13] L. Brown, R. Douglas, and P. Fillmore, Extensions of C^*-algebras and K-homology, Ann. of Math. (2) **105**, 265–324 (1977).
[14] T. Brzeziński and S. Majid, Quantum group gauge theory on quantum spaces, Commun. Math. Phys. **157**, 591–638 (1993).
[15] D. Burghelea, The cyclic homology of the group rings, Comment. Math. Helv. **60**, 354–365 (1985).
[16] N. Chriss and V. Ginzburg, Representation theory and complex geometry, Birkhäuser Boston, Boston, MA, 1997.
[17] A. Connes, A survey of foliations and operator algebras, Operator algebras and applications, Part I (Kingston, Ont., 1980), pp. 521–628, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982.
[18] A. Connes, C^* algébres et géométrie différentielle, (French) C. R. Acad. Sci. Paris Sér. A-B **290**, A599–A604 (1980).
[19] A. Connes, Spectral sequence and homology of currents for operator algebras, Oberwolfach Tagungsber., 41/81, Funktionalanalysis und C^*-Algebren, 27-9/3-10, 1981.
[20] A. Connes, An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of R, Adv. Math. **39**, 31–55 (1981).
[21] A. Connes, Cohomologie cyclique et foncteurs Ext^n , (French) (Cyclic cohomology and functors Ext^n) C. R. Acad. Sci. Paris Sr. I Math. **296**, 953–958 (1983).
[22] A. Connes, Noncommutative differential geometry. Chapter I: The Chern character in K homology, Preprint IHES octobre 1982; Chapter II: de Rham

homology and noncommutative algebra, Preprint IHES février 1983, Publ. Math. IHES no. 62, 41–144 (1985).
[23] A. Connes, Cyclic cohomology and the transverse fundamental class of a foliation, Geometric methods in operator algebras (Kyoto, 1983), 52–144, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, 1986.
[24] A. Connes, Noncommutative geometry, Academic Press, Inc., San Diego, CA, 1994.
[25] A. Connes, Noncommutative geometry year 2000, Highlights of mathematical physics (London, 2000), 49–110, Amer. Math. Soc., Providence, RI, 2002.
[26] A. Connes and M. Dubois-Violette, Moduli space and structure of noncommutative 3-spheres, Lett. Math. Phys. **66**, 91–121 (2003).
[27] A. Connes and M. Dubois-Violette, Yang–Mills and some related algebras, J. Alg. **261**, 172–185 (2003).
[28] A. Connes and M. Dubois-Violette, Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples, math.QA/0107070.
[29] A. Connes and D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Commun. Math. Phys. **199**, 203–242 (1998).
[30] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem I, Commun. Math. Phys. **210**, (2000).
[31] A. Connes and D. Kreimer, Lessons from quantum field theory, hep-th/9904044.
[32] A. Connes and G. Landi, Noncommutative manifolds, the instanton algebra and isospectral deformations, Commun. Math. Phys. **221**, 141–159 (2001).
[33] A. Connes and M. Marcolli, From physics to number theory via noncommutative geometry. Frontiers in number theory, physics, and geometry. I, 269–347, Springer, Berlin, 2006.
[34] A. Connes and M. Marcolli, A walk in the noncommutative garden, math.QA/0601054, in this volume.
[35] A. Connes and M. Marcolli, Quantum fields and motives, J. Geom. Phys. **56**, 55–85 (2006).
[36] A. Connes and M. Marcolli, book in preparation.
[37] A. Connes, C. Consani and M. Marcolli, Noncommutative geometry and motives: the thermodynamics of endomotives, math.QA/0512138.
[38] A. Connes and H. Moscovici, The local index formula in noncommutative geometry, Geom. Funct. Anal. **5**, 174–243 (1995).
[39] A. Connes and H. Moscovici, Hopf algebras, cyclic cohomology and the transverse index theorem, Commun. Math. Phys. **198**, 199–246 (1998).
[40] A. Connes and H. Moscovici, Cyclic cohomology and Hopf algebra symmetry. Conference Moshé Flato 1999 (Dijon), Lett. Math. Phys. **52**, 1–28 (2000).
[41] A. Connes and H. Moscovici, Rankin–Cohen brackets and the Hopf algebra of transverse geometry, Mosc. Math. J. **4**, 111–130 (2004).
[42] A. Connes and H. Moscovici, Modular Hecke algebras and their Hopf symmetry, Mosc. Math. J. **4**, 67–109, 310 (2004).

[43] A. Connes and H. Moscovici, Background independent geometry and Hopf cyclic cohomology, math.QA/0505475.
[44] A. Connes and M. Rieffel, Yang–Mills for noncommutative two-tori. Operator algebras and mathematical physics (Iowa City, Iowa, 1985), Contemp. Math. **62**, 237–266 (1987).
[45] M. Crainic, Cyclic cohomology of Hopf algebras, J. Pure Appl. Alg. **166**, 29–66 (2002).
[46] J. Cuntz, Cyclic theory, bivariant K-theory and the bivariant Chern–Connes character, Cyclic homology in non-commutative geometry, 1–71, Encyclopaedia Math. Sci., 121, Springer, Berlin, 2004.
[47] J. Cuntz and D. Quillen, Cyclic homology and nonsingularity, J. Amer. Math. Soc. **8**, 373–442 (1995).
[48] J. Cuntz and D. Quillen, Excision in bivariant periodic cyclic cohomology, Invent. Math. **127**, 67–98 (1997).
[49] K. Davidson, C^*-algebras by example, Fields Institute Monographs, 6, Amer. Math. Soc., Providence, RI, 1996.
[50] P. Dirac, The fundamental equations of quantum mechanics, Proc. Roy. Soc. A **109**, (1926).
[51] J. Dixmier and A. Douady, Champs continus d'espaces hilbertiens et de C^*-algébres, (French) Bull. Soc. Math. France **91**, 227–284 (1963).
[52] P. Donovon and M. Karoubi, Graded Brauer groups and K-theory with local coefficients, Inst. Hautes Études Sci. Publ. Math. No. 38, 5–25 (1970).
[53] R. Douglas, Banach algebra techniques in operator theory. Second edition. Graduate Texts in Mathematics, 179. Springer-Verlag, New York, 1998.
[54] V. Drinfeld, Quantum groups. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.
[55] J. Enock and M. Schwartz, Kac algebras and duality of locally compact groups. With a preface by Alain Connes. With a postface by Adrian Ocneanu, Springer-Verlag, Berlin, 1992.
[56] R. Feynman, Statistical mechanics. A set of lectures. Perseus Books, Advanced Book Program, Reading, MA, 1998.
[57] P. A. Fillmore, A user's guide to operator algebras. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley and Sons, Inc., New York, 1996.
[58] D. Freed, The Verlinde algebra is twisted equivariant K-theory, Turk. J. Math. **25**, 159–167 (2001).
[59] D. Feed, M. Hopkins and C. Teleman, Twisted K-theory and loop group representations, math.AT/0312155.
[60] B. Feĭgin and B. L. Tsygan, Additive K-theory. K-theory, arithmetic and geometry (Moscow, 1984–1986), 67–209, Lecture Notes in Math., 1289, Springer, Berlin, 1987.
[61] I. M. Gelfand and M. A. Naimark, On the embedding of normed rings into the ring of operators in Hilbert space, Mat. Sbornik **12**, 197–213 (1943).

[62] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. Math. **78**, 267–288 (1963).
[63] M. Gerstenhaber and S. Schack, Algebraic cohomology and deformation theory. Deformation theory of algebras and structures and applications (Il Ciocco, 1986), 11–264, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 247, Kluwer Acad. Publ., Dordrecht, 1988.
[64] J. M. Gracia-Bondia, J. C. Várilly and H. Figueroa, Elements of noncommutative geometry, Birkhäuser, 2000.
[65] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires. (French) Mem. Amer. Math. Soc. (1955), no. 16.
[66] V. Guillemin and S. Sternberg, Variations on a theme by Kepler. American Mathematical Society Colloquium Publications, 42. Amer. Math. Soc., Providence, RI, 1990.
[67] P. Hajac, M. Khalkhali, B. Rangipour and Y. Sommerhäuser, Stable anti-Yetter–Drinfeld modules, C. R. Math. Acad. Sci. Paris **338**, 587–590 (2004).
[68] P. Hajac, M. Khalkhali, B. Rangipour and Y. Sommerhäuser, Hopf-cyclic homology and cohomology with coefficients, C. R. Math. Acad. Sci. Paris **338**, 667–672 (2004).
[69] W. Heisenberg, Quantum-theoretical re-interpretation of Kinematic and mechanical relations, Z. Phys. **33**, 879–893 (1925).
[70] D. Hilbert, Über das Unendliche. Mathematische Annalen, 95, 1926 161–190. Translated as "On the infinite" in van Heijenoort, From Frege to Gödel: A source book in mathematical logic, 1879–1931, Harvard University Press.
[71] N. Higson and J. Roe, Analytic K-homology. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.
[72] G. Hochschild, B. Kostant and A. Rosenberg, Differential forms on regular affine algebras, Trans. Amer. Math. Soc. **102**, 383–408 (1962).
[73] H. Hopf, Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, (German) Ann. Math. (2) **42**, 22–52 (1941).
[74] D. Kaledin, Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie, math.KT/0611623.
[75] M. Karoubi, K-theory. An introduction. Grundlehren der Mathematischen Wissenschaften, Band 226. Springer-Verlag, Berlin, New York, 1978.
[76] M. Karoubi, Homologie cyclique et K-théorie, (French) [Cyclic homology and K-theory] Astrisque No. 149, 1987.
[77] C. Kassel, Quantum groups, Graduate Texts in Mathematics, 155. Springer-Verlag, Berlin, 1995.
[78] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture, Invent. Math. **91**, 147–201 (1988).
[79] M. Khalkhali, Very basic noncomutative geometry, math.KT/0408416.
[80] M. Khalkhali and B. Rangipour, A new cyclic module for Hopf algebras, K-Theory **27**(2), 111–131 (2002).
[81] M. Khalkhali and B. Rangipour, Invariant cyclic homology, K-Theory **28**, 183–205 (2003).

[82] M. Khalkhali and B. Rangipour, Introduction to Hopf cyclic cohomology. Noncommutative Geometry and Number Theory: Where Arithmetic meets Geometry and Physics, eds. Caterina Consani and Matilde Marcolli, Vieweg Verlag, 2006.

[83] A. Klimyk and K. Schmüdgen, Quantum groups and their representations, Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

[84] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. **66**, 157–216 (2003).

[85] M. Kontsevich, Operads and motives in deformation quantization, Lett. Math. Phys. **48**, 35–72 (1999).

[86] M. Kontsevich and Y. Soibelman, Deformations of algebras over operads and Deligne's conjecture, math.QA/0001151.

[87] O. Kravchenko and B. Khesin, A central extension of the algebra of pseudodifferential symbols, Funct. Anal. Appl. **25**, 152–154 (1991).

[88] J. Kustermans and S. Vaes, Locally compact quantum groups, Ann. Sci. École Norm. Sup. (4) **33**, 837–934 (2000).

[89] H. B. Lawson and M. Michelsohn, Spin geometry, Princeton Mathematical Series, 38, Princeton University Press, Princeton, NJ, 1989.

[90] G. Landi, An introduction to noncommutative spaces and their geometries, Lecture Notes in Physics. New Series m: Monographs, 51. Springer-Verlag, Berlin, 1997.

[91] G. Landi and W. van Suijlekom, Noncommutative bundles and instantons in Tehran, hep-th/0603053, in this volume.

[92] J. L. Loday, Cyclic homology, Springer-Verlag, second edition (1998).

[93] J. L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, Comment. Math. Helv. **59**, 569–591 (1984).

[94] S. Majid, Foundations of quantum group theory, Cambridge University Press, Cambridge, 1995.

[95] Yu. I. Manin, Algebraic aspects of nonlinear differential equations, Current problems in mathematics, Vol. 11, pp. 5–152. Moscow, 1978.

[96] Yu. I. Manin, Quantum groups and noncommutative geometry, Universit de Montral, Centre de Recherches Mathmatiques, Montreal, QC, 1988.

[97] Yu. I. Manin, Topics in noncommutative geometry, Princeton University Press, Princeton, NJ, 1991.

[98] M. Marcolli, Arithmetic noncommutative geometry, University Lecture Series, 36. Amer. Math. Soc., Providence, RI, 2005.

[99] J. Milnor and J. Moore, On the structure of Hopf algebras, Ann. Math. (2) **81**, 211–264 (1965).

[100] J. Milnor and J. Stasheff, Characteristic classes, Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N.J. (1974).

[101] C. Moore and M. Schochet, Global Analysis on Foliated Spaces (second revised edition), Cambridge University Press, 2005.

[102] F. J. Murray and J. von Neumann, On rings of operators, Ann. Math. (2) **37**, 116–229 (1936).

[103] R. Nest and B. Tsygan, Algebraic index theorem, Comm. Math. Phys. **172**, 223–262 (1995).
[104] V. Nistor, Group cohomology and the cyclic cohomology of crossed products, Invent. Math. **99**, 411–424 (1990).
[105] M. Pimsner and D. Voiculescu, K-groups of reduced crossed products by free groups, J. Operator Theory **8**, 131–156 (1982).
[106] A. Pressley and G. Segal, Loop groups, Oxford Mathematical Monographs. Oxford University Press, New York, 1986.
[107] I. Raeburn and D. Williams, Morita equivalence and continuous trace C^*-algebras. Mathematical Surveys and Monographs, 60, Amer. Math. Soc., Providence, RI, 1998.
[108] J. Renault, A groupoid approach to C^*-algebras, Lecture Notes in Mathematics, 793. Springer, Berlin, 1980.
[109] M. Rieffel, Morita equivalence for C^*-algebras and W^*-algebras, J. Pure Appl. Alg. **5**, 51–96 (1974).
[110] M. Rieffel, Deformation quantization for actions of R^d, Mem. Amer. Math. Soc. **106** (1993), no. 506.
[111] M. Rieffel, Deformation quantization of Heisenberg manifolds, Comm. Math. Phys. **122**, 531–562 (1989).
[112] M. Rieffel, Quantization and C^*-algebras. C^*-algebras: 1943–1993 (San Antonio, TX, 1993), 66–97, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994.
[113] J. Rosenberg, Continuous-trace algebras from the bundle theoretic point of view, J. Austral. Math. Soc. Ser. A 47 (1989), no. 3, 368–381.
[114] J. Rosenberg, Algebraic K-theory and its applications, Graduate Texts in Mathematics, 147. Springer-Verlag, New York, 1994.
[115] L. Schweitzer, A short proof that $M_n(A)$ is local if A is local and Frchet, Internat. J. Math. **3**, 581–589 (1992).
[116] J. P. Serre, Modules projectifs et espaces fibrés à fibre vectorielle. (French) 1958 Sminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58, Fasc. 2, Expos 23.
[117] R. Swan, Vector bundles and projective modules, Trans. Amer. Math. Soc. **105**, 264–277 (1962).
[118] M. Sweedler, Hopf algebras, Mathematics Lecture Note Series W. A. Benjamin, Inc., New York 1969.
[119] A. Valette, Introduction to the Baum–Connes conjecture, Lectures in Mathematics ETH Zürich, Birkhuser Verlag, Basel, 2002.
[120] J. C. Várilly, Hopf algebras in noncommutative geometry, hep-th/0109077.
[121] M. Wodzicki, Noncommutative residue. I. Fundamentals. K-theory, arithmetic and geometry (Moscow, 1984–1986), 320–399, Lecture Notes in Math., 1289, Springer, Berlin, 1987.
[122] S. L. Woronowicz, Compact quantum groups, Symétries quantiques (Les Houches, 1995), 845–884, North-Holland, Amsterdam, 1998.
[123] D. Yetter, Quantum groups and representations of monoidal categories. Math. Proc. Cambridge Philos. Soc. **108**, 261–290 (1990).

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An Invitation to Noncommutative Geometry

NONCOMMUTATIVE BUNDLES AND INSTANTONS IN TEHRAN

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We present an introduction to the use of noncommutative geometry for gauge theories by constructing instantons for a class of four-dimensional toric noncommutative manifolds. These instantons are solutions of self-duality equations and are critical points of an action functional. We explain the crucial role of twisted symmetries as well as methods from noncommutative index theorems.

AMS Subject Classification: 58B34, 46L87, 81R60

1. Introduction

These notes are intended to be a pedagogical introduction to noncommutative gauge theories. They report on attempts to explore the "Noncommutative Pardis" [24] seeking instantons. We started the journey with the guiding image of an instanton as "a rank two complex vector bundle on a four-dimensional manifold endowed with a self-dual connection", but we usually travel with a mind open to diversity. Descriptions of parts of the region where noncommutative instantons grow have already been reported [56] and have resulted in a host of interesting and current developing activities.

We concentrate on gauge theories on toric noncommutative manifolds [23] and in fact, working out explicit examples, we give a detailed construction of a family of SU(2) gauge instantons on a four-dimensional noncommutative sphere S^4_θ as constructed in [47, 48].

Here is a brief synopsis of these notes. We start in Sec. 2 with some elements of classical gauge theory on principal and vector bundles and review the theory of connections on modules – the algebraic substitute for bundles. Section 3 is devoted to toric noncommutative manifolds M_θ – where θ is an antisymmetric matrix of deformation parameters – which are deformations of Riemannian manifolds M along torus actions. In Sec. 4, we focus on two such manifolds, $S^7_{\theta'}$ and S^4_θ, and exhibit a one-parameter family of noncommutative SU(2) principal fibrations $S^7_{\theta'} \to S^4_\theta$, with θ' a simple function of θ.

In Sec. 5, we first develop gauge theories on S^4_θ by defining a Yang–Mills action functional in terms of the curvature of a connection on a projective module over $C^\infty(S^4_\theta)$. We derive the "absolute minima" of this functional as connections with (anti)self-dual curvature – the instantons. These are characterized by a "topological charge" – an integer which is the K-theory class of the projective module on which the instanton is given, and which is calculated by a noncommutative index theorem. We also sketch gauge theories on any four-dimensional toric noncommutative manifold.

The sphere S^4_θ carries an action of twisted rotations. This is shown in Sec. 6, after a description of the general procedure to twist Hopf algebras and their actions. The twisted infinitesimal symmetries of S^4_θ make up the Hopf algebra $\mathrm{U}_\theta(so(5))$, which also leaves invariant a basic instanton. The latter is introduced in Sec. 7 where we show how to obtain a family of gauge non-equivalent instantons by acting on the basic instanton with twisted infinitesimal conformal transformations, encoded in the Hopf algebra $\mathrm{U}_\theta(so(5,1))$. There we also prove – by using noncommutative index theoretical arguments – that this collection is the complete set of (infinitesimal) charge 1 instantons. We complete in Sec. 8, with a mention of alternative $\mathrm{SU}_q(2)$ bundles over quantum four-spheres.

2. Elements of Gauge Theories

Classically, Yang–Mills gauge theories are described by means of principal and vector bundles and connections on them. From a noncommutative point of view, there are suitable substitutes and an analogous theory can be developed.

2.1. *Connections on principal and vector bundles*

With P and X smooth manifolds and G a Lie group, the surjection $\pi : P \to X$ is a principal G-bundle on X if it is a fiber bundle with typical fiber G

and G acts freely and transitively on P, so that $X = M/G$, the space of orbits. The group G is called the structure (or sometime the gauge) group; we denote by R_g the right action of the element $g \in G$ on P and also write $R_g(p) = pg$. If $\mathfrak{g}$ is the Lie algebra of G, the fundamental vector field $\xi^\#$ associated to $\xi \in \mathfrak{g}$ acts on a smooth function f on P by

$$\xi^\# f(p) = \frac{d}{dt}_{|_{t=0}} f(p \exp(t\xi)).$$

A connection on a principal bundle is mostly easily given via a connection form.

Definition 1. *A* connection one-form *on P is a one-form ω taking values in $\mathfrak{g}$ and satisfying the conditions*:

(i) $\omega(\xi^\#) = \xi$,

(ii) $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$,

where R_g^* is the induced action of G on the one-forms $\Omega^1(P)$ on P and

$$\left(\mathrm{Ad}_{g^{-1}} \omega\right)_p (Y_p) := g^{-1} \omega_p(Y_p) g,$$

with $Y_p \in T_p P$, the tangent bundle at the point $p \in P$.

Given a connection one-form ω, the corresponding horizontal subspace HP is the kernel of ω in the tangent bundle TP,

$$H_p P = \{Y_p \in T_p P : \omega(Y_p) = 0\},$$

and $Y^H = Y - \omega(Y)$ will denote the projection of $Y \in \Gamma(TP)$ onto $\Gamma(HP)$.

With W a vector space (later to carry a representation of G), the covariant derivative of any vector-valued form on P, $\varphi \in \Omega^r(P) \otimes W$, is then defined as,

$$\begin{aligned} &\mathrm{D} : \Omega^r(P) \otimes W \to \Omega^{r+1}(P) \otimes W, \\ &\mathrm{D}\varphi(Y_1, \ldots, Y_{r+1}) = \mathrm{d}_P \varphi(Y_1^H, \ldots, Y_{r+1}^H), \end{aligned}$$

where d_P is the exterior derivative on P.

Definition 2. *The* curvature Ω *of ω is the covariant derivative of ω,*

$$\Omega = \mathrm{D}\omega.$$

A more explicit form of the curvature is given in terms of Cartan's structure equation,

$$\Omega(X, Y) = \mathrm{d}_P\omega(X, Y) + [\omega(X), \omega(Y)],$$

which is usually written as $\Omega = \mathrm{d}_P\omega + \omega \wedge \omega$. The Bianchi identity is the statement that the covariant derivative of the curvature vanishes automatically:

$$\mathrm{D}\Omega = 0.$$

If ρ is a (finite-dimensional) representation of G on the vector space W, the associated bundle to P by W is defined to be the vector bundle $E := P \times_G W$ having typical fiber W. It is a classical result in differential geometry that the space of sections $\Gamma(E)$ can be given as the collection of G-equivariant maps from P to W:

$$C_G(P, W) := \{\varphi \in C(P, W) := C(P) \otimes W : \varphi(p \cdot g) = \rho_g(\varphi(p))\}.$$

This identification is as (right) $C(X)$-modules: one multiplies sections (or equivariant maps with $C(X)$ realized as a subalgebra of $C(P)$) by functions pointwise.

With the module identification, $\Gamma(E) \simeq C_G(P, W)$, a *connection* or *covariant derivative* on E is defined as the map

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes_{C(X)} \Omega^1(X), \qquad \nabla(\varphi) := \mathrm{d}_P\varphi + \omega\varphi,$$

and is a particular case of the above definition of the covariant derivative of vector-valued forms on P. Then the curvature of the connection is also the map ∇^2 and one finds that

$$\nabla^2(\varphi) \simeq \Omega(\varphi),$$

with both maps ∇^2 and Ω being $C(X)$-linear.

There is also an equivalent description of connections in terms of local charts of X. Choose a local section of $P \to X$ and define A to be the pull-back of ω under this section. Then A is a (locally defined) one-form on X, taking values in $\mathfrak{g}$, and is called the *gauge potential*. The pull-back F of the curvature (the *field strength*) is, in terms of A,

$$F = \mathrm{d}A + A \wedge A,$$

with Bianchi identity $\mathrm{d}F + A \wedge F = 0$. It turns out that F is a two-form taking values in the adjoint bundle $\mathrm{ad}(P) := P \times_G \mathfrak{g}$, where G acts on $\mathfrak{g}$ with the adjoint representation.

We are at the crucial notion of a gauge transformation: it is just a section of the bundle of automorphisms of E. More precisely, the infinite dimensional group $\mathcal{G}$ of gauge transformations consists of sections of the bundle $P \times_G G$ where G acts on itself by conjugation. A gauge transformation f acts on a connection as

$$\nabla \mapsto f^{-1} \nabla f,$$

inducing the familiar transformation rule for the connection one-form A,

$$A \mapsto f^{-1} A f + f^{-1} \mathrm{d} f,$$

together with

$$F \mapsto f^{-1} F f$$

for its curvature. This transformation make evident the invariance under gauge transformations of the Yang–Mills action functional defined (up to a constant factor) by

$$S[A] = \|F\|^2 := - \int_X \mathrm{tr}(F \wedge *F),$$

where $*$ is the Hodge star operator of a Riemannian metric on X – that, from now on we take to be four-dimensional. The corresponding critical points are solutions of the equation

$$\mathrm{D} * F = 0.$$

If we decompose $F = F_+ \oplus F_-$ into its self-dual and anti-self-dual part, i.e. $*F_\pm = \pm F_\pm$, we can relate this action to the second Chern number

$$c_2 = \int_X \mathrm{ch}_2(\nabla) = \frac{1}{2} \Big(\frac{\mathrm{i}}{2\pi} \Big)^2 \int \mathrm{tr}\, F \wedge F.$$

In fact, this is a topological quantity – a topological charge in physicists' parlance – depending only on the bundle and not on the connection and taking integral values. If $c_2 = k \in \mathbb{Z}$, one has that

$$S[A] = \|F_+\| + \|F_-\|, \qquad 8\pi^2 k = \|F_+\| - \|F_-\|$$

from which we deduce the lower bound $S \geq 8\pi^2 |k|$. Equality holds if $*F = \pm F$. Connections with self-dual or anti-self-dual curvature are called instantons (or anti-instantons) and are absolute minima of the Yang–Mills action; for them the Bianchi identity automatically implies the field equations.

Instantons can be used to obtain an approximation for the path integral

$$Z(t) = \int \mathcal{D}[A] e^{-tS[A]},$$

with the (formal) integral taken over the space of all gauge potentials. One is really interested in integrating over the moduli space of gauge connections modulo gauge transformations. As $t \to 0$, the path integral essentially reduces to the integral over the moduli space of instantons.

Instantons are most elegantly described via the so-called ADHM construction [7, 6]. These ideas have culminated in Donaldson's construction of invariants of smooth four-dimensional manifolds [30, 31]. In [68] it is shown how to recover the Donaldson invariants from the path integral $Z(t)$ as t tends to 0.

The generalization in [56] of the ADHM method to the noncommutative space $\mathbb{R}^4$ has found several important applications, notably in brane and superconformal theories.

Let us briefly illustrate the above structure with the crucial example of the principal Hopf fibration $S^7 \to S^4$ with SU(2) as structure group – more on this bundle is in Sec. 4.2 below. Let E be the vector bundle associated to the fundamental representation of SU(2): $E = S^7 \times_{\mathrm{SU}(2)} \mathbb{C}^2$. In this case, an instanton is a connection on this rank two complex vector bundle on S^4 – which we can define by a gauge potential A – having self-dual curvature. The basic instanton constructed in [13] can be given on the local chart $\mathbb{R}^4$ – with local coordinates $\{\zeta_\mu, \zeta_\mu^*\}$ coming from stereographic projection – by the connection,

$$A := \frac{1}{1+|\zeta|^2}\Big((\zeta_1 \mathrm{d}\zeta_1^* - \zeta_1^* \mathrm{d}\zeta_1 - \zeta_2 \mathrm{d}\zeta_2^* + \zeta_2^* \mathrm{d}\zeta_2)\sigma_3 \\ + 2(\zeta_1 \mathrm{d}\zeta_2^* - \overline{\lambda}\zeta_2^* \mathrm{d}\zeta_1)\sigma_+ + 2(\zeta_2 \mathrm{d}\zeta_1^* - \lambda\zeta_1^* \mathrm{d}\zeta_2)\sigma_-\Big),$$

with $\sigma_3, \sigma_\pm$ generators of the Lie algebra $su(2)$. The corresponding self-dual curvature is

$$F = \frac{1}{(1+|\zeta|^2)^2}\Big((\mathrm{d}\zeta_1 \mathrm{d}\zeta_1^* - \mathrm{d}\zeta_2^* \mathrm{d}\zeta_2)\sigma_3 + 2(\mathrm{d}\zeta_1 \mathrm{d}\zeta_2^*)\sigma_+ + 2(\mathrm{d}\zeta_2 \mathrm{d}\zeta_1^*)\sigma_-\Big),$$

and the value of the topological number is computed to be 1.

By acting with the conformal group $SL(2,\mathbb{H})$ of S^4 on the above gauge potential, one obtains different gauge inequivalent instantons [6]. The conformal group leaves both the (anti-)self-dual equation $*F = \pm F$ and the Yang–Mills action invariant, thus transforming instantons into instantons, with the subgroup $\mathrm{Spin}(5) \simeq \mathrm{Sp}(2,\mathbb{H}) \subset SL(2,\mathbb{H})$ yielding gauge equivalent instantons (in fact leaving invariant the basic instanton). Thus, there is a five-parameter family of instantons up to gauge transformations. On the local chart $\mathbb{R}^4$ of S^4, these five parameters correspond to one scaling ρ and four 'translations' of the basic instanton. They form the five-dimensional moduli space of charge 1 instantons.

2.2. *Connections on noncommutative vector bundles*

We now review the notion of a (gauge) connection on a (finite projective) module $\mathcal{E}$ over an algebra $\mathcal{A}$ with respect to a given calculus; we take a right module structure. Also, we recall gauge transformations in this setting.

Let us suppose we have an algebra $\mathcal{A}$ with a differential calculus $(\Omega\mathcal{A} = \oplus_p \Omega^p\mathcal{A}, \mathrm{d})$. A *connection* on the right $\mathcal{A}$-module $\mathcal{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1}\mathcal{A},$$

defined for any $p \geq 0$, and satisfying the Leibniz rule

$$\nabla(\omega\rho) = (\nabla\omega)\rho + (-1)^p \omega \mathrm{d}\rho, \quad \forall\, \omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A},\ \rho \in \Omega\mathcal{A}.$$

A connection is completely determined by its restriction

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}, \tag{2.1}$$

which satisfies

$$\nabla(\eta a) = (\nabla\eta)a + \eta \otimes_{\mathcal{A}} \mathrm{d}a, \quad \forall\, \eta \in \mathcal{E},\ a \in \mathcal{A}, \tag{2.2}$$

and which is extended by the Leibniz rule. It is again the latter property that implies the $\Omega\mathcal{A}$-linearity of the composition,

$$\nabla^2 = \nabla \circ \nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+2}\mathcal{A}.$$

Indeed, for any $\omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A}, \rho \in \Omega\mathcal{A}$ one has $\nabla^2(\omega\rho) = \nabla\left((\nabla\omega)\rho + (-1)^p\omega\mathrm{d}\rho\right) = (\nabla^2\omega)\rho + (-1)^{p+1}(\nabla\omega)\mathrm{d}\rho + (-1)^p(\nabla\omega)\mathrm{d}\rho + \omega\mathrm{d}^2\rho = (\nabla^2\omega)\rho$. The restriction of ∇^2 to $\mathcal{E}$ is the *curvature*

$$F : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2\mathcal{A}, \tag{2.3}$$

of the connection. It is $\mathcal{A}$-linear, $F(\eta a) = F(\eta)a$ for any $\eta \in \mathcal{E}, a \in \mathcal{A}$, and satisfies

$$\nabla^2(\eta \otimes_{\mathcal{A}} \rho) = F(\eta)\rho, \quad \forall\, \eta \in \mathcal{E},\ \rho \in \Omega\mathcal{A}. \tag{2.4}$$

Thus, $F \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2\mathcal{A})$, the latter being the collection of (right) $\mathcal{A}$-linear endomorphisms of $\mathcal{E}$, with values in the two-forms $\Omega^2\mathcal{A}$.

In order to have the notion of a Bianchi identity we need some generalization. Let $\mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ be the collection of all $\Omega\mathcal{A}$-linear endomorphisms of $\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}$. It is an algebra under composition. The curvature F can be thought of as an element of $\mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$. There is then a well-defined map

$$\begin{aligned} [\nabla, \cdot\,] \ &: \ \mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}) \longrightarrow \mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}) \\ [\nabla, T] &:= \nabla \circ T - (-1)^{|T|}\, T \circ \nabla, \end{aligned} \tag{2.5}$$

where $|T|$ denotes the degree of T with respect to the $\mathbb{Z}^2$-grading of $\Omega\mathcal{A}$. Indeed, for any $\omega \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A},\ \rho \in \Omega\mathcal{A}$, it follows that

$$\begin{aligned}
[\nabla, T](\omega\rho) &= \nabla(T(\omega\rho)) - (-1)^{|T|}\, T(\nabla(\omega\rho)) \\
&= \nabla\big(T(\omega)\rho\big) - (-1)^{|T|}\, T\big((\nabla\omega)\rho + (-1)^p \omega \mathrm{d}\rho\big) \\
&= \big(\nabla(T(\omega))\big)\rho + (-1)^{p+|T|}\, T(\omega)\mathrm{d}\rho \\
&\quad\ - (-1)^{|T|}\, T(\nabla\omega)\rho - (-1)^{p+|T|}\, T(\omega)\mathrm{d}\rho \\
&= \big(\nabla(T(\omega)) - (-1)^{|T|}\, T(\nabla\omega)\big)\rho = \big([\nabla, T](\omega)\big)\rho,
\end{aligned}$$

proving $\Omega\mathcal{A}$-linearity. It is easily checked that $[\nabla, \cdot\,]$ is a graded derivation for the algebra $\mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$:

$$[\nabla, S \circ T] = [\nabla, S] \circ T + (-1)^{|S|} S \circ [\nabla, T]. \tag{2.6}$$

Proposition 3. *The curvature F satisfies the* Bianchi identity,

$$[\nabla, F] = 0. \tag{2.7}$$

Proof. Since $F \in \mathrm{End}^0_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$, the map $[\nabla, F]$ makes sense. Furthermore,

$$[\nabla, F] = \nabla \circ \nabla^2 - \nabla^2 \circ \nabla = \nabla^3 - \nabla^3 = 0. \qquad \square$$

In Sec. II.2 of [19], such a Bianchi identity was implicitly used in the construction of a so-called canonical cycle from a connection on a finite projective $\mathcal{A}$-module $\mathcal{E}$.

Connections always exist on a projective module. On the module $\mathcal{E} = \mathbb{C}^N \otimes_{\mathbb{C}} \mathcal{A} \simeq \mathcal{A}^N$, which is free, a connection is given by the operator

$$\nabla_0 = \mathbb{I} \otimes \mathrm{d} : \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^p\mathcal{A} \longrightarrow \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^{p+1}\mathcal{A}.$$

With the canonical identification $\mathbb{C}^N \otimes_{\mathbb{C}} \Omega\mathcal{A} = (\mathbb{C}^N \otimes_{\mathbb{C}} \mathcal{A}) \otimes_{\mathcal{A}} \Omega\mathcal{A} \simeq (\Omega\mathcal{A})^N$, one thinks of ∇_0 as acting on $(\Omega\mathcal{A})^N$ as the operator $\nabla_0 = (\mathrm{d}, \mathrm{d}, \ldots, \mathrm{d})$ (N-times). Next, take a finite projective module $\mathcal{E}$ with inclusion map, $\lambda : \mathcal{E} \to \mathcal{A}^N$, which identifies $\mathcal{E}$ as a direct summand of the free module $\mathcal{A}^N$, and idempotent $p : \mathcal{A}^N \to \mathcal{E}$ which allows one to identify $\mathcal{E} = p\mathcal{A}^N$. Using these maps and their natural extensions to $\mathcal{E}$-valued forms, a connection ∇_0 on $\mathcal{E}$ (called *Levi–Civita* or *Grassmann*) is the composition,

$$\mathcal{E} \otimes_{\mathcal{A}} \Omega^p\mathcal{A} \ \xrightarrow{\lambda}\ \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^p\mathcal{A} \ \xrightarrow{\mathbb{I}\otimes\mathrm{d}}\ \mathbb{C}^N \otimes_{\mathbb{C}} \Omega^{p+1}\mathcal{A} \ \xrightarrow{p}\ \mathcal{E} \otimes_{\mathcal{A}} \Omega^{p+1}\mathcal{A},$$

that is

$$\nabla_0 = p \circ (\mathbb{I} \otimes \mathrm{d}) \circ \lambda. \tag{2.8}$$

One indicates it simply by $\nabla_0 = p\mathrm{d}$.

Remark 4. For the universal calculus $(\Omega\mathcal{A}, \mathrm{d}) = (\Omega\mathcal{A}_{\mathrm{un}}, \delta)$, the existence of a connection on the module $\mathcal{E}$ is equivalent to it being projective [26].

The collection of all connections on $\mathcal{E}$, denoted $C(\mathcal{E})$, is an affine space modeled on $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A})$; if ∇_1, ∇_2 are two connections on $\mathcal{E}$, their difference is $\mathcal{A}$-linear,

$$(\nabla_1 - \nabla_2)(\eta a) = ((\nabla_1 - \nabla_2)(\eta))a, \quad \forall\, \eta \in \mathcal{E},\ a \in \mathcal{A},$$

so that $\nabla_1 - \nabla_2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A})$. Thus, any connection can be written as

$$\nabla = p\mathrm{d} + \alpha, \tag{2.9}$$

where α is any element in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A})$. The "matrix of 1-forms" α as in (2.9) is called the *gauge potential* of the connection ∇. The corresponding curvature F of ∇ is

$$F = p\mathrm{d}p\mathrm{d}p + p\mathrm{d}\alpha + \alpha^2. \tag{2.10}$$

Next, let the algebra $\mathcal{A}$ have an involution * extended to the whole of $\Omega\mathcal{A}$ by the requirement $(\mathrm{d}a)^* = \mathrm{d}a^*$ for any $a \in \mathcal{A}$. A *Hermitian structure* on the module $\mathcal{E}$ is a map $\langle\cdot,\cdot\rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{A}$ with the properties

$$\begin{aligned} &\langle \eta a, \xi\rangle = a^* \langle \xi, \eta\rangle\,, \qquad \langle \eta, \xi\rangle^* = \langle \xi, \eta\rangle\,,\\ &\langle \eta, \eta\rangle \geq 0, \langle \eta, \eta\rangle = 0 \iff \eta = 0, \end{aligned} \tag{2.11}$$

for any $\eta, \xi \in \mathcal{E}$ and $a \in \mathcal{A}$ (an element $a \in \mathcal{A}$ is positive if it is of the form $a = b^*b$ for some $b \in \mathcal{A}$). We shall also require the Hermitian structure to be *self-dual*, i.e. every right $\mathcal{A}$-module homomorphism $\phi : \mathcal{E} \to \mathcal{A}$ is represented by an element of $\eta \in \mathcal{E}$, by the assignment $\phi(\cdot) = \langle \eta, \cdot\rangle$, the latter having the correct properties by the first of (2.11). With a self-dual Hermitian structure, any $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ is adjointable, that is it admits an adjoint, an $\mathcal{A}$-linear map $T^* : \mathcal{E} \to \mathcal{E}$ such that,

$$\langle T^*\eta, \xi\rangle = \langle \eta, T\xi\rangle\,, \quad \forall\, \eta, \xi \in \mathcal{E}.$$

The Hermitian structure is naturally extended to an $\Omega\mathcal{A}$-valued linear map on the product $\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A} \times \mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}$ by

$$\langle \eta \otimes_{\mathcal{A}} \omega, \xi \otimes_{\mathcal{A}} \rho\rangle = (-1)^{|\eta||\omega|}\omega^* \langle \eta, \xi\rangle\, \rho, \quad \forall\, \eta, \xi \in \mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A},\ \omega, \rho \in \Omega\mathcal{A}. \tag{2.12}$$

A connection ∇ on $\mathcal{E}$ and a Hermitian structure $\langle \cdot, \cdot \rangle$ on $\mathcal{E}$ are said to be compatible if the following condition is satisfied [20],

$$\langle \nabla \eta, \xi \rangle + \langle \eta, \nabla \xi \rangle = \mathrm{d} \langle \eta, \xi \rangle , \quad \forall \, \eta, \xi \in \mathcal{E}. \tag{2.13}$$

It follows directly from the Leibniz rule and (2.12) that this extends to

$$\langle \nabla \eta, \xi \rangle + (-1)^{|\eta|} \langle \eta, \nabla \xi \rangle = \mathrm{d} \langle \eta, \xi \rangle , \quad \forall \, \eta, \xi \in \mathcal{E} \otimes_{\mathcal{A}} \Omega \mathcal{A}. \tag{2.14}$$

On the free module $\mathcal{A}^N$ there is a canonical Hermitian structure given by

$$\langle \eta, \xi \rangle = \sum_{j=1}^{N} \eta_j^* \xi_j , \tag{2.15}$$

with $\eta = (\eta_1, \ldots, \eta_N)$ and $\xi = (\xi_1, \ldots, \xi_N)$ any two elements of $\mathcal{A}^N$. Under suitable regularity conditions on the algebra $\mathcal{A}$ all Hermitian structures on a given finite projective module $\mathcal{E}$ over $\mathcal{A}$ are isomorphic to each other and are obtained from the canonical structure (2.15) on $\mathcal{A}^N$ by restriction [20, II.1]. Moreover, if $\mathcal{E} = p\mathcal{A}^N$, then p is self-adjoint: $p = p^*$, with p^* obtained by the composition of the involution * in the algebra $\mathcal{A}$ with the usual matrix transposition. The Grassmann connection (2.8) is easily seen to be compatible with this Hermitian structure,

$$\mathrm{d} \langle \eta, \xi \rangle = \langle \nabla_0 \eta, \xi \rangle + \langle \eta, \nabla_0 \xi \rangle . \tag{2.16}$$

For a general connection (2.9), the compatibility with the Hermitian structure reduces to

$$\langle \alpha \eta, \xi \rangle + \langle \eta, \alpha \xi \rangle = 0, \quad \forall \, \eta, \xi \in \mathcal{E}, \tag{2.17}$$

which just says that the gauge potential is skew-Hermitian,

$$\alpha^* = -\alpha. \tag{2.18}$$

We still use the symbol $C(\mathcal{E})$ to denote the space of compatible connections on $\mathcal{E}$.

Let $\mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ denote the space of elements T in $\mathrm{End}_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ which are skew-Hermitian with respect to the Hermitian structure (2.12), i.e. satisfying

$$\langle T\eta, \xi \rangle + \langle \eta, T\xi \rangle = 0, \qquad \forall \, \eta, \xi \in \mathcal{E}. \tag{2.19}$$

Proposition 5. *The map* $[\nabla, \cdot \,]$ *in* (2.5) *restricts to* $\mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ *as a derivation*

$$[\nabla, \cdot \,] \ : \ \mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}) \longrightarrow \mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}). \tag{2.20}$$

Proof. Let $T \in \mathrm{End}^s_{\Omega\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A})$ be of order $|T|$; it then satisfies

$$\langle T\eta, \xi\rangle + (-1)^{|\eta||T|} \langle \eta, T\xi\rangle = 0, \tag{2.21}$$

for $\eta, \xi \in \mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}$. Since $[\nabla, T]$ is $\Omega\mathcal{A}$-linear, it is enough to show that

$$\langle [\nabla, T]\eta, \xi\rangle + \langle \eta, [\nabla, T]\xi\rangle = 0, \qquad \forall\, \eta, \xi \in \mathcal{E}.$$

This follows from Eqs. (2.21) and (2.14),

$$\begin{aligned}\langle [\nabla, T]\eta, \xi\rangle + \langle \eta, [\nabla, T]\xi\rangle &= \langle \nabla T\eta, \xi\rangle - (-1)^{|T|} \langle T\nabla\eta, \xi\rangle \\ &\quad + \langle \eta, \nabla T\xi\rangle - (-1)^{|T|} \langle \eta, T\nabla\xi\rangle \\ &= \langle \nabla T\eta, \xi\rangle - \langle \nabla\eta, T\xi\rangle + \langle \eta, \nabla T\xi\rangle \\ &\quad - (-1)^{|T|} \langle T\eta, \nabla\xi\rangle \\ &= \mathrm{d}\big(\langle T\eta, \xi\rangle + \langle \eta, T\xi\rangle \big) = 0. \end{aligned}$$ □

2.3. *Gauge transformations*

The group $\mathcal{U}(\mathcal{E})$ of unitary endomorphisms of $\mathcal{E}$ is given by

$$\mathcal{U}(\mathcal{E}) := \{u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \mid uu^* = u^*u = \mathrm{id}_{\mathcal{E}}\}. \tag{2.22}$$

This group plays the role of the *infinite dimensional group of gauge transformations.* It naturally acts on compatible connections by

$$(u, \nabla) \mapsto \nabla^u := u^*\nabla u, \quad \forall\, u \in \mathcal{U}(\mathcal{E}),\ \nabla \in C(\mathcal{E}), \tag{2.23}$$

where u^* is really $u^* \otimes \mathrm{id}_{\Omega\mathcal{A}}$; this will always be understood in the following. Then the curvature transforms in a covariant way

$$(u, F) \mapsto F^u = u^*Fu, \tag{2.24}$$

since, evidently, $F^u = (\nabla^u)^2 = u^*\nabla uu^*\nabla u^* = u^*\nabla^2 u = u^*Fu$.

As for the gauge potential, one has the usual affine transformation

$$(u, \alpha) \mapsto \alpha^u := u^*p\mathrm{d}u + u^*\alpha u. \tag{2.25}$$

Indeed, $\nabla^u(\eta) = u^*(p\mathrm{d} + \alpha)u\eta = u^*p\mathrm{d}(u\eta) + u^*\alpha u\eta = u^*pu\mathrm{d}\eta + u^*p(\mathrm{d}u)\eta + u^*\alpha u\eta = p\mathrm{d}\eta + (u^*p\mathrm{d}u + u^*\alpha u)\eta$ for any $\eta \in \mathcal{E}$, which yields (2.25) for the transformed potential.

In order to proceed, we add the additional requirement that the algebra $\mathcal{A}$ is a Fréchet algebra and that $\mathcal{E}$ a right Fréchet module. That is, both $\mathcal{A}$ and $\mathcal{E}$ are complete in the topology defined by a family of seminorms

$\|\cdot\|_i$ such that the following condition is satisfied: for all j there exists a constant c_j and an index k such that

$$\|\eta a\|_j \leq c_j \|\eta\|_k \|a\|_k. \tag{2.26}$$

These are also used to make $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ a Fréchet algebra with corresponding family of seminorms: for $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ given by,

$$\|T\|_i = \sup_{\eta} \{\|T\eta\|_i : \|\eta\|_i \leq 1\}. \tag{2.27}$$

The "tangent vectors" to the gauge group $\mathcal{U}(\mathcal{E})$ constitute the vector space of infinitesimal gauge transformations. Suppose $\{u_t\}_{t\in\mathbb{R}}$ is a differentiable family of elements in $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ (in the topology defined by the above sup-norms) and define $X := (\partial u_t/\partial t)_{t=0}$. Unitarity of u_t then implies that $X = -X^*$. In other words, for u_t to be a gauge transformation, X should be a skew-Hermitian endomorphism of $\mathcal{E}$. In this way, we understand $\mathrm{End}^s_{\mathcal{A}}(\mathcal{E})$ as the collection of *infinitesimal gauge transformations*. It is a real vector space whose complexification $\mathrm{End}^s_{\mathcal{A}}(\mathcal{E})\otimes_{\mathbb{R}}\mathbb{C}$ can be identified with $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$.

Infinitesimal gauge transformations act on a connection in a natural way. Let the gauge transformation u_t, with $X = (\partial u_t/\partial t)_{t=0}$, act on ∇ as in (2.23). From the fact that $(\partial(u_t\nabla u_t^*)/\partial t)_{t=0} = [\nabla, X]$, we conclude that an element $X \in \mathrm{End}^s_{\mathcal{A}}(\mathcal{E})$ acts infinitesimally on a connection ∇ by the addition of $[\nabla, X]$,

$$(X, \nabla) \mapsto \nabla^X = \nabla + t[\nabla, X] + \mathcal{O}(t^2), \quad \forall\, X \in \mathrm{End}^s_{\mathcal{A}}(\mathcal{E}),\ \nabla \in C(\mathcal{E}). \tag{2.28}$$

As a consequence, for the transformed curvature one has

$$(X, F) \mapsto F^X = F + t[F, X] + \mathcal{O}(t^2), \tag{2.29}$$

since $F^X = (\nabla + t[\nabla, X]) \circ (\nabla + t[\nabla, X]) = \nabla^2 + t[\nabla^2, X] + \mathcal{O}(t^2)$.

2.4. *Noncommutative principal bundles*

The datum of a Hopf–Galois extension is an efficient encoding of the notion of a noncommutative principal bundle (see e.g. [15, 39]). Let us recall some relevant definitions [41, 55]. Recall that we work over the field $k = \mathbb{C}$.

Definition 6. *Let H be a Hopf algebra and P a right H-comodule algebra with multiplication $m : P \otimes P \to P$ and coaction $\Delta_R : P \to P \otimes H$. Let $B \subseteq P$ be the subalgebra of coinvariants, i.e. $B := \{p \in P \mid \Delta_R(p) = p\otimes 1\}$.*

The extension $B \hookrightarrow P$ is called an H Hopf–Galois extension if the canonical map

$$\begin{aligned} &\chi : P \otimes_B P \longrightarrow P \otimes H, \\ &\chi := (m \otimes id) \circ (id \otimes_B \Delta_R), \quad p' \otimes_B p \mapsto \chi(p' \otimes_B p) = p' p_{(0)} \otimes p_{(1)}, \end{aligned} \tag{2.30}$$

is bijective.

We use Sweedler-like notation $\Delta_R p = p_{(0)} \otimes p_{(1)}$. By construction, the canonical map is left P-linear and right H-colinear and is a morphism (an isomorphism for Hopf–Galois extensions) of left P-modules and right H-comodules. It is also clear that P is both a left and a right B-module.

Classically, the above requirement corresponds to freeness and transitivity of the action of a Lie group G on a manifold X. The injectivity of the canonical map dualizes the condition of a group action $X \times G \to X$ to be free: if α is the map $\alpha : X \times G \to X \times_M X$, $(x, g) \mapsto (x, x \cdot g)$ then $\alpha^* = \chi$ with P, H the algebras of functions on X, G respectively and the action is free if and only if α is injective. Here $M := X/G$ is the space of orbits with projection map $\pi : X \to M$, $\pi(x \cdot g) = \pi(x)$, for all $x \in X, g \in G$. Moreover, α is surjective if and only if for all $x \in X$, the fiber $\pi^{-1}(\pi(x))$ of $\pi(x)$ is equal to the orbit $x \cdot G$, that is, if and only if G acts transitively on the fibers of π.

In differential geometry a principal bundle is more than just a free and effective action of a Lie group. However, for a structure Hopf algebra H which is cosemisimple and has bijective antipode – as is clearly the case for the example $H = \mathcal{A}(\mathrm{SU}(2))$ of the present paper – from Th. I of [62] further nice properties can be established. In particular the surjectivity of the canonical map implies bijectivity and faithful flatness of the extension.

An additional useful result [61] is that the map χ is surjective whenever, for any generator h of H, the element $1 \otimes h$ is in its image. This follows from the left P-linearity and right H-colinearity of the map χ. Indeed, let h, k be two elements of H and $\sum p'_i \otimes p_i$, $\sum q'_j \otimes q_j \in P \otimes P$ be such that $\chi(\sum p'_i \otimes_B p_i) = 1 \otimes h$, $\chi(\sum q'_j \otimes_B q_j) = 1 \otimes k$. Then $\chi(\sum p'_i q'_j \otimes_B q_j p_i) = 1 \otimes kh$, that is, $1 \otimes kh$ is in the image of χ. Surjectivity of the map χ then follows from its left P-linearity. It is also easy to write down an explicit expression for the inverse of the canonical map: in the above notation one has $\chi^{-1}(1 \otimes kh) = \sum p'_i q'_j \otimes_B q'_i p'_j$ and its general form follows again from left P-linearity.

We are only interested in H-Hopf–Galois extensions $B \hookrightarrow P$ for a cosemisimple Hopf algebra H with bijective antipode.

An important consequence of being a faithfully flat Hopf–Galois extension is the existence of a so-called *strong connection*. Constructing a strong connection is an alternative way to prove that one has a Hopf–Galois extension [38, 27]. If H is cosemisimple and has a bijective antipode, then a H-Hopf–Galois extension $B \hookrightarrow P$ is *equivariantly projective*, that is, there exists a left B-linear right H-colinear splitting $s : P \to B \otimes P$ of the multiplication map $m : B \otimes P \to P$, $m \circ s = \mathrm{id}_P$. Such a map characterizes a strong connection. If H has an invertible antipode S, a possible description of a strong connection is given in terms of a map $\ell : H \to P \otimes P$ satisfying a list of conditions [53, 14]:

$$\begin{aligned} \ell(1) &= 1 \otimes 1, \\ \overline{\chi}(\ell(h)) &= 1 \otimes h, \\ (\ell \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \Delta_R) \circ \ell, \\ (\mathrm{id} \otimes \ell) \circ \Delta &= (\Delta_L \otimes \mathrm{id}) \circ \ell, \end{aligned} \tag{2.31}$$

Here the map $\overline{\chi} : P \otimes P \to P \otimes H$ is the lift of the canonical map as $\overline{\chi}(p' \otimes p) = p' p_{(0)} \otimes p_{(1)}$; Δ is the comultiplication on H and $\Delta_L : P \to H \otimes P$, $p \mapsto S^{-1} p_{(1)} \otimes p_{(0)}$. Then, one defines a "strong connection one-form" with respect to the universal calculus, $\omega : H \to \Omega^1_{\mathrm{un}} P$ by

$$\omega : h \mapsto \ell(h) - \epsilon(h) 1 \otimes 1.$$

Indeed, if one writes $\ell(h) = h^{\langle 1 \rangle} \otimes h^{\langle 2 \rangle}$ (summation is understood) and applies $\mathrm{id} \otimes \epsilon$ to the second formula in (2.31), one has $h^{\langle 1 \rangle} h^{\langle 2 \rangle} = \epsilon(h)$. Therefore,

$$\omega(h) = h^{\langle 1 \rangle} \delta h^{\langle 2 \rangle}, \tag{2.32}$$

where $\delta : P \to \Omega^1_{\mathrm{un}} P$, $p \mapsto 1 \otimes p - p \otimes 1$ is the universal differential. Equivariant projectivity of $B \hookrightarrow P$ follows by taking as splitting of the multiplication the map $s : P \to B \otimes P$, $p \to p_{(0)} \ell(p_{(1)})$. The form (2.32) enjoys a list of properties which could be given as its definition,

1. $\overline{\chi} \circ \omega = 1 \otimes (\mathrm{id} - \epsilon)$, (*fundamental vector field condition*)
2. $\Delta_{\Omega^1_{\mathrm{un}}(P)} \circ \omega = (\omega \otimes \mathrm{id}) \circ \mathrm{Ad}_R$, (*right adjoint colinearity*)
3. $\delta p - p_{(0)} \omega(p_{(1)}) \in (\Omega^1_{\mathrm{un}} B) P$, $\forall\, p \in P$, (*strongness condition*).

Here $\Omega_{\mathrm{un}}(B) \subset \Omega_{\mathrm{un}}(P)$ are the universal calculi on B and P with differential δ; the comultiplication $\Delta_R : P \to P \otimes H$, is extended to $\Delta_{\Omega^1_{\mathrm{un}}(P)}$ on $\Omega^1_{\mathrm{un}} P \subset P \otimes P$ in a natural way by $\Delta_{\Omega^1_{\mathrm{un}}(P)}(p' \otimes p) \mapsto p'_{(0)} \otimes p_{(0)} \otimes p'_{(1)} p_{(1)}$; and $\mathrm{Ad}_R(h) = h_{(2)} \otimes S(h_{(1)}) h_{(3)}$ is the right adjoint coaction of H on itself.

A strong connection on the extension $B \hookrightarrow P$ induces connections – in the sense of Sec. 2.2 and with the universal calculi – on the associated modules [39].

Definition 7. *Let $\rho : W \to H \otimes W$ be a left H-comodule and denote $\rho(v) = v_{(0)} \otimes v_{(1)}$. The right B-module* $\mathrm{Hom}^\rho(W, P)$ *associated to $B \hookrightarrow P$ by (ρ, W) consists of H-coequivariant maps $\varphi : W \to P$, i.e. they satisfy*

$$\varphi(v_{(1)}) \otimes Sv_{(0)} = \Delta_R \varphi(v), \qquad v \in W.$$

For $\varphi \in \mathrm{Hom}^\rho(W, P)$, we set

$$\nabla_\omega(\varphi)(v) \mapsto \delta\varphi(v) + \omega(v_{(0)})\varphi(v_{(1)}),$$

where ω is the form defined in (**??**). Using the right adjoint colinearity of ω and a little algebra one shows that $\nabla_\omega(\varphi)$ satisfies the following coequivariance condition:

$$\nabla_\omega(\varphi)(v_{(1)}) \otimes Sv_{(0)} = \Delta_{\Omega^1_{\mathrm{un}}(P)}\big(\nabla_\omega(\varphi)(v)\big)$$

so that

$$\nabla_\omega : \mathrm{Hom}^\rho(W, P) \to \mathrm{Hom}^\rho(W, \Omega^1_{\mathrm{un}}(P)).$$

In fact, from properties of the form ω, one establishes that ∇_ω is a map

$$\nabla_\omega : \mathrm{Hom}^\rho(W, P) \to \mathrm{Hom}^\rho(W, P) \otimes_B \Omega^1_{\mathrm{un}}(B).$$

Classically, a notion that is close – although not equivalent – to triviality of a principal bundle is the one of being cleft. In general one says that a Hopf–Galois extension is cleft if there exists a (unital) convolution-invertible colinear map $\phi : H \to P$, called a cleaving map. Furthermore, if a Hopf–Galois extension is cleft, its associated modules are trivial, i.e. isomorphic to the free module B^N for some N [15, 39].

3. Toric Noncommutative Manifolds

We start by recalling the general construction of toric noncommutative manifolds given in [23] where they were called isospectral deformations. These are deformations of a classical Riemannian manifold and satisfy all the properties of noncommutative spin geometry [21]. They represent the content of the following result taken from [23].

Theorem 8. *Let M be a compact spin Riemannian manifold whose isometry group has rank $r \geq 2$. Then M admits a natural one-parameter isospectral deformation to noncommutative geometries M_θ.*

The idea of the construction is to deform the standard spectral triple describing the Riemannian geometry of M along a torus embedded in the isometry group, thus obtaining a family of spectral triples describing noncommutative geometries.

3.1. *Deforming along a torus*

Let M be an m-dimensional compact Riemannian manifold equipped with an isometric smooth action σ of an n-torus $\mathbb{T}^n$, $n \geq 2$. We denote by σ also the corresponding action of $\mathbb{T}^n$ by automorphisms on the algebra $C^\infty(M)$ of smooth functions on M, obtained by pull-back. The algebra $C^\infty(M)$ may be decomposed into spectral subspaces which are indexed by the dual group $\mathbb{Z}^n = \widehat{\mathbb{T}}^n$. Now, with $s = (s_1, \ldots, s_n) \in \mathbb{T}^n$, each $r \in \mathbb{Z}^n$ labels a character $e^{2\pi i s} \mapsto e^{2\pi i r \cdot s}$ of $\mathbb{T}^n$, with the scalar product $r \cdot s := r_1 s_1 + \cdots + r_n s_n$. The rth spectral subspace for the action σ of $\mathbb{T}^n$ on $C^\infty(M)$ consists of those smooth functions f_r for which

$$\sigma_s(f_r) = e^{2\pi \mathrm{i} r \cdot s} f_r, \tag{3.1}$$

and each $f \in C^\infty(M)$ is the sum of a unique (rapidly convergent) series $f = \sum_{r \in \mathbb{Z}^n} f_r$. Let now $\theta = (\theta_{jk} = -\theta_{kj})$ be a real antisymmetric $n \times n$ matrix. The θ-deformation of $C^\infty(M)$ may be defined by replacing the ordinary product by a deformed product, given on spectral subspaces by

$$f_r \times_\theta g_{r'} := f_r \ \sigma_{\frac{1}{2} r \cdot \theta}(g_{r'}) = e^{\pi \mathrm{i} r \cdot \theta \cdot r'} f_r g_{r'}, \tag{3.2}$$

where $r \cdot \theta = (r_j \theta_{j1}, \ldots, r_j \theta_{jn}) \in \mathbb{R}^n$. This product is then extended linearly to all functions in $C^\infty(M)$. We denote $C^\infty(M_\theta) := (C^\infty(M), \times_\theta)$ and note that the action σ of $\mathbb{T}^n$ on $C^\infty(M)$ extends to an action on $C^\infty(M_\theta)$ given again by (3.1) on the homogeneous elements.

Next, let us take (M, g) to be a Riemannian spin manifold with $\mathcal{H} := L^2(M, \mathcal{S})$ the Hilbert space of spinors and $D\!\!\!/$ the usual Dirac operator of the metric g of M. Smooth functions act on spinors by pointwise multiplication, thus giving a representation $\pi : C^\infty(M) \to \mathcal{B}(\mathcal{H})$, the latter being the algebra of bounded operators on $\mathcal{H}$. There is a double cover $c : \widetilde{\mathbb{T}}^n \to \mathbb{T}^n$ and a representation of $\widetilde{\mathbb{T}}^n$ on $\mathcal{H}$ by unitary operators $U(s), s \in \widetilde{\mathbb{T}}^n$, so that

$$U(s) D\!\!\!/ U(s)^{-1} = D\!\!\!/, \tag{3.3}$$

since the torus action is assumed to be isometric, and such that for all $f \in C^\infty(M)$,

$$U(s) \pi(f) U(s)^{-1} = \pi(\sigma_{c(s)}(f)). \tag{3.4}$$

Recall that an element $T \in \mathcal{B}(\mathcal{H})$ is called smooth for the action of $\widetilde{\mathbb{T}}^n$ if the map

$$\widetilde{\mathbb{T}}^n \ni s \mapsto \alpha_s(T) := U(s) T U(s)^{-1},$$

is smooth for the norm topology. From its very definition, the map α_s coincides on $\pi(C^\infty(M)) \subset \mathcal{B}(\mathcal{H})$ with the automorphism $\sigma_{c(s)}$. Much as it was done before for the smooth functions, we shall use the torus action to give a spectral decomposition of smooth elements of $\mathcal{B}(\mathcal{H})$. Any such smooth element T is written as a (rapidly convergent) series $T = \sum T_r$ with $r \in \mathbb{Z}$ and each T_r is homogeneous of degree r under the action of $\widetilde{\mathbb{T}}^n$, i.e.

$$\alpha_s(T_r) = e^{2\pi \mathrm{i} r\cdot s} T_r, \quad \forall\, s \in \widetilde{\mathbb{T}}^n. \tag{3.5}$$

Let $P = (P_1, P_2, \ldots, P_n)$ be the infinitesimal generators of the action of $\widetilde{\mathbb{T}}^n$ so that we can write $U(s) = \exp 2\pi \mathrm{i} s \cdot P$. Now, with θ a real $n \times n$ anti-symmetric matrix as above, one defines a twisted representation of the smooth elements $\mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ by

$$L_\theta(T) := \sum_r T_r U(\tfrac{1}{2} r \cdot \theta) = \sum_r T_r \exp\left\{\pi \mathrm{i}\, r_j \theta_{jk} P_k\right\}. \tag{3.6}$$

The twist L_θ commutes with the action α_s of $\widetilde{\mathbb{T}}^n$ and preserves the spectral components of smooth operators:

$$\begin{aligned} \alpha_s(L_\theta(T_r)) &= U(s)\ TU\left(\frac{1}{2} r \cdot \theta\right) U(s)^{-1} = U(s) T U(s)^{-1} U\left(\frac{1}{2} r \cdot \theta\right) \\ &= e^{2\pi i r\cdot s} L_\theta(T_r). \end{aligned} \tag{3.7}$$

In particular, taking smooth functions on M as elements of $\mathcal{B}(\mathcal{H})$, via the representation π, the previous definition gives an algebra $L_\theta(C^\infty(M))$ which we may think of as a representation (as bounded operators on $\mathcal{H}$) of the algebra $C^\infty(M_\theta)$. Indeed, by the very definition of the product $\times_\theta$ in (3.2) one establishes that

$$L_\theta(f \times_\theta g) = L_\theta(f) L_\theta(g), \tag{3.8}$$

proving that the algebra $C^\infty(M)$ equipped with the product $\times_\theta$ is isomorphic to the algebra $L_\theta(C^\infty(M))$. It was shown in [23] that the datum $(L_\theta(C^\infty(M)), \mathcal{H}, D)$, with the operator $D = \not{D}$ just the "undeformed Dirac operator" on the "undeformed Hilbert space of spinors" $\mathcal{H}$, satisfies all the requirements for a noncommutative spin geometry [21, 37]. Since $\mathbb{T}^n$ acts by isometries, each generator P_k commutes with D and the latter is of degree 0. This yields in particular, boundedness of the commutators $[D, L_\theta(f)]$ for $f \in C^\infty(M)$, being then $[D, L_\theta(f)] = L_\theta([D, f])$. There is also a grading γ (for the even case) and a real structure J obtained by "twisting" the undeformed one.

Thus the noncommutative geometry of M_θ is an isospectral deformation of the classical Riemannian geometry of M – in that the spectrum of the operator D on M_θ coincides with that of the Dirac operator $\not{D}$ on M – and all spectral properties are unchanged. In particular, the triple is m^+-summable and there is a noncommutative integral as a Dixmier trace [29],

$$\fint L_\theta(f) := \mathrm{Tr}_\omega\big(f|D|^{-m}\big), \tag{3.9}$$

with $f \in C^\infty(M_\theta)$ understood in its representation on $\mathcal{H}$. A drastic simplification of this noncommutative integral is given by the lemma [35].

Lemma 9. *If $f \in C^\infty(M)$ then*

$$\fint L_\theta(f) = \int_M f \mathrm{d}\mu_g,$$

with $\mathrm{d}\mu_g$ the Riemannian measure of the metric g.

Proof. Any element $f \in C^\infty(M)$ is given as an infinite sum of functions that are homogeneous under the action of $\mathbb{T}^n$. Let us therefore assume that f is homogeneous of degree k so that $\sigma_s(L_\theta(f)) = L_\theta(\sigma_s(f)) = e^{2\pi i k\cdot s}L_\theta(f)$. From the tracial property of the noncommutative integral and the invariance of D under the action of $\mathbb{T}^n$, we see that

$$\begin{aligned}\mathrm{Tr}_\omega\big(\sigma_s(L_\theta(f))|D|^{-m}\big) &= \mathrm{Tr}_\omega\big(U(s)L_\theta(f)U(s)^{-1}|D|^{-m}\big)\\ &= \mathrm{Tr}_\omega(L_\theta(f)|D|^{-m}).\end{aligned}$$

In other words, $e^{2\pi i k\cdot s}\mathrm{Tr}_\omega(L_\theta(f)|D|^{-m}) = \mathrm{Tr}_\omega(L_\theta(f)|D|^{-m})$ from which we infer that this trace vanishes if $k \neq 0$. If $k = 0$, then $L_\theta(f) = f$, leading to the desired result. □

As shown in [59] that there is a natural completion of the algebra $C^\infty(M_\theta)$ to a C^*-algebra $C(M_\theta)$ whose smooth subalgebra – under the extended action of $\mathbb{T}^n$ – is precisely $C^\infty(M_\theta)$. Thus, we can understand L_θ as a *quantization map*

$$L_\theta : C^\infty(M) \to C^\infty(M_\theta), \tag{3.10}$$

which provides a strict deformation quantization in the sense of Rieffel. More generally, in [59] one considers a (not necessarily commutative) C^*-algebra A carrying an action of $\mathbb{R}^n$. For an anti-symmetric $n \times n$ matrix θ, one defines a star product $\times_\theta$ between elements in A much as we did before. The algebra A equipped with the product $\times_\theta$ gives rise to a C^*-algebra denoted by A_θ. Then the collection $\{A_{\hbar\theta}\}_{\hbar\in[0,1]}$ is a continuous family of C^*-algebras providing a strict deformation quantization in the direction of the Poisson structure on A defined by the matrix θ.

Our cases correspond to the choice $A = C(M)$ with an action of $\mathbb{R}^n$ that is periodic or, in other words, an action of $\mathbb{T}^n$. The smooth elements in the deformed algebra make up the algebra $C^\infty(M_\theta)$. The quantization map plays a key role in the following, allowing us to extend differential geometric techniques from M to the noncommutative space M_θ.

3.2. *Examples: planes and spheres*

For $\lambda_{\mu\nu} = e^{2\pi i\theta_{\mu\nu}}$, where $\theta_{\mu\nu}$ is an antisymmetric real-valued matrix, the algebra $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ of polynomial functions on the noncommutative $2n$-plane is defined to be the unital $*$-algebra generated by $2n$ elements $\{z_\mu, z^*_\mu, \mu = 1, \ldots, n\}$ with relations

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu, \quad z^*_\mu z_\nu = \lambda_{\nu\mu} z_\nu z^*_\mu, \quad z^*_\mu z^*_\nu = \lambda_{\mu\nu} z^*_\nu z^*_\mu.$$

The involution $*$ is defined by putting $(z_\mu)^* = z^*_\mu$. For $\theta = 0$ one recovers the commutative $*$-algebra of complex polynomial functions on $\mathbb{R}^{2n}$.

For any value of the index μ, the element $z^*_\mu z_\mu = z_\mu z^*_\mu$ is central. Let $\mathcal{A}(S^{2n-1}_\theta)$ be the $*$-quotient of $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ by the two-sided ideal generated by the central element $\sum_\mu z^*_\mu z_\mu - 1$. We will denote the images of z_μ under the quotient map again by z_μ.

The abelian group $\mathbb{T}^n$ acts on $\mathcal{A}(\mathbb{R}^{2n}_\theta)$ by $*$-automorphisms. For $s = (s_\mu) \in \mathbb{T}^n$, σ_s is defined on the generators by $\sigma_s(z_\mu) = e^{2\pi i s_\mu} z_\mu$. Clearly, $s \mapsto \sigma_s$ is a group-homomorphism from $\mathbb{T}^n \to \mathrm{Aut}(\mathcal{A}(\mathbb{R}^{2n}_\theta))$. At the special case $\theta = 0$, the map σ is induced by a smooth action of $\mathbb{T}^n$ on the manifold $\mathbb{R}^{2n}$. Since the ideal generating the algebra $\mathcal{A}(S^{2n-1}_\theta)$ is invariant under the action, σ induces a group homomorphism from $\mathbb{T}^n$ into the group of automorphisms on the quotient $\mathcal{A}(S^{2n-1}_\theta)$ as well.

We continue by defining the unital $*$-algebra $\mathcal{A}(\mathbb{R}^{2n+1}_\theta)$ of polynomial functions on the noncommutative $(2n+1)$-plane which is given by adjoining a central self-adjoint generator z_0 to the algebra $\mathcal{A}(\mathbb{R}^{2n}_\theta)$, i.e. $z^*_0 = z_0$ and $z_0 z_\mu = z_\mu z_0$, for $\mu = 1, \ldots, n$. The action of $\mathbb{T}^n$ is extended trivially by $\sigma_s(z_0) = z_0$. Let $\mathcal{A}(S^{2n}_\theta)$ be the $*$-quotient of $\mathcal{A}(\mathbb{R}^{2n+1}_\theta)$ by the ideal generated by the central element $\sum z^*_\mu z_\mu + z_0^2 - 1$. As before, we will denote the canonical images of z_μ and z_0 again by z_μ and z_0, respectively. Since $\mathbb{T}^n$ leaves this ideal invariant, it induces an action by $*$-automorphisms on the quotient $\mathcal{A}(S^{2n}_\theta)$.

Next, we construct a differential calculus on $\mathbb{R}^m_\theta$. For $m = 2n$, the complex unital associative graded $*$-algebra $\Omega(\mathbb{R}^{2n}_\theta)$ of forms is generated by $2n$ elements z_μ, z^*_μ of degree 0 and $2n$ elements dz_μ, dz^*_μ of degree 1 with relations

$$\begin{gathered} dz_\mu dz_\nu + \lambda_{\mu\nu} dz_\nu dz_\mu = 0, \quad dz^*_\mu dz_\nu + \lambda_{\nu\mu} dz_\nu dz^*_\mu = 0, \\ dz^*_\mu dz^*_\nu + \lambda_{\mu\nu} dz^*_\nu dz^*_\mu = 0, \quad z_\mu dz_\nu = \lambda_{\mu\nu} dz_\nu z_\mu, \\ z^*_\mu dz_\nu = \lambda_{\nu\mu} dz_\nu z^*_\mu, \quad z^*_\mu dz^*_\nu = \lambda_{\mu\nu} dz^*_\nu z^*_\mu. \end{gathered} \tag{3.11}$$

There is a unique differential d on $\Omega(\mathbb{R}^{2n}_\theta)$ such that $\mathrm{d} : z_\mu \mapsto dz_\mu$. The involution $\omega \mapsto \omega^*$ for $\omega \in \Omega(\mathbb{R}^{2n}_\theta)$ is the graded extension of $z_\mu \mapsto z^*_\mu$, i.e. it is such that $(\mathrm{d}\omega)^* = \mathrm{d}\omega^*$ and $(\omega_1\omega_2)^* = (-1)^{p_1p_2}\omega^*_2\omega^*_1$ for $\omega_i \in \Omega^{p_i}(\mathbb{R}^{2n}_\theta)$. For $m = 2n+1$, we adjoin to $\Omega(\mathbb{R}^{2n}_\theta)$ one generator z_0 of degree 0 and one generator dz_0 of degree 1 with commutations

$$z_0 dz_0 = dz_0 z_0, \quad z_0\omega = \omega z_0, \quad dz_0\omega = (-1)^{|\omega|}\omega dz_0.$$

We extend the differential d and the graded involution $\omega \mapsto \omega^*$ of $\Omega(\mathbb{R}^{2n}_\theta)$ to $\Omega(\mathbb{R}^{2n+1}_\theta)$ by setting $z^*_0 = z_0$ and $(\mathrm{d}z_0)^* = \mathrm{d}z_0$, so that $(dz_0)^* = dz_0$.

The differential calculi $\Omega(S^m_\theta)$ on the noncommutative spheres S^m_θ are defined to be the quotients of $\Omega(\mathbb{R}^{m+1}_\theta)$ by the differential ideals generated by the central elements $\sum_\mu z^*_\mu z_\mu - 1$ and $\sum z^*_\mu z_\mu + z_0^2 - 1$, for $m = 2n-1$ and $m = 2n$ respectively.

The action of $\mathbb{T}^n$ by $*$-automorphisms on $\mathcal{A}(M_\theta)$ is extended to the differential calculi $\Omega(M_\theta)$, for $M_\theta = \mathbb{R}^m_\theta$ and $M = S^m_\theta$, by imposing that $\sigma_s \circ \mathrm{d} = \mathrm{d} \circ \sigma_s$.

3.3. *Vector bundles on M_θ*

We will presently give vector bundles on M_θ in terms of a deformed or $*$-product.

Let E be a *σ-equivariant* vector bundle M, that is a bundle which carries an action V of $\mathbb{T}^n$ by automorphisms, covering the action σ of $\mathbb{T}^n$ on M,

$$V_s(f\psi) = \sigma_s(f)V_s(\psi), \quad \forall\, f \in C^\infty(M),\, \psi \in \Gamma^\infty(M, E). \tag{3.12}$$

The $C^\infty(M_\theta)$-bimodule $\Gamma^\infty(M_\theta, E)$ is defined as the vector space $\Gamma^\infty(M, E)$ but with the bimodule structure given by

$$f \triangleright_\theta \psi = \sum_k f_k V_{\frac{1}{2}k\cdot\theta}(\psi), \qquad \psi \triangleleft_\theta f = \sum_k V_{-\frac{1}{2}k\cdot\theta}(\psi) f_k, \tag{3.13}$$

where $f = \sum_k f_k$, with $f_k \in C^\infty(M)$ homogeneous of degree k under the action of $\mathbb{T}^n$ – as in (3.5) – and ψ is a smooth section of E. By using the explicit expression (3.2) for the star product and Eq. (3.12), one checks that these are indeed actions of $C^\infty(M_\theta)$.

The $C^\infty(M_\theta)$-bimodule $\Gamma^\infty(M_\theta, E)$ is finite projective [22] and still carries an action of $\mathbb{T}^n$ by V with equivariance as in (3.12) for both the left and right action of $C^\infty(M_\theta)$. Indeed, the group $\mathbb{T}^n$ being abelian, one establishes that

$$V_s(f \triangleright_\theta \psi) = \sigma_s(f) \triangleright_\theta V_s(\psi), \quad \forall f \in C^\infty(M_\theta), \, \psi \in \Gamma^\infty(M_\theta, E), \tag{3.14}$$

and a similar property for the right structure $\triangleleft_\theta$. Indeed, due to the fact that the category of σ-equivariant finite projective modules over $C^\infty(M)$ is equivalent to the category of σ-equivariant finite projective modules over $C^\infty(M_\theta)$ [40], all equivariant finite projective modules on $C^\infty(M_\theta)$ are of the above type. This also reflects the result in [60] that the K-groups of a C^*-algebra deformed by an action of $\mathbb{R}^n$ are isomorphic to the K-groups of the original C^*-algebra: as mentioned above, the noncommutative manifolds M_θ are a special case – in which the starting algebra is commutative and the action periodic – of the general formulation in [59] of deformations of C^*-algebras under an action of $\mathbb{R}^n$.

From the very definition of $\Gamma^\infty(M_\theta, E)$ the following lemma is true.

Lemma 10. *If E, F are isomorphic as σ-equivariant vector bundles, then $\Gamma^\infty(M_\theta, E)$ and $\Gamma^\infty(M_\theta, F)$ are isomorphic as $C^\infty(M_\theta)$-bimodules.*

Although we defined the above left and right actions on the sections with respect to an action of $\mathbb{T}^n$ on E, the same construction can be done for vector bundles carrying instead an action of the double cover $\widetilde{\mathbb{T}}^n$. We have already seen an example of this in the case of the spinor bundle, where we defined a left action of $C^\infty(M_\theta)$ in terms of (3.6).

3.4. *Differential calculus on M_θ*

It is straightforward to construct a differential calculus on M_θ. This can be done in two equivalent manners, either by extending to forms the quantization maps, or by using the general construction in [20] by means of the Dirac operator.

Firstly, let $(\Omega(M), \mathrm{d})$ be the usual differential calculus on M, with d the exterior derivative. The quantization map $L_\theta : C^\infty(M) \to C^\infty(M_\theta)$ is extended to $\Omega(M)$ by imposing that it commutes with d. The image $L_\theta(\Omega(M))$ will be denoted $\Omega(M_\theta)$. Equivalently, $\Omega(M_\theta)$ could be defined to be $\Omega(M)$ as a vector space but equipped with an "exterior deformed product" which is the extension of the product (3.2) to $\Omega(M)$ by the requirement that it commutes with d. Indeed, since the action of $\mathbb{T}^n$ commutes with d, an element in $\Omega(M)$ can be decomposed into a sum of homogeneous elements for the action of $\mathbb{T}^n$ – as was done for $C^\infty(M)$. Then one defines a star product $\times_\theta$ on homogeneous elements in $\Omega(M)$ as in (3.2) and denotes

$\Omega(M_\theta) = (\Omega(M), \times_\theta)$. This construction is in concordance with the previous section, when $\Omega(M)$ is considered as the $C^\infty(M)$-bimodule of sections of the cotangent bundle. The extended action of $\mathbb{T}^n$ from $C^\infty(M)$ to $\Omega(M)$ is used to endow the space $\Omega(M_\theta)$ with the structure of a $C^\infty(M_\theta)$-bimodule with the left and right action given in (3.13).

As mentioned, a differential calculus $\Omega_D(C^\infty(M_\theta))$ on $C^\infty(M_\theta)$ can also be obtained from a general procedure [20] by means of the isospectral Dirac operator D on M_θ. The $C^\infty(M_\theta)$-bimodule $\Omega^p_D(C^\infty(M_\theta))$ of p-forms is made of classes of operators of the form

$$\omega = \sum_j a_0^j [D, a_1^j] \cdots [D, a_p^j], \quad a_i^j \in C^\infty(M_\theta), \tag{3.15}$$

modulo the sub-bimodule of operators (the so-called "junk forms")

$$\left\{ \sum_j [D, b_0^j][D, b_1^j] \cdots [D, b_{p-1}^j] : \ b_i^j \in C^\infty(M_\theta), \ b_0^j[D, b_1^j] \cdots [D, b_{p-1}^j] = 0 \right\}. \tag{3.16}$$

With brackets $[\ \cdot\ ]$ denoting the corresponding equivalence classes, the exterior differential d_D is

$$\mathrm{d}_D \left[\sum_j a_0^j [D, a_1^j] \cdots [D, a_p^j] \right] := \left[\sum_j [D, a_0^j][D, a_1^j] \cdots [D, a_p^j] \right], \tag{3.17}$$

and satisfies $\mathrm{d}_D^2 = 0$. One introduces an inner product on forms by declaring that forms of different degree are orthogonal, while the inner product of two p-forms ω_1, ω_2 is defined to be

$$(\omega_1, \omega_2)_D := \int\!\!\!\!\!\!- \ \omega_1^* \omega_2. \tag{3.18}$$

Here the noncommutative integral is the natural extension of the one in (3.9),

$$\int\!\!\!\!\!\!- \ T := \mathrm{Tr}_\omega \big(T |D|^{-m} \big), \tag{3.19}$$

with T an element in a suitable class of operators. Not surprisingly, these two constructions of forms agree [22], that is, the differential calculi $\Omega(M_\theta)$ and $\Omega_D(C^\infty(M_\theta))$ are isomorphic. This allows us in particular to integrate forms of top dimension, by defining

$$\int_{M_\theta} \omega := \int\!\!\!\!\!\!- \ \omega_D, \qquad \text{for } \omega \in \Omega^m(M_\theta), \tag{3.20}$$

where ω_D denotes the element in $\Omega_D^m(C^\infty(M_\theta))$ corresponding to ω (replacing every d in ω by d_D). We have the following noncommutative Stokes theorem.

Lemma 11. *If $\omega \in \Omega^{m-1}(M_\theta)$, then*

$$\int_{M_\theta} \mathrm{d}\omega = 0.$$

Proof. From the definition of the noncommutative integral,

$$\int_{M_\theta} \mathrm{d}\omega = ⨍ \mathrm{d}_D \omega_D = ⨍ \mathrm{d}_D L_\theta(\omega_D^{(0)}),$$

with $\omega_D^{(0)}$ the classical analogue of ω, i.e. $\omega = L_\theta(\omega_D^{(0)})$. At this point one remembers that D commutes with L_θ (see Sec. 3.1), and realizes that there is an analogue of Lemma 9 for forms, i.e. $⨍ L_\theta(T) = \int_M T$. One concludes that the above integral vanishes since it vanishes in the classical case. $\square$

The next ingredient is a Hodge star operator on $\Omega(M_\theta)$. Classically, the Hodge star operator is a map $* : \Omega^p(M) \to \Omega^{m-p}(M)$ depending only on the conformal class of the metric on M. On the one hand, since $\mathbb{T}^n$ acts by isometries, it leaves the conformal structure invariant and therefore, it commutes with $*$. On the other hand, with the isospectral deformation one does not change the metric. Thus it makes sense to define a map $*_\theta : \Omega^p(M_\theta) \to \Omega^{m-p}(M_\theta)$ by

$$*_\theta L_\theta(\omega) = L_\theta(*\omega), \quad \text{for } \omega \in \Omega(M_\theta). \tag{3.21}$$

With this Hodge operator, there is an alternative definition of an inner product on $\Omega(M_\theta)$. Given that $*_\theta$ maps $\Omega^p(M_\theta)$ to $\Omega^{m-p}(M_\theta)$, we can define for $\alpha, \beta \in \Omega^p(M_\theta)$

$$(\alpha, \beta)_2 = ⨍ *_\theta(\alpha^* *_\theta \beta), \tag{3.22}$$

since $*_\theta(\alpha^* *_\theta \beta)$ is an element in $C^\infty(M_\theta)$.

Lemma 12. *Under the isomorphism $\Omega_D(C^\infty(M_\theta)) \simeq \Omega(M_\theta)$, the inner product $(\cdot,\cdot)_2$ coincides with $(\cdot,\cdot)_D$.*

Proof. Let ω_1, ω_2 be two forms in $\Omega_D(C^\infty(M))$, so that $L_\theta(\omega_i)$ are two generic forms in $\Omega_D(C^\infty(M_\theta)) \simeq L_\theta(\Omega_D(C^\infty(M))$. Then, using Lemma 9 it follows that

$$⨍ L_\theta(\omega_1)^* L_\theta(\omega_2) = ⨍ L_\theta(\omega_1^* \times_\theta \omega_2) = ⨍ \omega_1^* \times_\theta \omega_2. \tag{3.23}$$

Now, the inner product $(\ ,\ )_D$ coincides with $(\ ,\ )_2$ as defined by (3.22) in the classical case – under the above isomorphism $\Omega_D(C^\infty(M)) \simeq \Omega(M)$; see for example [20, VI.1]. It follows that the above expression equals

$$\fint *(\omega_1^* \times_\theta (*\omega_2)) = \fint *_\theta(L_\theta(\omega_1)^*(*_\theta L_\theta(\omega_2))), \tag{3.24}$$

using Lemma 9 for forms once more, together with the defining property of $*_\theta$. □

Lemma 13. *The formal adjoint* d^* *of* d *with respect to the inner product* $(\cdot,\cdot)_2$, *i.e. so that* $(\mathrm{d}^*\alpha,\beta)_2 = (\alpha,\mathrm{d}\beta)_2$, *is given on* $\Omega^p(M_\theta)$ *by*

$$\mathrm{d}^* = (-1)^{m(p+1)+1} *_\theta \mathrm{d} *_\theta$$

Proof. Just as in the classical case, this follows from Lemma 11, together with

$$\int_{M_\theta} \omega = \fint *_\theta \omega, \qquad \omega \in \Omega^m(M_\theta),$$

a result again established from the classical case using the mentioned analogue of Lemma 9 for forms. □

3.5. *Local index formula on toric noncommutative manifolds*

For toric noncommutative manifolds the local index formula of Connes and Moscovici [25] – that we shall use later on – simplifies drastically.

We recall the general form of the local index formula; and we limit ourselves to the "even" case, the only relevant one for the present paper. Suppose in general that $(\mathcal{A},\mathcal{H},D,\gamma)$ is an even p-summable spectral triple with discrete and simple dimension spectrum. For a projection $e \in M_N(A)$, the operator

$$D_e = e(D \otimes \mathbb{I}_N)e$$

is a Fredholm operator, thought of as the Dirac operator with coefficients in the module determined by e. Then the local index formula of Connes and Moscovici [25] provides a method to compute its index via the pairing of suitable cyclic cycles and cocycles.

Let $C_*(\mathcal{A})$ be the chain complex over the algebra $\mathcal{A}$; in degree n, $C_n(\mathcal{A}) := \mathcal{A}^{\otimes(n+1)}$. On this complex there are defined the Hochschild operator $b : C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A})$ and the boundary operator $B : C_n(\mathcal{A}) \to C_{n+1}(\mathcal{A})$, satisfying $b^2 = 0, B^2 = 0, bB + Bb = 0$; thus $(b+B)^2 = 0$. From general homological theory, one defines a bicomplex $CC_*(\mathcal{A})$ by $CC_{(n,m)}(\mathcal{A}) := CC_{n-m}(\mathcal{A})$ in bi-degree (n,m). Dually, one defines $CC^*(\mathcal{A})$ as functionals on $CC_*(\mathcal{A})$, equipped with the dual Hochschild operator b and coboundary operator B (we refer to [20] and [51] for more details on this).

Theorem 14. (Connes–Moscovici [25]). (a) *An even cocycle* $\phi^* = \sum_{k\geq 0} \phi^{2k}$ *in* $CC^*(\mathcal{A})$, $(b+B)\phi^* = 0$, *is defined by the following formulae. For* $k=0$,

$$\phi^0(a) := \operatorname*{Res}_{z=0} z^{-1} \operatorname{tr}(\gamma a |D|^{-2z});$$

whereas for $k > 0$

$$\begin{aligned}\phi^k(a^0, \ldots, a^{2k}) \\ := \sum_{\alpha} c_{k,\alpha} \operatorname*{Res}_{z=0} \operatorname{tr}\left(\gamma a^0 [D, a^1]^{(\alpha_1)} \cdots [D, a^{2k}]^{(\alpha_{2k})} |D|^{-2(|\alpha|+k+z)}\right),\end{aligned}$$

with

$$c_{k,\alpha} = (-1)^{|\alpha|}\Gamma(k+|\alpha|)\big(\alpha!(\alpha_1+1)(\alpha_1+\alpha_2+2)\cdots(\alpha_1+\cdots+\alpha_{2k}+2k)\big)^{-1},$$

and $T^{(j)}$ *denotes the jth iteration of the derivation* $T \mapsto [D^2, T]$.

(b) *For* $e \in K_0(\mathcal{A})$, *the Chern character* $\operatorname{ch}_*(e) = \sum_{k\geq 0} \operatorname{ch}_k(e)$ *is the even cycle in* $CC_*(\mathcal{A})$, $(b+B)\operatorname{ch}_*(e) = 0$, *defined by the following formulae. For* $k = 0$,

$$\operatorname{ch}_0(e) := \operatorname{tr}(e),$$

whereas for $k > 0$

$$\operatorname{ch}_k(e) := (-1)^k \frac{(2k)!}{k!} \sum \left(e_{i_0 i_1} - \frac{1}{2}\delta_{i_0 i_1}\right) \otimes e_{i_1 i_2} \otimes e_{i_2 i_3} \otimes \cdots \otimes e_{i_{2k} i_0}.$$

(c) *The index is given by the natural pairing between cycles and cocycles*

$$\operatorname{index}(D_e) = \langle \phi^*, \operatorname{ch}_*(e) \rangle. \tag{3.25}$$

For toric noncommutative manifolds, the above local index formula simplifies drastically [47].

Theorem 15. *For a projection* $p \in M_N(C^\infty(M_\theta))$, *we have*

$$\begin{aligned}\operatorname{index} D_p &= \operatorname*{Res}_{z=0} z^{-1} \operatorname{tr}\left(\gamma p |D|^{-2z}\right) \\ &\quad + \sum_{k\geq 1} \frac{(-1)^k}{k} \operatorname*{Res}_{z=0} \operatorname{tr}\left(\gamma\left(p - \frac{1}{2}\right)[D, p]^{2k} |D|^{-2(k+z)}\right),\end{aligned}$$

and now the trace tr *comprises a matrix trace as well.*

Proof. Recall that the quantization map L_θ preserves the spectral decomposition, for the toric action of $\widetilde{\mathbb{T}}^n$, of smooth operators (see Eq. (3.7)). Then, having extended the deformed $\times_\theta$-product to $C^\infty(M_\theta) \cup [D, C^\infty(M_\theta)]$ – unambiguously since D is of degree 0 – we write the local cocycles ϕ^k in Theorem 14 in terms of the quantization map L_θ:

$$\begin{aligned}&\phi^k\big(L_\theta(f^0), L_\theta(f^1), \ldots, L_\theta(f^{2k})\big)\\ &\quad = \operatorname*{Res}_{z=0} \operatorname{tr}\Big(\gamma L_\theta\big(f^0 \times_\theta [D, f^1]^{(\alpha_1)} \times_\theta \cdots \times_\theta [D, f^{2k}]^{(\alpha_{2k})}\big)|D|^{-2(|\alpha|+k+z)}\Big).\end{aligned} \tag{3.26}$$

Suppose now that $f^0, \ldots, f^{2k} \in C^\infty(M)$ are homogeneous of degree $r^0, \ldots, r^{2k}$ respectively, so that the operator $f^0 \times_\theta [D, f^1]^{(\alpha_1)} \times_\theta \cdots \times_\theta [D, f^{2k}]$ is a homogeneous element of degree $r = \sum_{j=0}^{2k} r^j$. It is in fact a multiple of $f^0[D, f^1] \cdots [D, f^{2k}]$ as can be established by working out the $\times_\theta$-products. Forgetting about this factor – which is a power of the deformation parameter λ – we obtain from (3.6) that

$$L_\theta(f^0 \times_\theta [D, f^1]^{(\alpha_1)} \times_\theta \cdots \times_\theta [D, f^{2k}]) = f^0[D, f^1] \cdots [D, f^{2k}]U\left(\frac{1}{2} r \cdot \theta\right). \tag{3.27}$$

Each term in the local index formula for $(C^\infty(M_\theta), \mathcal{H}, D)$ then takes the form

$$\operatorname*{Res}_{z=0} \operatorname{tr}\big(\gamma f^0[D, f^1]^{(\alpha_1)} \cdots [D, f^{2k}]^{(\alpha_{2k})}|D|^{-2(|\alpha|+k+z)}U(s)\big)$$

for $s = \frac{1}{2} r \cdot \theta \in \widetilde{\mathbb{T}}^n$. The appearance of the operator $U(s)$ here is a consequence of the close relation with the index formula for $\mathbb{T}^n$-equivariant Dirac spectral triples. In [18] an even-dimensional compact spin manifold M on which a (connected compact) Lie group G acts by isometries was studied. The equivariant Chern character was defined as an equivariant version of the JLO-cocycle, the latter being an element in equivariant entire cyclic cohomology. The essential point is that an explicit formula for the above residues was obtained. In the case of a $\mathbb{T}^n$-action on M this is

$$\begin{aligned}&\operatorname*{Res}_{z=0} \operatorname{tr}\big(\gamma f^0[D, f^1]^{(\alpha_1)} \cdots [D, f^{2k}]^{(\alpha_{2k})}|D|^{-2(|\alpha|+k+z)}U(s)\big)\\ &\quad = \Gamma^\infty(|\alpha| + k) \lim_{t \to 0} t^{|\alpha|+k} \operatorname{tr}\big(\gamma f^0[D, f^1]^{(\alpha_1)} \cdots [D, f^{2k}]^{(\alpha_{2k})} e^{-tD^2} U(s)\big),\end{aligned} \tag{3.28}$$

for every $s \in \widetilde{\mathbb{T}}^n$. Moreover, from Theorem 2 in [18] the limit vanishes when $|\alpha| \neq 0$. This finishes the proof of our theorem. $\square$

By inserting the symbol π for the algebra representation, the components of the Chern character are represented as operators on the Hilbert space $\mathcal{H}$ by explicit formulae,

$$\pi_D(\mathrm{ch}_k(e)) := (-1)^k \frac{(2k)!}{k!} \sum \left(\pi(e_{i_0 i_1}) - \frac{1}{2}\delta_{i_0 i_1} \right) [D, \pi(e_{i_1 i_2})][D, \pi(e_{i_1 i_2})] \cdots [D, \pi(e_{i_{2k} i_0})], \tag{3.29}$$

for $k > 0$, while $\pi_D(\mathrm{ch}_0(e)) = \sum \pi(e_{i_0 i_0})$.

4. The Hopf Fibration on S^4_θ

We now focus on the two noncommutative spheres S^4_θ and $S^7_{\theta'}$ starting from the algebras $\mathcal{A}(S^4_\theta)$ and $\mathcal{A}(S^7_{\theta'})$ of polynomial functions on them. The latter algebra carries an action of the (classical) group $\mathrm{SU}(2)$ by automorphisms in such a way that its invariant elements are exactly the polynomials on S^4_θ. The antisymmetric 2×2 matrix θ is given by a single real number also denoted by θ. On the other hand, the requirements that $\mathrm{SU}(2)$ acts by automorphisms and that S^4_θ is the algebra of invariant functions, give the matrix θ' in terms of θ as well. This yields a one-parameter family of noncommutative Hopf fibrations $S^7_{\theta'} \to S^4_\theta$. Moreover, there is an inclusion of the differential calculi $\Omega(S^4_\theta) \subset \Omega(S^7_{\theta'})$, defined in Sec. 3.4.

For each irreducible representation $W^{(n)} := \mathrm{Sym}^n(\mathbb{C}^2)$ of $\mathrm{SU}(2)$ we construct the noncommutative vector bundles $E^{(n)}$ associated to the fibration $S^7_{\theta'} \to S^4_\theta$. These bundles are described by the $C^\infty(S^4_\theta)$-bimodules of "equivariant maps from $S^7_{\theta'}$ to $W^{(n)}$". As expected, these modules are finite projective and we construct explicitly the projections $p_{(n)} \in M_{4^n}(\mathcal{A}(S^4_\theta))$ such that these modules are isomorphic to the image of $p_{(n)}$ in $\mathcal{A}(S^4_\theta)^{4^n}$. In the special case of the defining representation, we recover the basic instanton projection on the sphere S^4_θ constructed in [23]. Then, one defines connections $\nabla = p_{(n)}\mathrm{d}$ as maps from $\Gamma^\infty(S^4_\theta, E^{(n)})$ to $\Gamma^\infty(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta)$. The corresponding connection one-form A turns out to be valued in a representation of the Lie algebra $su(2)$. By using the projection $p_{(n)}$, the Dirac operator with coefficients in the noncommutative vector bundle $E^{(n)}$ is given by $D_{p_{(n)}} := p_{(n)} D p_{(n)}$. We compute its index by using the very simple form of the local index theorem of Connes and Moscovici [25] in the case of toric noncommutative manifolds as obtained in Th. 15.

Finally, we show that the fibration $S^7_{\theta'} \to S^4_\theta$ is a "non-trivial principal bundle with structure group $\mathrm{SU}(2)$". This means that the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a non-cleft Hopf–Galois extension [41, 55]; in fact, it is a principal extension [14]. We find an explicit form of the so-called strong connection [38] which induces connections on the associated bundles

$E^{(n)}$ as maps from $\Gamma^\infty(S^4_\theta, E^{(n)})$ to $\Gamma^\infty(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$, where $\Omega^*_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$ is the universal differential calculus on $\mathcal{A}(S^4_\theta)$. We show that these connections coincide with the Grassmann connections $\nabla = p_{(n)}\mathrm{d}$ on $\Omega(S^4_\theta)$.

4.1. *Let us roll noncommutative spheres*

With θ a real parameter, the algebra $\mathcal{A}(S^4_\theta)$ of polynomial functions on the sphere S^4_θ is generated by elements $z_0 = z_0^*, z_j, z_j^*$, $j = 1, 2$, subject to relations

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu, \quad z_\mu z_\nu^* = \lambda_{\nu\mu} z_\nu z_\mu^*, \quad z_\mu^* z_\nu^* = \lambda_{\mu\nu} z_\nu^* z_\mu^*, \quad \mu, \nu = 0, 1, 2, \tag{4.1}$$

together with the spherical relation $\sum_\mu z_\mu^* z_\mu = 1$. The deformation parameters are given by $\lambda_{\mu\mu} = 1$ and

$$\lambda_{12} = \overline{\lambda}_{21} =: \lambda = e^{2\pi \mathrm{i}\theta}, \quad \lambda_{j0} = \lambda_{0j} = 1, \quad j = 1, 2. \tag{4.2}$$

At $\theta = 0$ one recovers the $*$-algebra of complex polynomial functions on the usual S^4.

The differential calculus $\Omega(S^4_\theta)$ is generated as a graded differential $*$-algebra by the elements z_μ, z_μ^* of degree 0 and elements dz_μ, dz_μ^* of degree 1 satisfying the relations

$$\begin{aligned} &dz_\mu dz_\nu + \lambda_{\mu\nu} dz_\nu dz_\mu = 0, \qquad &&dz_\mu^* dz_\nu + \lambda_{\nu\mu} dz_\nu dz_\mu^* = 0, \\ &z_\mu dz_\nu = \lambda_{\mu\nu} dz_\nu z_\mu, &&z_\mu^* dz_\nu = \lambda_{\nu\mu} dz_\nu z_\mu^*. \end{aligned} \tag{4.3}$$

with $\lambda_{\mu\nu}$ as before. There is a unique differential d on $\Omega(S^4_\theta)$ such that $\mathrm{d} : z_\mu \mapsto dz_\mu$ and the involution on $\Omega(S^4_\theta)$ is the graded extension of $z_\mu \mapsto z_\mu^*$.

With $\lambda'_{ab} = e^{2\pi \mathrm{i}\theta'_{ab}}$ and $(\theta'_{ab} = -\theta'_{ba})$ a real antisymmetric matrix, the algebra $\mathcal{A}(S^7_{\theta'})$ of polynomial functions on the sphere $S^7_{\theta'}$ is generated by elements ψ_a, ψ_a^*, $a = 1, \ldots, 4$, subject to relations

$$\psi_a \psi_b = \lambda'_{ab} \psi_b \psi_a, \quad \psi_a \psi_b^* = \lambda'_{ba} \psi_b^* \psi_a, \quad \psi_a^* \psi_b^* = \lambda'_{ab} \psi_b^* \psi_a^*, \tag{4.4}$$

and with the spherical relation $\sum_a \psi_a^* \psi_a = 1$. The above is a deformation of the $*$-algebra of complex polynomial functions on the sphere S^7. As before, a differential calculus $\Omega(S^7_{\theta'})$ can be defined to be generated by the elements ψ_a, ψ_a^* of degree 0 and elements $d\psi_a, d\psi_a^*$ of degree 1 satisfying relations similar to the ones in (4.3).

4.2. *The* SU(2) *principal fibration on* S^4

We review the classical construction of the instanton bundle on S^4, taking the approach of [43]. We generalize slightly and construct complex vector bundles on S^4 associated to all finite-dimensional irreducible representations of SU(2). Let

$$S^7 := \{\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{C}^4 : |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 = 1\},$$
$$S^4 := \{z = (z_1, z_2, z_0) \in \mathbb{C}^2 \oplus \mathbb{R} : z_1^* z_1 + z_2^* z_2 + z_0^2 = 1\},$$

$$\begin{aligned}\mathrm{SU}(2) &:= \{w \in GL(2, \mathbb{C}) : w^* w = w w^* = 1, \det\, w = 1\} \\ &= \left\{ w = \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} : w^1 \overline{w}^1 + w^2 \overline{w}^2 = 1 \right\}.\end{aligned}$$

The space S^7 carries a right SU(2)-action, $S^7 \times \mathrm{SU}(2) \to S^7$, given on generators by

$$\big((\psi_1, -\psi_2^*, \psi_3, -\psi_4^*), w\big) \mapsto (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}.$$

It might seem unnatural to define an action that mixes the ψ's and ψ^*'s. However, this is only a more convenient labelling for the left action of $Spin(5)$ on S^7, as we will see later on (cf. Eq. (6.19)). The Hopf projection map is defined as a map $\pi : (\psi) \mapsto (z)$ with

$$z_0 = \psi_1^* \psi_1 + \psi_2^* \psi_2 - \psi_3^* \psi_3 - \psi_4^* \psi_4,$$
$$z_1 = 2(\psi_1 \psi_3^* + \psi_2^* \psi_4), \quad z_2 = 2(-\psi_1^* \psi_4 + \psi_2 \psi_3^*),$$

and one computes that $z_1^* z_1 + z_2^* z_2 + z_0^2 = (\sum_a \psi_a^* \psi_a)^2 = 1$.

The finite-dimensional irreducible representations of SU(2) are labeled by a positive integer n with $(n+1)$-dimensional representation space $W^{(n)} \simeq \mathrm{Sym}^n(\mathbb{C}^2)$. The collection of smooth SU(2)-equivariant maps from S^7 to $W^{(n)}$,

$$C^\infty_{\mathrm{SU}(2)}(S^7, W^{(n)}) := \{\varphi : S^7 \to W^{(n)} : \varphi(\psi \cdot w) = w^{-1} \cdot \varphi(\psi)\}, \tag{4.5}$$

is identified with the $C^\infty(M)$-module of smooth sections of the associated vector bundle $S^7 \times_{\mathrm{SU}(2)} W^{(n)} \to S^4$. We will now construct projections $p_{(n)}$ as $N \times N$ matrices taking values in $C^\infty(S^4)$), such that $\Gamma^\infty(S^4, E^{(n)}) := p_{(n)} C^\infty(S^4)^N$ is isomorphic to $C^\infty_{\mathrm{SU}(2)}(S^7, W^{(n)})$ as right $C^\infty(S^4)$-modules. As the notation suggests, $E^{(n)}$ is the vector bundle over S^4 associated with the corresponding representation. Let us first recall the case $n = 1$ from

[43] and then use this to generate the vector bundles for any n. The SU(2)-equivariant maps from S^7 to $W^{(1)} \simeq \mathbb{C}^2$ are of the form

$$\varphi_{(1)}(\psi) = \begin{pmatrix} \psi_1^* \\ -\psi_2 \end{pmatrix} f_1 + \begin{pmatrix} \psi_2^* \\ \psi_1 \end{pmatrix} f_2 + \begin{pmatrix} \psi_3^* \\ -\psi_4 \end{pmatrix} f_3 + \begin{pmatrix} \psi_4^* \\ \psi_3 \end{pmatrix} f_4, \tag{4.6}$$

where $f_1, \ldots, f_4$ are smooth functions that are invariant under the action of SU(2), i.e. they are functions on the base space S^4.

A description of the equivariant maps is given in terms of ket-valued functions $|\xi\rangle$ on S^7, which are then elements in the free module $\mathcal{E} := \mathbb{C}^N \otimes C^\infty(S^7)$. The $C^\infty(S^7)$-valued Hermitian structure on $\mathcal{E}$ given by $\langle \xi, \eta\rangle = \sum_b \xi_b^* \eta_b$ allows one to associate dual elements $\langle \xi| \in \mathcal{E}^*$ to each $|\xi\rangle \in \mathcal{E}$ by $\langle \xi|(\eta) := \langle \xi, \eta\rangle,\ \forall\, \eta \in \mathcal{E}$. If we define $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{A}(S^7)^4$ by

$$|\psi_1\rangle = (\psi_1, \psi_2, \psi_3, \psi_4)^{\mathrm{t}}, \quad |\psi_2\rangle = (-\psi_2^*, \psi_1^*, -\psi_4^*, \psi_3^*)^{\mathrm{t}},$$

with t denoting transposition, the equivariant maps in (4.6) are given by

$$\varphi_{(1)}(\psi) = \begin{pmatrix} \langle \psi_1 | f\rangle \\ \langle \psi_2 | f\rangle \end{pmatrix}, \tag{4.7}$$

where $|f\rangle \in (C^\infty(S^4))^4 := \mathbb{C}^4 \otimes C^\infty(S^4)$. Since $\langle \psi_k | \psi_l\rangle = \delta_{kl}$ as is easily seen, we get a projection $p_{(1)}^2 = p_{(1)} = p_{(1)}^*$ in $M_4(C^\infty(S^4))$ by

$$p_{(1)} = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|.$$

Indeed, by explicit computation we find a matrix with entries in $C^\infty(S^4)$ which is the limit of the projection (4.19) at $\theta = 0$. Denoting the right $C^\infty(S^4)$-module $p_{(1)}(C^\infty(S^4))^4$ by $\Gamma^\infty(S^4, E^{(1)})$, we have that

$$\Gamma^\infty(S^4, E^{(1)}) \simeq C^\infty_{\mathrm{SU}(2)}(S^7, \mathbb{C}^2), \quad \sigma_{(1)} = p_{(1)}|f\rangle \leftrightarrow \varphi_{(1)},$$

with $\varphi_{(1)}$ the generic equivariant map given in (4.7). For a general representation, SU(2)-equivariant maps from S^7 to $W^{(n)}$ are of the form

$$\varphi_{(n)}(\psi) = \begin{pmatrix} \langle \phi_1 | f\rangle \\ \vdots \\ \langle \phi_{n+1} | f\rangle \end{pmatrix}, \tag{4.8}$$

where $|f\rangle \in C^\infty(S^4)^{4^n}$ and $|\phi_k\rangle$ is the completely symmetrized form of the tensor product $|\psi_1\rangle^{\otimes n-k+1} \otimes |\psi_2\rangle^{\otimes k-1}$ for $k = 1, \ldots, n+1$, normalized to have norm 1. For example, for the adjoint representation $n = 2$, we have

$$|\phi_1\rangle := |\psi_1\rangle \otimes |\psi_1\rangle,$$

$$|\phi_2\rangle := \frac{1}{\sqrt{2}}\big(|\psi_1\rangle \otimes |\psi_2\rangle + |\psi_2\rangle \otimes |\psi_1\rangle\big),$$

$$|\phi_3\rangle := |\psi_2\rangle \otimes |\psi_2\rangle.$$

Since in general $\langle\phi_k|\phi_l\rangle = \delta_{kl}$, the matrix-valued function

$$p_{(n)} = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_{n+1}\rangle\langle\phi_{n+1}| \in M_{4^n}(C^\infty(S^4))$$

is a projection with entries in $C^\infty(S^4)$, since each element $(p_{(n)})_{ab} = \sum_k |\phi_k\rangle_a \langle\phi_k|_b$ is SU(2)-invariant as can be easily seen. We conclude that

$$p_{(n)}(C^\infty(S^4)^{4^n}) \simeq C^\infty_{\mathrm{SU}(2)}(S^7, W^{(n)}), \quad \sigma_{(n)} = p_{(n)}|f\rangle \leftrightarrow \varphi_{(n)},$$

with $\varphi_{(n)}$ the generic equivariant map given in (4.8).

4.3. *The* SU(2) *principal fibration on* S^4_θ

Firstly, we recall that while there is a θ-deformation of the manifold $S^3 \simeq \mathrm{SU}(2)$ to a sphere S^3_θ, on the latter there is no compatible group structure so that there is no θ-deformation of the group SU(2) [22]. Therefore, we must choose the matrix $\theta'_{\mu\nu}$ in such a way that the noncommutative 7-sphere $S^7_{\theta'}$ carries a classical SU(2) action, which in addition is such that the subalgebra of $\mathcal{A}(S^7_{\theta'})$ consisting of SU(2)-invariant polynomials is exactly $\mathcal{A}(S^4_\theta)$. The action of SU(2) on the generators of $\mathcal{A}(S^7_{\theta'})$ is simply defined by

$$\begin{gathered}\alpha_w : (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \mapsto (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \\ w = \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix}. \end{gathered} \tag{4.9}$$

Here w^1 and w^2, satisfying $w^1\overline{w}^1 + w^2\overline{w}^2 = 1$, are the coordinate functions on SU(2). By imposing that the map $w \mapsto \alpha_w$ embeds SU(2) in $\mathrm{Aut}(\mathcal{A}(S^7_{\theta'}))$ we find that $\lambda'_{12} = \lambda'_{34} = 1$ and $\lambda'_{14} = \lambda'_{23} = \overline{\lambda'_{24}} = \overline{\lambda'_{13}} =: \lambda'$.

The subalgebra of SU(2)-invariant elements in $\mathcal{A}(S^7_{\theta'})$ can be found in the following way. From the diagonal nature of the action of SU(2) on $\mathcal{A}(S^7_{\theta'})$ and the above formulae for λ'_{ab}, we see that the action of SU(2) commutes with the action of $\mathbb{T}^2$ on $\mathcal{A}(S^7_{\theta'})$. Since $\mathcal{A}(S^7_{\theta'})$ coincides with

$\mathcal{A}(S^7)$ as vector spaces, we see that the subalgebra of SU(2)-invariant elements in $\mathcal{A}(S^7_{\theta'})$ is completely determined by the classical subalgebra of SU(2)-invariant elements in $\mathcal{A}(S^7)$. From Sec. 4.2 we can conclude that

$$\mathrm{Inv}_{\mathrm{SU}(2)}(\mathcal{A}(S^7_\theta)) = \mathbb{C}[\ 1, \psi_1\psi_3^* + \psi_2^*\psi_4, -\psi_1^*\psi_4 + \psi_2\psi_3^*, \psi_1\psi_1^* + \psi_2^*\psi_2\ ]$$

modulo the relations in the algebra $\mathcal{A}(S^7_{\theta'})$. We identify

$$\begin{aligned} z_0 &= \psi_1^*\psi_1 + \psi_2^*\psi_2 - \psi_3^*\psi_3 - \psi_4^*\psi_4 = 2(\psi_1^*\psi_1 + \psi_2^*\psi_2) - 1 \\ &= 1 - 2(\psi_3^*\psi_3 + \psi_4^*\psi_4), \\ z_1 &= 2(\mu\psi_3^*\psi_1 + \psi_2^*\psi_4) = 2(\psi_1\psi_3^* + \psi_2^*\psi_4), \\ z_2 &= 2(-\mu\psi_4\psi_1^* + \psi_2\psi_3^*) = 2(-\psi_1^*\psi_4 + \psi_2\psi_3^*) \end{aligned} \tag{4.10}$$

and compute that $z_1z_1^* + z_2z_2^* + z_0^2 = 1$. By imposing the commutation rules $z_1z_2 = \lambda z_2z_1$ and $z_1z_2^* = \overline{\lambda} z_2^*z_1$, we infer that $\lambda'_{14} = \lambda'_{23} = \overline{\lambda'_{24}} = \overline{\lambda'_{13}} = \sqrt{\lambda} =: \mu$ on $S^7_{\theta'}$. (Compatibility requires that $\mu^2 = \lambda$; we drop the case $\mu = -\sqrt{\lambda}$ since its "classical" limit would correspond to "anti-commuting" coordinates.) We conclude that $\mathrm{Inv}_{\mathrm{SU}(2)}(\mathcal{A}(S^7_{\theta'})) = \mathcal{A}(S^4_\theta)$ for a matrix of deformation parameters $\lambda'_{ab} = e^{2\pi\mathrm{i}\theta'_{ab}}$ of the following form:

$$\lambda'_{ab} = \begin{pmatrix} 1 & 1 & \overline{\mu} & \mu \\ 1 & 1 & \mu & \overline{\mu} \\ \mu & \overline{\mu} & 1 & 1 \\ \overline{\mu} & \mu & 1 & 1 \end{pmatrix}, \quad \mu = \sqrt{\lambda}, \quad \text{or} \quad \theta'_{ab} = \frac{\theta}{2}\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}. \tag{4.11}$$

The relations (4.10) can be also expressed in the form,

$$z_\alpha = \sum_{ab} \psi_a^*(\gamma_\alpha)_{ab}\psi_b, \qquad z_\alpha^* = \sum_{ab} \psi_a^*(\gamma_\alpha^*)_{ab}\psi_b, \qquad \alpha = 0, 1, 2, \tag{4.12}$$

with γ_α the following twisted 4×4 Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \gamma_1 = 2\begin{pmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \\ \begin{smallmatrix} \mu & 0 \\ 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}, \quad \gamma_2 = 2\begin{pmatrix} 0 & \begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & \overline{\mu} \\ 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}. \tag{4.13}$$

Note that γ_0 is the usual grading given by

$$\gamma_0 = -\frac{1}{4}[\gamma_1, \gamma_1^*][\gamma_2, \gamma_2^*].$$

These matrices satisfy the following relations of the twisted Clifford algebra [22]:

$$\gamma_j\gamma_k + \lambda_{jk}\gamma_k\gamma_j = 0, \qquad \gamma_j\gamma_k^* + \lambda_{kj}\gamma_k^*\gamma_j = 4\delta_{jk}, \quad j, k = 1, 2. \tag{4.14}$$

There are compatible toric actions on S^4_θ and $S^7_{\theta'}$. The torus $\mathbb{T}^2$ acts on $\mathcal{A}(S^4_\theta)$ as

$$\sigma_s(z_0, z_1, z_2) = (z_0, e^{2\pi i s_1} z_1, e^{2\pi i s_2} z_2), \quad s \in \mathbb{T}^2. \tag{4.15}$$

This action lifts to a double cover action on $\mathcal{A}(S^7_{\theta'})$. The double cover map $p : \widetilde{\mathbb{T}}^2 \to \mathbb{T}^2$ is given explicitly by $p : (s_1, s_2) \mapsto (s_1 + s_2, -s_1 + s_2)$. Then $\widetilde{\mathbb{T}}^2$ acts on the ψ_a's as

$$\widetilde{\sigma} : (\psi_1, \psi_2, \psi_3, \psi_4) \mapsto \left(e^{2\pi i s_1} \psi_1,\ e^{-2\pi i s_1} \psi_2,\ e^{-2\pi i s_2} \psi_3,\ e^{2\pi i s_2} \psi_4\right). \tag{4.16}$$

Equation (4.10) shows that $\widetilde{\sigma}$ is indeed a lifting to $S^7_{\theta'}$ of the action of $\mathbb{T}^2$ on S^4_θ. Clearly, this compatibility is built into the construction of the Hopf fibration $S^7_{\theta'} \to S^4_\theta$ as a deformation of the classical Hopf fibration $S^7 \to S^4$ with respect to an action of $\mathbb{T}^2$, a fact that also dictated the form of the deformation parameter λ' in (4.11). As we shall see, the previous double cover of tori comes from a spin cover $\mathrm{Spin}_\theta(5)$ of $\mathrm{SO}_\theta(5)$ deforming the usual action of $\mathrm{Spin}(5)$ on S^7 as a double cover of the action of $\mathrm{SO}(5)$ on S^4.

4.4. *The noncommutative instanton bundle*

There is a description of the instanton projection constructed in [23] in terms of ket-valued polynomial functions on $S^7_{\theta'}$. The latter are elements in the right $\mathcal{A}(S^7_{\theta'})$-module $\mathcal{E} := \mathbb{C}^4 \otimes \mathcal{A}(S^7_{\theta'}) =: \mathcal{A}(S^7_{\theta'})^4$ with a $\mathcal{A}(S^7_{\theta'})$-valued Hermitian structure $\langle \xi, \eta \rangle = \sum_b \xi_b^* \eta_b$. To any $|\xi\rangle \in \mathcal{E}$ one associates its dual $\langle \xi | \in \mathcal{E}^*$ by setting $\langle \xi | (\eta) := \langle \xi, \eta \rangle$, $\forall\, \eta \in \mathcal{E}$.

Similarly to the classical case (see Sec. 4.2), we define a 2×4 matrix Ψ in terms of two ket-valued polynomials $|\psi_1\rangle$ and $|\psi_2\rangle$ by

$$\Psi = (|\psi_1\rangle, |\psi_2\rangle) = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \\ \psi_3 & -\psi_4^* \\ \psi_4 & \psi_3^* \end{pmatrix}. \tag{4.17}$$

Then $\Psi^* \Psi = \mathbb{I}_2$ so that the 4×4 matrix,

$$p = \Psi \Psi^* = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|,$$

is a projection, $p^2 = p = p^*$, with entries in $\mathcal{A}(S^4_\theta)$. The action (4.9) becomes

$$\alpha_w(\Psi) = \Psi w, \tag{4.18}$$

from which the invariance of the entries of p follows at once. Explicitly one finds

$$p = \frac{1}{2}\begin{pmatrix} 1+z_0 & 0 & z_1 & -\overline{\mu} z_2^* \\ 0 & 1+z_0 & z_2 & \mu z_1^* \\ z_1^* & z_2^* & 1-z_0 & 0 \\ -\mu z_2 & \overline{\mu} z_1 & 0 & 1-z_0 \end{pmatrix}. \tag{4.19}$$

The projection p is easily seen to be equivalent to the projection describing the instanton on S^4_θ constructed in [23]. Indeed, if one defines

$$|\widetilde{\psi}_1\rangle = (\psi_1, \psi_2, \psi_3, \mu\psi_4)^{\mathrm{t}}, \quad |\widetilde{\psi}_2\rangle = (-\psi_2^*, \psi_1^*, -\psi_4^*, \mu\psi_3^*)^{\mathrm{t}},$$

one obtains after a substitution $z_2 \mapsto -\overline{\lambda} z_2^*$ exactly the projection obtained therein:

$$\widetilde{p} = \frac{1}{2}\begin{pmatrix} 1+z_0 & 0 & z_1 & -\overline{\lambda} z_2^* \\ 0 & 1+z_0 & z_2 & -z_1^* \\ z_1^* & z_2^* & 1-z_0 & 0 \\ -\lambda z_2 & -z_1 & 0 & 1-z_0 \end{pmatrix}.$$

One shows by direct calculation that the first component of the Chern character – defined in Th. 14 – of p vanishes,

$$\mathrm{ch}_1(p) = 0.$$

It follows that $\mathrm{ch}_2(p)$ is a Hochschild cycle, i.e. $b\mathrm{ch}_2(p) = 0$, and plays the role of the round volume form on S^4_θ. Indeed, it was shown in [23] that with the isospectral geometry $(C^\infty(S^4_\theta), D, \mathcal{H}, \gamma_5)$, the image of $\mathrm{ch}_2(p)$ via the map π_D in (3.29), satisfies the following quartic equation in D

$$\pi_D(\mathrm{ch}_2(p)) = 3\gamma_5. \tag{4.20}$$

As we shall see in Sec. 7.1, this means that the "bundle" p has the correct "topological numbers". In that section we shall also show that the canonical connection $\nabla = p \circ d$ has self-dual curvature, hence the name instanton bundle for the module determined by (the class of) p.

The projection (4.19) can be written in terms of the Dirac matrices defined in (4.13).

Lemma 16. *The matrices* $\widetilde{\gamma}_0 := \gamma_0$, $\widetilde{\gamma}_1 := \overline{\mu}\gamma_1^t$ *and* $\widetilde{\gamma}_2 := \mu\gamma_2^t$ *satisfy the relations*

$$\widetilde{\gamma}_j\widetilde{\gamma}_k + \lambda_{kj}\widetilde{\gamma}_k\widetilde{\gamma}_j = 0, \quad \widetilde{\gamma}_j\widetilde{\gamma}_k^* + \lambda_{jk}\widetilde{\gamma}_k^*\widetilde{\gamma}_j = 4\delta_{jk}, \quad j,k = 1,2, \tag{4.21}$$

and the above projection (4.19) *can be expressed as*

$$p = \frac{1}{2}\Big(1 + \widetilde{\gamma}_0 z_0 + \sum_i \widetilde{\gamma}_i z_i + \sum_i \widetilde{\gamma}_i^* z_i^*\Big).$$

Note the difference of (4.21) with (4.14) in the exchange of λ_{jk} by λ_{kj}.

The image of p in $\mathcal{A}(S^4_\theta)^4$, denoted by $\Gamma^\infty(S^4_\theta, E) = p\mathcal{A}(S^4_\theta)^4$, is clearly a right $\mathcal{A}(S^4_\theta)$-module. An equivalent description of the module $\Gamma^\infty(S^4_\theta, E)$ comes from considering "equivariant maps" from $S^7_{\theta'}$ to $\mathbb{C}^2$, in a way similar to the undeformed case in (4.5).

Let $\rho : \mathrm{SU}(2) \times \mathbb{C}^2 \to \mathbb{C}^2, (w, v) \mapsto \rho(w)v = w \cdot v$, be the defining left representation of SU(2). A corresponding equivariant map from $S^7_{\theta'}$ to $\mathbb{C}^2$ is an element $\varphi \in \mathcal{A}(S^7_{\theta'}) \otimes \mathbb{C}^2$, satisfying

$$(\alpha_w \otimes \mathrm{id})(\varphi) = (\mathrm{id} \otimes \rho(w)^{-1})(\varphi). \tag{4.22}$$

The collection of all such equivariant maps is denoted by $A(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$. It is a right $\mathcal{A}(S^4_\theta)$-module (it is in fact a $\mathcal{A}(S^4_\theta)$-bimodule) since multiplication by an element in $\mathcal{A}(S^4_\theta)$ does not affect the equivariance condition (4.22). Since SU(2) acts classically on $\mathcal{A}(S^7_{\theta'})$, one sees from (4.18) that the equivariant maps are given by elements of the form $\varphi := \Psi^* f$ for some $f \in \mathcal{A}(S^4_\theta) \otimes \mathbb{C}^4$. In terms of the canonical basis $\{e_1, e_2\}$ of $\mathbb{C}^2$, we can write $\varphi = \sum_k \langle \psi_k | f \rangle \otimes e_k$ for $|f\rangle = |f_1, f_2, f_3, f_4\rangle^{\mathrm{t}}$, with $f_a \in \mathcal{A}(S^4_\theta)$. We then have the following isomorphism

$$\Gamma^\infty(S^4_\theta, E) \simeq \mathcal{A}(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2, \qquad \sigma = p|f\rangle \leftrightarrow \varphi = \Psi^* f = \sum \langle \psi_k | f \rangle \otimes e_k. \tag{4.23}$$

4.5. *Associated modules and their properties*

More generally, one can define the right $\mathcal{A}(S^4_\theta)$-module $\Gamma^\infty(S^4_\theta, E^{(n)})$ associated with any irreducible representation $\rho_{(n)} : \mathrm{SU}(2) \to \mathrm{GL}(W^{(n)})$, with $W^{(n)} = \mathrm{Sym}^n(\mathbb{C}^2)$ for a positive integer n. The module of SU(2)-equivariant maps from $S^7_{\theta'}$ to $W^{(n)}$ is defined as

$$\mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho_{(n)}} W := \big\{\varphi \in \mathcal{A}(S^7_{\theta'}) \otimes W : (\alpha_w \otimes \mathrm{id})(\varphi) = (\mathrm{id} \otimes \rho_{(n)}(w)^{-1})(\varphi)\big\}.$$

It is easy to see that these maps are of the form $\varphi_{(n)} = \sum_k \langle \phi_k | f \rangle \otimes e_k$ on the basis $\{e_1, \ldots, e_{n+1}\}$ of $W^{(n)}$, where now $|f\rangle \in \mathcal{A}(S^4_\theta)^{4^n}$ and

$$|\phi_k\rangle = \frac{1}{a_k} |\psi_1\rangle^{\otimes(n-k+1)} \otimes_S |\psi_2\rangle^{\otimes(k-1)}, \quad k = 1, \ldots, n+1,$$

with $\otimes_S$ denoting symmetrization, and a_k are suitable normalization constants. These vectors $|\phi_k\rangle \in \mathbb{C}^{4^n} \otimes \mathcal{A}(S^7_{\theta'}) =: \mathcal{A}(S^7_{\theta'})^{4^n}$ are orthogonal (with the natural Hermitian structure), and with $a_k^2 = \binom{n}{k-1}$ they are also normalized. Then

$$p_{(n)} := |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_{n+1}\rangle\langle\phi_{n+1}| \in \mathrm{Mat}_{4^n}(\mathcal{A}(S^4_\theta)) \quad (4.24)$$

defines a projection $p^2 = p = p^*$. That its entries are in $\mathcal{A}(S^4_\theta)$ and not in $\mathcal{A}(S^7_{\theta'})$ is easily seen. Indeed, much as it happens for the vector Ψ in Eq. (4.18), for every $i = 1, \ldots, 4^n$, the vector $\big(|\phi_1\rangle_i, |\phi_2\rangle_i, \ldots, |\phi_{n+1}\rangle_i\big)$ transforms under the action of SU(2) to the vector $\big(|\phi_1\rangle_i, \ldots, |\phi_{n+1}\rangle_i\big) \cdot \rho_{(n)}(w)$ so that each entry $\sum_k |\phi_k\rangle_i\langle\phi_k|_j$ of $p_{(n)}$ is SU(2)-invariant and hence an element in $\mathcal{A}(S^4_\theta)$. With this we have proved the following.

Proposition 17. *The right $\mathcal{A}(S^4_\theta)$-module of equivariant maps $\mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho_{(n)}} W^{(n)}$ is isomorphic to the right $\mathcal{A}(S^4_\theta)$-module $\Gamma^\infty(S^4_\theta, E^{(n)}) := p_{(n)}(\mathcal{A}(S^4_\theta)^{4^n})$ with the isomorphism given explicitly by*

$$\Gamma^\infty(S^4_\theta, E^{(n)}) \simeq \mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho_{(n)}} W^{(n)},$$

$$\sigma_{(n)} = p_{(n)}|f\rangle \leftrightarrow \varphi_{(n)} = \sum_k \langle\phi_k|f\rangle \otimes e_k.$$

With the projections $p_{(n)}$ one associates (Grassmann) connections on the right $C^\infty(S^4_\theta)$-modules $\Gamma^\infty(S^4_\theta, E^{(n)})$ in a canonical way:

$$\nabla = p_{(n)} \circ \mathrm{d} : \Gamma^\infty(S^4_\theta, E^{(n)}) \to \Gamma^\infty(S^4_\theta, E^{(n)}) \otimes_{\mathcal{A}(S^4_\theta)} \Omega^1(S^4_\theta), \quad (4.25)$$

where $(\Omega^*(S^4_\theta), \mathrm{d})$ is the differential calculus defined in the previous section. An expression for these connections as acting on coequivariant maps can be obtained using the above isomorphism and results in

$$\nabla(\phi_k) = \mathrm{d}(\phi_k) + A_{kl}\phi_l, \quad (4.26)$$

where $A_{kl} = \langle\phi_k|\mathrm{d}\phi_l\rangle \in \Omega^1(S^7_{\theta'})$. The corresponding matrix A is called the connection one-form; it is clearly anti-Hermitian, and it is valued in the derived representation space, $\rho'_n : su(2) \to \mathrm{End}(W^{(n)})$, of the Lie algebra $su(2)$.

Let us now discuss some properties of the associated modules, like Hermitian structures and the structure of the algebra of endomorphisms on them.

We observe that one can lift the construction of the previous section to the smooth level by replacing polynomial algebras by their smooth completions as defined in Sec. 3. Then, with ρ any representation of SU(2) on an

n-dimensional vector space W, the $C^\infty(S^4_\theta)$ bimodule associated to W is defined by

$$\mathcal{E} := C^\infty(S^7_{\theta'}) \boxtimes_\rho W := \left\{\eta \in C^\infty(S^7_{\theta'}) \otimes W : (\alpha_w \otimes \mathrm{id})(\eta) = (\mathrm{id} \otimes \rho(w)^{-1})(\eta)\right\}. \tag{4.27}$$

Along the lines of the above proof, the module $\mathcal{E}$ is a finite projective $C^\infty(S^4_\theta)$ module. Note that the choice of a projection for a finite projective module requires the choice of one of the two (left or right) module structures. Similarly, the definition of a Hermitian structure requires the choice of a left or right module structure. In the following, we will always work with the right structure for the associated modules. On $C^\infty(S^7_{\theta'}) \boxtimes_\rho W$ there is a natural (right) Hermitian structure defined in terms of a fixed inner product of W as

$$\langle \eta, \eta' \rangle := \sum_i \overline{\eta}_i \eta'_i, \tag{4.28}$$

where, given an orthonormal basis $\{e_i,\ i = 1, \ldots, \dim W\}$ of W, we denote $\eta = \sum_i \eta_i \otimes e_i$ and $\eta' = \sum_i \eta'_i \otimes e_i$. One quickly checks that $\langle \eta, \eta' \rangle$ is an element in $C^\infty(S^4_\theta)$, and that $\langle\ ,\ \rangle$ satisfies all conditions of a right Hermitian structure.

The bimodules $C^\infty(S^7_{\theta'}) \boxtimes_\rho W$ are of the type described in Sec. 3.3. The associated vector bundle $E = S^7 \times_\rho W$ on S^4 carries an action V of $\widetilde{\mathbb{T}}^2$ induced from its action on S^7, which is obviously σ-equivariant. By the very definition of $C^\infty(S^7_{\theta'})$ and of $\Gamma^\infty(S^4_\theta, E)$ in Sec. 3.3, it follows that $C^\infty(S^7_{\theta'}) \boxtimes_\rho W \simeq \Gamma^\infty(S^4_\theta, E)$. Indeed, from the undeformed isomorphism, $\Gamma^\infty(S^4, E) \simeq C^\infty(S^7) \boxtimes_\rho W$, the quantization map $L_{\theta'}$ of $C^\infty(S^7)$, acting only on the first leg of the tensor product, establishes this isomorphism,

$$L_{\theta'} : C^\infty(S^7) \boxtimes_\rho W \to C^\infty(S^7_{\theta'}) \boxtimes_\rho W. \tag{4.29}$$

The above is well defined since the action of $\widetilde{\mathbb{T}}^2$ commutes with the action of SU(2). Also, it is such that $L_{\theta'}(f \triangleright_{\theta'} \eta) = L_{\theta'}(f) L_{\theta'}(\eta) = L_\theta(f) L_{\theta'}(\eta)$ for $f \in C^\infty(S^4)$ and $\eta \in C^\infty(S^7) \boxtimes_\rho W$, due to the identity $L_{\theta'} = L_\theta$ on $C^\infty(S^4) \subset C^\infty(S^7)$. A similar result holds for the action $\triangleleft_{\theta'}$.

Proposition 18. *The right $C^\infty(S^4_\theta)$-module $C^\infty(S^7_{\theta'}) \boxtimes_\rho W$ admits a homogeneous module basis $\{e_\alpha,\ \alpha = 1, \ldots, N\}$ – with a suitable N – that is, under the action V of the torus $\widetilde{\mathbb{T}}^2$, its elements transform as*

$$V_s(e_\alpha) = e^{2\pi i s \cdot r_\alpha} e_\alpha, \quad s \in \widetilde{\mathbb{T}}^2, \tag{4.30}$$

with $r_\alpha \in \mathbb{Z}^2$ the degree of e_α.

Proof. The vector space W is a direct sum of irreducible representation spaces of $\mathrm{SU}(2)$ and the module $C^\infty(S^7_{\theta'}) \boxtimes_\rho W$ decomposes accordingly. Thus, we can restrict to irreducible representations. The latter are labeled by an integer n with $W \simeq \mathbb{C}^{n+1}$.

Consider first the case $W = \mathbb{C}^2$. Using [47, Prop. 2], a basis $\{e_1, \ldots, e_4\}$ of the right module $C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$ is given by the columns of $\Psi^\dagger$ where Ψ is the matrix in (4.17):

$$e_1 := \begin{pmatrix} \psi_1^* \\ -\psi_2 \end{pmatrix}, \quad e_2 := \begin{pmatrix} \psi_2^* \\ \psi_1 \end{pmatrix}, \quad e_3 := \begin{pmatrix} \psi_3^* \\ -\psi_4 \end{pmatrix}, \quad e_4 := \begin{pmatrix} \psi_4^* \\ \psi_3 \end{pmatrix}. \tag{4.31}$$

Using the explicit action (4.16) it is immediate to compute the corresponding degrees,

$$r_1 = (-1, 0), \quad r_2 = (1, 0), \quad r_3 = (0, 1), \quad r_4 = (0, -1). \tag{4.32}$$

More generally, with $W = \mathbb{C}^{n+1}$, a homogeneous basis $\{e_\alpha, \alpha = 1, \ldots, 4^n\}$ for the right module $C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^{n+1}$ can be constructed from the columns of a similar $(n+1) \times 4^n$ matrix $\Psi^\dagger_{(n)}$ given in [47]. □

The above property allows us to prove a useful result for the associated modules.

Proposition 19. *Let ρ_1 and ρ_2 be two finite dimensional representations of $\mathrm{SU}(2)$ on the vector spaces W_1 and W_2, respectively. There is an isomorphism of right $C^\infty(S^4_\theta)$-modules:*

$$(C^\infty(S^7_{\theta'}) \boxtimes_{\rho_1} W_1) \overline{\otimes}_{C^\infty(S^4_\theta)} (C^\infty(S^7_{\theta'}) \boxtimes_{\rho_2} W_2) \simeq C^\infty(S^7_{\theta'}) \boxtimes_{\rho_1 \otimes \rho_2} (W_1 \otimes W_2).$$

Proof. Let $\{e^1_\alpha,\ \alpha = 1, \ldots, N_1\}$ and $\{e^2_\beta,\ \beta = 1, \ldots, N_2\}$ be homogeneous bases for the right modules $C^\infty(S^7_{\theta'}) \boxtimes_{\rho_1} W_1$ and $C^\infty(S^7_{\theta'}) \boxtimes_{\rho_2} W_2$ respectively. Then the right module $C^\infty(S^7_{\theta'}) \boxtimes_{\rho_1 \otimes \rho_2} (W_1 \otimes W_2)$ has a homogeneous basis given by $\{e^1_\alpha \otimes e^2_\beta\}$. We define a map

$$\phi : (C^\infty(S^7_{\theta'}) \boxtimes_{\rho_1} W_1) \overline{\otimes}_{C^\infty(S^4_\theta)} (C^\infty(S^7_{\theta'}) \boxtimes_{\rho_2} W_2) \to C^\infty(S^7_{\theta'}) \boxtimes_{\rho_1 \otimes \rho_2} (W_1 \otimes W_2)$$

by

$$\phi(e^1_\alpha \times_\theta f^1_\alpha \otimes e^2_\beta \times_\theta f^2_\beta) = (e^1_\alpha \otimes e^2_\beta) \times_\theta \sigma_{r_\beta \theta}(f^1_\alpha) \times_\theta f^2_\beta,$$

with summation over α and β understood. Here $r_\beta \in \mathbb{Z}^2$ is the degree of e^2_β under the action of $\widetilde{\mathbb{T}}^2$, so that $e^2_\beta \times_\theta \sigma_{r_\beta \theta}(f) = f \times_\theta e^2_\beta$ for any $f \in C^\infty(S^4_\theta)$. Note that this map is well-defined since

$$\phi\left(e^1_\alpha \times_\theta f^1_\alpha \times_\theta f \otimes e^2_\beta \times_\theta f^2_\beta - e^1_\alpha \times_\theta f^1_\alpha \otimes f \times_\theta e^2_\beta \times_\theta f^2_\beta\right) = 0.$$

Moreover, it is clearly a map of right $C^\infty(S^4_\theta)$-modules. In fact, it is an isomorphism with its inverse given explicitly by

$$\phi^{-1}\left((e^1_\alpha \otimes e^2_\beta) \times_\theta f_{\alpha\beta}\right) = e^1_1 \otimes (e^2_\beta \times_\theta f_{1\beta}) + \cdots + e^1_{N_1} \otimes (e^2_\beta \times_\theta f_{N_1\beta})$$

with $f_{\alpha\beta} \in C^\infty(S^4_\theta)$. □

4.6. *The adjoint bundle*

Given a right $C^\infty(S^4_\theta)$-module $\mathcal{E}$, its *dual module* is defined by

$$\mathcal{E}' := \left\{\phi : \mathcal{E} \to C^\infty(S^4_\theta) : \phi(\eta f) = \phi(\eta) f, \forall\, f \in C^\infty(S^4_\theta)\right\}, \tag{4.33}$$

and is naturally a left $C^\infty(S^4_\theta)$-module. In the case that $\mathcal{E}$ is also a left $C^\infty(S^4_\theta)$-module, then $\mathcal{E}'$ is also a right $C^\infty(S^4_\theta)$-module. If $\mathcal{E} := C^\infty(S^7_{\theta'}) \boxtimes_\rho W$ comes from the SU(2)-representation (W, ρ), by using the induced dual representation ρ' on the dual vector space W' given by

$$\left(\rho'(w)v'\right)(v) := v'\left(\rho(w)^{-1}v\right), \qquad \forall\, v' \in W', v \in W, \tag{4.34}$$

we have that

$$\begin{aligned}\mathcal{E}' &\simeq C^\infty(S^7_{\theta'}) \boxtimes_{\rho'} W' \\ &:= \left\{\phi \in C^\infty(S^7_{\theta'}) \otimes W' : (\alpha_w \otimes \mathrm{id})(\phi) = (\mathrm{id} \otimes \rho'(w)^{-1})(\phi), \forall\, w \in \mathrm{SU}(2)\right\}.\end{aligned} \tag{4.35}$$

Next, let $L(W)$ denote the space of linear maps on W, so that $L(W) = W \otimes W'$. The adjoint action of SU(2) on $L(W)$ is the tensor product representation $\mathrm{ad} := \rho \otimes \rho'$ on $W \otimes W'$. We define

$$\begin{aligned}C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(W) := \big\{T \in C^\infty(S^7_{\theta'}) \otimes L(W) :& \\ : (\alpha_w \otimes \mathrm{id})(T) = (\mathrm{id} \otimes \mathrm{ad}(w)^{-1})(T), \forall\, w \in \mathrm{SU}(2)\big\},&\end{aligned} \tag{4.36}$$

and write $T = T_{ij} \otimes e_{ij}$ with respect to the basis $\{e_{ij}\}$ of $L(W)$ induced from the basis $\{e_i\}_{i=1}^{\dim W}$ of W and the dual one $\{e'_i\}_{i=1}^{\dim W}$ of W'.

On the other hand, there is the endomorphism algebra

$$\mathrm{End}_{C^\infty(S^4_\theta)}(\mathcal{E}) := \left\{T : \mathcal{E} \to \mathcal{E} : T(\eta f) = T(\eta) f, \forall\, f \in C^\infty(S^4_\theta)\right\}. \tag{4.37}$$

We will suppress the subscript $C^\infty(S^4_\theta)$ from End in the following. As a corollary to the previous proposition, we have the following.

Proposition 20. *Let $\mathcal{E} := C^\infty(S^7_{\theta'}) \boxtimes_\rho W$ for a finite-dimensional representation ρ. Then there is an isomorphism of algebras*

$$\mathrm{End}(\mathcal{E}) \simeq C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(W).$$

Proof. By Proposition 19, we have that $C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(W) \equiv C^\infty(S^7_{\theta'}) \boxtimes_{\rho\otimes\rho'} (W \otimes W')$ is isomorphic to $\mathcal{E}\overline{\otimes}_{C^\infty(S^4_\theta)}\mathcal{E}'$ as a right $C^\infty(S^4_\theta)$-module. Since $\mathcal{E}$ is a finite projective $C^\infty(S^4_\theta)$-module, there is an isomophism $\mathrm{End}(\mathcal{E}) \simeq \mathcal{E}\overline{\otimes}_{C^\infty(S^4_\theta)}\mathcal{E}'$, whence the result. □

We see that the algebra of endomorphisms of the module $\mathcal{E}$ can be understood as the space of sections of the noncommutative vector bundle associated to the adjoint representation on $L(W)$ – exactly as happens in the classical case. This also allows an identification of skew-Hermitian endomorphisms $\mathrm{End}^s(\mathcal{E})$ – which were defined in general in (2.19) – for the toric deformations at hand.

Corollary 21. *There is an identification*

$$\mathrm{End}^s(\mathcal{E}) \simeq C^\infty_{\mathbb{R}}(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} u(n),$$

with $C^\infty_{\mathbb{R}}(S^7_{\theta'})$ denoting the subspace of self-adjoint elements in $C^\infty(S^7_{\theta'})$ and $u(n)$ consisting of skew-adjoint matrices in $M_n(\mathbb{C}) \simeq L(W)$, with $n = \dim W$.

Proof. Note that the involution $T \mapsto T^*$ in $\mathrm{End}(\mathcal{E})$ reads in components $T_{ij} \mapsto \overline{T_{ji}}$ so that, using the identification of Proposition 20, the space $\mathrm{End}^s(\mathcal{E})$ consists of elements $X \in C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} L(W)$ satisfying $\overline{X_{ji}} = -X_{ij}$. Since any element in $C^\infty(S^7_{\theta'})$ can be written as the sum of two self-adjoint elements, $X_{ij} = X^{\Re}_{ij} + iX^{\Im}_{ij}$, we can write

$$X = \sum_i X^{\Im}_{ii} \otimes ie_{ii} + \sum_{i\neq j} X^{\Re}_{ij} \otimes (e_{ij} - e_{ji}) + X^{\Im}_{ij} \otimes (ie_{ij} + ie_{ji}) = \sum_a X_a \otimes \sigma^a,$$

with X_a generic elements in $C^\infty_{\mathbb{R}}(S^7_{\theta'})$ and $\{\sigma^a, a = 1, \ldots, n^2\}$ the generators of $u(n)$. □

Example 22. Let us return to the instanton bundle $\mathcal{E} = C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$. In this case, $\mathrm{End}(\mathcal{E}) \simeq C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} M_2(\mathbb{C}))$. Since the matrix algebra $M_2(\mathbb{C})$ decomposes into the adjoint representation $su(2)$ and the trivial representation $\mathbb{C}$ and $C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{id}} \mathbb{C} \simeq C^\infty(S^4_\theta)$, we conclude that

$$\mathrm{End}(\mathcal{E}) \simeq \Gamma^\infty(\mathrm{ad}(S^7_{\theta'})) \oplus C^\infty(S^4_\theta), \tag{4.38}$$

where we have set $\Gamma^\infty(\mathrm{ad}(S^7_{\theta'})) := C^\infty(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} su(2)$. The latter $C^\infty(S^4_\theta)$-bimodule will be understood as the space of (complex) sections of the adjoint bundle. It is the complexification of the traceless skew-Hermitian endomorphisms $C^\infty_{\mathbb{R}}(S^7_{\theta'}) \boxtimes_{\mathrm{ad}} su(2)$.

4.7. *Index of twisted Dirac operators*

In this section, we shall compute explicitly the index of the Dirac operator with coefficients in the bundles $E^{(n)}$, that is the index of the operator $D_{p_{(n)}} := p_{(n)}(D \otimes \mathbb{I}_{4^n})p_{(n)}$. We compute this index using the special form of the Connes-Moscovici local index formula, as given in Theorem 15. For the present case, the index of the Dirac operator on S^4_θ twisted by $p \in K_0(C(S^4_\theta))$ is given by

$$\begin{aligned}\mathrm{index}(D_p) &= \langle \phi^*, \mathrm{ch}_*(p)\rangle \\ &= \operatorname*{Res}_{z=0} z^{-1}\,\mathrm{tr}\left(\gamma\pi_D(\mathrm{ch}_0(p))|D|^{-2z}\right) \\ &\quad + \frac{1}{2!}\operatorname*{Res}_{z=0}\mathrm{tr}\left(\gamma\pi_D(\mathrm{ch}_1(p))|D|^{-2-2z}\right) \\ &\quad + \frac{1}{4!}\operatorname*{Res}_{z=0}\mathrm{tr}\left(\gamma\pi_D(\mathrm{ch}_2(p))|D|^{-4-2z}\right).\end{aligned}$$

The Chern character classes and their realization as operators are described in Sec. 3.5. In particular, π_D represents the universal differential calculus as operators by the map

$$\pi_D : \Omega^p_{\mathrm{un}}(\mathcal{A}(S^4_\theta)) \to \mathcal{B}(\mathcal{H}), \quad a^0\delta a^1\cdots\delta a^p \mapsto a^0[D,a^1]\cdots[D,a^p].$$

We know from Sec. 3.4 that these operators are noncommutative forms (see Eq. (3.15)) provided we quotient them by junk forms, that is, operators of the type in Eq. (3.16). We will avoid a discussion on junk forms and introduce instead a different quotient of $\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$. We define $\Omega_D(S^4_\theta)$ to be $\Omega_{\mathrm{un}}(\mathcal{A}(S^4_\theta))$ modulo the relations

$$\begin{aligned}&\alpha\delta\beta - \lambda(\delta\beta)\alpha = 0, \quad (\delta\alpha)\beta - \lambda\beta\delta\alpha = 0,\\ &\alpha\delta\beta^* - \overline{\lambda}(\delta\beta^*)\alpha = 0, \quad (\delta\alpha^*)\beta - \overline{\lambda}\beta\delta\alpha^* = 0,\\ &a\delta x - (\delta x)a = 0, \qquad \forall\, a \in \mathcal{A}(S^4_\theta),\end{aligned}$$

avoiding the second order relations that define $\Omega(S^4_\theta)$. One proves that the above relations are in the kernel of π_D: for instance, $\alpha[D,\beta]-\lambda[D,\beta]\alpha = 0$, so that π_D is well-defined on $\Omega_D(S^4_\theta)$. The differential calculus $\Omega_D(S^7_{\theta'})$ is the quotient of $\Omega_{\rm un}(\mathcal{A}(S^7_{\theta'}))$ by only the relations in (3.11) of order one, that is, by the relations

$$z_\mu \delta z_\nu = \lambda_{\mu\nu}(\delta z_\nu) z_\mu, \quad z_\mu \delta z_\nu^* = \lambda_{\nu\mu}\delta z_\nu^* z_\mu.$$

For a proof of the following result we refer to [47].

Lemma 23. *The images under π_D of the Chern characters of the projections $p_{(n)}$ are:*

$$\begin{aligned}
\pi_D(\mathrm{ch}_0(p_{(n)})) &= n+1,\\
\pi_D(\mathrm{ch}_1(p_{(n)})) &= 0,\\
\pi_D(\mathrm{ch}_2(p_{(n)})) &= \frac{1}{6}n(n+1)(n+2)\pi_D(\mathrm{ch}_2(p_{(1)})),
\end{aligned}$$

up to the coefficients $\mu_k = (-1)^k \frac{(2k)!}{k!}$.

Combining this result with the simple form of the index formula given above, while taking the proper coefficients, we find that

$$\mathrm{index}(D_{p_{(n)}}) = \frac{1}{4!}\frac{4!}{2!}\frac{1}{6}n(n+1)(n+2)\operatorname*{Res}_{z=0}\mathrm{tr}\left(\gamma_5\pi_D(\mathrm{ch}_2(p_{(1)}))|D|^{-4-2z}\right),$$

where the vanishing of the first term – the one involving $\pi_D(\mathrm{ch}_0(p_{(n)}))$ – follows from the fact that $\mathrm{index}\, D = 0$ on S^4. Th. I.2 in [25] allows one to express the residue as a Dixmier trace. Combining this with $\pi_D(\mathrm{ch}_2(p_{(1)})) = 3\gamma_5$ (as computed in [23]), we obtain

$$3\cdot \operatorname*{Res}_{z=0}\mathrm{tr}(|D|^{-4-2z}) = 6\cdot \mathrm{Tr}_\omega(|D|^{-4}) = 2$$

since the Dixmier trace of $|D|^{-m}$ on the m-sphere equals $8/m!$ (cf. for instance [37, 42]). This combines to give:

Proposition 24. *The index of the Dirac operator on S^4_θ with coefficients in $E^{(n)}$ is*

$$\mathrm{index}(D_{p_{(n)}}) = \frac{1}{6}n(n+1)(n+2).$$

This coincides with the classical result.

4.8. *The noncommutative principal bundle structure*

In this section, we apply the general theory of Hopf–Galois extensions to the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$. As explained in Sec. 2.4, such extensions can be understood as noncommutative principal bundles. We will first dualize the construction of the previous section, i.e. replace the action of SU(2) on $\mathcal{A}(S^7_{\theta'})$ by a coaction of $\mathcal{A}(\mathrm{SU}(2))$. Then, we show that $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a non-cleft (i.e. non-trivial) Hopf–Galois extension and compare the connections on the associated bundles, induced from the strong connection, with the Grassmann connection defined in Sec. 4.5.

The action of SU(2) on $\mathcal{A}(S^7_{\theta'})$ by automorphisms can be easily dualized to a coaction $\Delta_R : \mathcal{A}(S^7_{\theta'}) \to \mathcal{A}(S^7_{\theta'}) \otimes \mathcal{A}(\mathrm{SU}(2))$, where now $\mathcal{A}(\mathrm{SU}(2))$ is the unital complex $*$-algebra generated by $w^1, \overline{w}^1, w^2, \overline{w}^2$ with relation $w^1\overline{w}^1 + w^2\overline{w}^2 = 1$. Clearly, $\mathcal{A}(\mathrm{SU}(2))$ is a Hopf algebra with comultiplication

$$\Delta : \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} \mapsto \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix} \otimes \begin{pmatrix} w^1 & w^2 \\ -\overline{w}^2 & \overline{w}^1 \end{pmatrix},$$

antipode $S(w^1) = \overline{w}^1, S(w^2) = -w^2$ and counit $\epsilon(w^1) = \epsilon(\overline{w}^1) = 1, \epsilon(w^2) = \epsilon(\overline{w}^2) = 0$. The coaction of $\mathcal{A}(\mathrm{SU}(2))$ on $\mathcal{A}(S^7_{\theta'})$ is given by

$$\Delta_R : (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \mapsto (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \otimes \begin{pmatrix} w^1 & w^2 & 0 & 0 \\ -\overline{w}^2 & \overline{w}^1 & 0 & 0 \\ 0 & 0 & w^1 & w^2 \\ 0 & 0 & -\overline{w}^2 & \overline{w}^1 \end{pmatrix}.$$

The algebra of coinvariants in $\mathcal{A}(S^7_{\theta'})$, which consists of elements $z \in \mathcal{A}(S^7_{\theta'})$ satisfying $\Delta_R(z) = z \otimes 1$, can be identified with $\mathcal{A}(S^4_\theta)$ for the particular values of θ'_{ij} found before, in the same way as in Sec. 4.3.

Theorem 25. *The inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a non-cleft faithfully flat $\mathcal{A}(\mathrm{SU}(2))$-Hopf–Galois extension.*

Proof. Since $\mathcal{A}(\mathrm{SU}(2))$ is cosemisimple, from the general considerations of Sec. 2.4, we simply need to prove that $1 \otimes h$ is in the image of χ, for h a generator of $\mathcal{A}(\mathrm{SU}(2))$. But it is straightforward to check that using the ket-valued polynomials in (4.17) we have

$$\chi(\textstyle\sum_a \langle\psi_1|_a \otimes_{\mathcal{A}(S^4_\theta)} |\psi_1\rangle_a) = 1 \otimes w^1, \quad \chi\left(\displaystyle\sum_a \langle\psi_1|_a \otimes_{\mathcal{A}(S^4_\theta)} |\psi_2\rangle_a\right) = 1 \otimes w^2,$$

$$\chi(\textstyle\sum_a \langle\psi_2|_a \otimes_{\mathcal{A}(S^4_\theta)} |\psi_1\rangle_a) = -1 \otimes \overline{w}^2, \quad \chi\left(\displaystyle\sum_a \langle\psi_2|_a \otimes_{\mathcal{A}(S^4_\theta)} |\psi_2\rangle_a\right) = 1 \otimes \overline{w}^1.$$

Non-cleftness is a simple consequence of the nontriviality of the Chern characters of the projections $p_{(n)}$ as seen in Sec. 4.7. Indeed, this implies that the associated modules are nontrivial, which would not be true were the extension cleft. $\square$

The existence of a strong connection follows from general properties. However, one can easily write an explicit expression in terms of the inverse of the canonical map. If we denote the latter when lifted to $P \otimes P$ by τ it follows that the map $\ell : H \to P \otimes P$ defined by $\ell(h) = \tau(1 \otimes h)$ enjoys the properties in (2.31), hence determines a strong connection. Furthermore, this map satisfies recursive relations similar to the one for χ^{-1}: if $\ell(h) = h_l \otimes h'_l$ and $\ell(g) = g_k \otimes g'_k$, then

$$\ell(hg) = g_k h_l \otimes h'_l g'_k. \tag{4.39}$$

As mentioned in Sec. 2.4, such a map $\ell : H \to P \otimes P$ yields a strong connection.

Proposition 26. *On the Hopf–Galois extension $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$, the following formulae on the generators of $\mathcal{A}(\mathrm{SU}(2))$,*

$$\begin{aligned} \ell(w^1) &= \sum_a \langle \psi_1|_a \otimes |\psi_1\rangle_a, \quad \ell(w^2) = \sum_a \langle \psi_1|_a \otimes |\psi_2\rangle_a, \\ \ell(\overline{w}^2) &= -\sum_a \langle \psi_2|_a \otimes |\psi_1\rangle_a, \quad \ell(\overline{w}^1) = \sum_a \langle \psi_2|_a \otimes |\psi_2\rangle_a, \end{aligned} \tag{4.40}$$

define a strong connection.

Proof. The expressions (4.40) are extended to all of $\mathcal{A}(\mathrm{SU}(2))$ by the recursive formula (4.39). Recall the usual vector basis $\{r^{klm} : k \in \mathbb{Z}, m, n \geq 0\}$ in $\mathcal{A}(\mathrm{SU}(2))$ given by

$$r^{klm} := \begin{cases} (-1)^n (w^1)^k (w^2)^m (\overline{w}^2)^n & k \geq 0, \\ (-1)^n (w^2)^m (\overline{w}^2)^n (\overline{w}^1)^{-k} & k < 0. \end{cases} \tag{4.41}$$

The recursive expressions on this basis are explicitly given by

$$\begin{aligned} \ell(r^{k+1,mn}) &= \psi_1^* \ell(r^{kmn}) \psi_1 + \psi_2^* \ell(r^{kmn}) \psi_2 + \psi_3^* \ell(r^{kmn}) \psi_3 \\ &\quad + \psi_4^* \ell(r^{kmn}) \psi_4, \quad k \geq 0, \\ \ell(w^{k-1,mn}) &= \psi_2 \ell(r^{kmn}) \psi_2^* + \psi_1 \ell(r^{kmn}) \psi_1^* + \psi_4 \ell(r^{kmn}) \psi_4^* \\ &\quad + \psi_3 \ell(r^{kmn}) \psi_3^*, \quad k < 0, \\ \ell(w^{k,m+1,n}) &= -\psi_1^* \ell(r^{kmn}) \psi_2^* + \psi_2^* \ell(r^{kmn}) \psi_1^* \\ &\quad - \psi_3^* \ell(r^{kmn}) \psi_4^* + \psi_4^* \ell(r^{kmn}) \psi_3^*, \\ \ell(w^{km,n+1}) &= -\psi_2 \ell(r^{kmn}) \psi_1 + \psi_1 \ell(r^{kmn}) \psi_2 \\ &\quad - \psi_4 \ell(r^{kmn}) \psi_3 + \psi_3 \ell(r^{kmn}) \psi_4, \end{aligned} \tag{4.42}$$

while setting $\ell(1) = 1 \otimes 1$. One systematically prove that the map ℓ defined by the above recursive relations indeed satisfies all conditions of a strong connection. $\square$

The associated modules $\Gamma^\infty(S^4_\theta, E^{(n)})$ are described in the following way. Given an irreducible corepresentation of $\mathcal{A}(\mathrm{SU}(2))$, $\rho_{(n)} : W^{(n)} \to \mathcal{A}(\mathrm{SU}(2)) \otimes W^{(n)}$, we denote $\rho_{(n)}(v) = v_{(0)} \otimes v_{(1)}$. Then the associated (right $\mathcal{A}(S^4_\theta)$)-module of coequivariant maps $\mathrm{Hom}^{\rho_{(n)}}(W^{(n)}, \mathcal{A}(S^7_{\theta'}))$ consists of maps $\varphi : W^{(n)} \to \mathcal{A}(S^7_{\theta'})$ satisfying

$$\varphi(v_{(1)}) \otimes S v_{(0)} = \Delta_R \varphi(v), \quad v \in W^{(n)}.$$

Again, such maps are $\mathbb{C}$-linear maps written – in the basis $\{e_i\}_{i=1,\cdots,n+1}$ of $W^{(n)}$ – as $\varphi_{(n)}(e_k) = \langle \phi_k | f \rangle$, in the notation of the previous section. With the projections defined in Eq. (4.24), Proposition 17 translates straightforwardly into an isomorphism

$$\mathrm{Hom}^{\rho_{(n)}}(W^{(n)}, \mathcal{A}(S^7_{\theta'})) \simeq p_{(n)}(\mathcal{A}(S^4_\theta))^{4^n} \simeq \mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho_{(n)}} W^{(n)}$$

with a slight abuse of notation – $\rho_{(n)}$ being the corepresentation of $\mathcal{A}(\mathrm{SU}(2))$ in the last expression and the representation of $\mathrm{SU}(2)$ in the first expression.

As explained in Sec. 2.4, the strong connection on the extension $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ induces connections on $\mathrm{Hom}^{\rho_{(n)}}(W^{(n)}, \mathcal{A}(S^7_{\theta'}))$, with respect to the universal calculus. Recall that for $\varphi \in \mathrm{Hom}^{\rho_{(n)}}(W^{(n)}, \mathcal{A}(S^7_{\theta'}))$, we have

$$\nabla_\omega(\varphi)(v) \mapsto \delta\varphi(v) + \omega(v_{(0)})\varphi(v_{(1)}),$$

which is in fact a map

$$\nabla_\omega : \mathrm{Hom}^{\rho_{(n)}}(W^{(n)}, \mathcal{A}(S^7_{\theta'})) \to \mathrm{Hom}^{\rho_{(n)}}(W^{(n)}, \Omega^1_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'}))).$$

It turns out that the connection one-form ω coincides with the connection one-form A of Eq. (4.26), on the quotient $\Omega^1(S^7_{\theta'})$ of $\Omega^1_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'}))$. More precisely, let $\{e^{(n)}_k\}$ be a basis of $W^{(n)}$, and $e^{(n)}_{kl}$ the corresponding matrix coefficients of $\mathcal{A}(\mathrm{SU}(2))$ in the representation $\rho_{(n)}$. An explicit expression for $\omega(e^{(n)}_{kl})$ can be obtained from Eqs. (4.42); for example $\omega(e^{(1)}_{kl}) = \langle \psi_k | \delta \psi_l \rangle$, $k, l = 1, 2$. Writing $\langle \phi_k | \mathrm{d}\phi_l \rangle$ in terms of $\langle \psi_k | \mathrm{d}\psi_l \rangle$, this indeed leads to

$$\pi(\omega(e^{(n)}_{kl})) = A^{(n)}_{kl} = \langle \phi_k | \mathrm{d}\phi_l \rangle,$$

where $\pi : \Omega_{\mathrm{un}}(\mathcal{A}(S^7_{\theta'})) \to \Omega(S^7_{\theta'})$ is the quotient map.

5. Yang–Mills Theory on Toric Manifolds

We now introduce the Yang–Mills action functional together with the corresponding equations of motion. We will show how instantons naturally arise. We first work out in detail the case of S^4_θ and then sketch the theory on a generic four-dimensional manifold.

5.1. *Yang–Mills theory on* S^4_θ

Before we proceed we recall the noncommutative spin structure $(C^\infty(S^4_\theta), \mathcal{H}, D, \gamma_5)$ of S^4_θ with $\mathcal{H} = L^2(S^4, \mathcal{S})$ the Hilbert space of spinors, $D = \not{D}$ the undeformed Dirac operator, and γ_5 – the even structure – the fifth Dirac matrix.

Let $\mathcal{E} = \Gamma^\infty(S^4_\theta, E)$ for some σ-equivariant vector bundle E on S^4, so that there exists a projection $p \in M_N(C^\infty(S^4_\theta))$ such that $\mathcal{E} \simeq p(C^\infty(S^4_\theta))^N$. Recall from Sec. 2.2 that a connection ∇ on $\mathcal{E} = \Gamma^\infty(S^4_\theta, E)$ for some vector bundle E on S^4, is a map from $\mathcal{E}$ to $\mathcal{E} \otimes \Omega(S^4_\theta)$. The Yang–Mills action functional is defined in terms of the curvature of a connection on $\mathcal{E}$, which is an element in $\mathrm{Hom}_{C^\infty(S^4_\theta)}(\mathcal{E}, \mathcal{E} \otimes \Omega^2(S^4_\theta))$; equivalently, it is an element in $\mathrm{End}_{\Omega(S^4_\theta)}(\mathcal{E} \otimes \Omega(S^4_\theta))$ of degree 2 (see Sec. 2.2). We define an inner product on the latter algebra as follows [20, III.3]. An element $T \in \mathrm{End}_{\Omega(S^4_\theta)}(\mathcal{E} \otimes \Omega(S^4_\theta))$ of degree k can be understood as an element in $pM_N(\Omega^k(S^4_\theta))p$, since $\mathcal{E} \otimes \Omega(S^4_\theta)$ is a finite projective module over $\Omega(S^4_\theta)$. A trace over internal indices, together with the inner product defined in (3.22), defines the inner product $(\cdot, \cdot)_2$ on $\mathrm{End}_{\Omega(S^4_\theta)}(\mathcal{E} \otimes \Omega(S^4_\theta))$. In particular, we can give the following definition.

Definition 27. *The* Yang–Mills action functional *on the collection $C(\mathcal{E})$ of compatible connections ∇ on the module $\mathcal{E}$ is defined by*

$$\mathrm{YM}(\nabla) = \big(F, F\big)_2 = \fint *_\theta \operatorname{tr}(F *_\theta F),$$

for any connection ∇ with curvature F.

Recall from Sec. 2.2 that gauge transformations are given by unitary endomorphisms $\mathcal{U}(\mathcal{E})$ of the module $\mathcal{E}$.

Lemma 28. *The Yang–Mills action functional is gauge invariant, positive and quartic.*

Proof. From Eq. (2.24), under a gauge transformation $u \in \mathcal{U}(\mathcal{E})$ the curvature F transforms as $F \mapsto u^* F u$. Since $\mathcal{U}(\mathcal{E})$ can be identified with the unitary elements in $pM_N(\mathcal{A})p$, it follows that

$$\mathrm{YM}(\nabla^u) = \int\!\!\!\!\!\!- \sum_{i,j,k,l} *_\theta(\overline{u_{ji}} F_{jk} *_\theta F_{kl}, u_{li}) = \mathrm{YM}(\nabla)$$

using the tracial property of the Dixmier trace and the fact that $u_{li}\overline{u_{ji}} = \delta_{lj}$.

Positivity of the Yang–Mills action functional follows from Lemma 12, giving

$$(F, F)_2 = (F_D, F_D)_D = \int\!\!\!\!\!\!- F_D^* F_D,$$

which is clearly positive. □

The Yang–Mills equations (equations for critical points) are obtained from the Yang–Mills action functional by a variational principle. Let us describe how this principle works in our case. We consider a linear perturbation $\nabla_t = \nabla + t\alpha$ of a connection ∇ on $\mathcal{E}$ by an element $\alpha \in \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega^1(S^4_\theta))$. The curvature F_t of ∇_t is readily computed as $F_t = F + t[\nabla, \alpha] + \mathcal{O}(t^2)$. If we suppose that ∇ is an extremum of the Yang–Mills action functional, this linear perturbation should not affect the action. In other words, we should have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \mathrm{YM}(\nabla_t) = 0. \tag{5.1}$$

If we substitute the explicit formula for F_t, we obtain

$$\big([\nabla, \alpha], F\big)_2 + \overline{\big([\nabla, \alpha], F\big)_2} = 0. \tag{5.2}$$

using properties of the complex scalar product $(\cdot,\cdot)_2$ on $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \Omega(S^4_\theta))$. Then, its positive definiteness implies that $(F_t, F_t) = \overline{(F_t, F_t)}$, which when differentiated with respect to t, at $t = 0$, yields $\big([\nabla, \alpha], F\big)_2 = \overline{\big([\nabla, \alpha], F\big)_2}$; hence $([\nabla, \alpha], F)_2 = 0$. Since α was arbitrary, we derive the following equations of motion

$$[\nabla^*, F] = 0, \tag{5.3}$$

where the adjoint of $[\nabla, \cdot\,]$ is defined with respect to the scalar product by

$$\big([\nabla^*, \alpha], \beta\big)_2 = \big(\alpha, [\nabla, \beta]\big)_2 \tag{5.4}$$

for $\alpha \in \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \Omega^3(S^4_\theta))$ and $\beta \in \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \Omega^1(S^4_\theta))$. From Lemma 13, it follows that $[\nabla^*, F] = *_\theta[\nabla, *_\theta F]$, and the equations of motion can also be written as the more familiar *Yang–Mills equations*:

$$[\nabla, *_\theta F] = 0. \tag{5.5}$$

Note that connections with a self-dual or anti-self-dual curvature $*_\theta F = \pm F$ are special solutions of the Yang–Mills equation. Indeed, in this case the latter equation follows directly from the Bianchi identity $[\nabla, F] = 0$, given in Proposition 3.

We will next establish a relation between the Yang–Mills action functional and the so-called topological action [20, VI.3] on S^4_θ. Suppose $\mathcal{E}$ is a finite projective module over $C^\infty(S^4_\theta)$ defined by a projection $p \in M_N(C^\infty(S^4_\theta))$. The topological action for $\mathcal{E}$ is given by the pairing between the class of p in K-theory and the cyclic cohomology of $C^\infty(S^4_\theta)$. For computational purposes, we give the following definition in terms of the curvature of a connection on the module $\mathcal{E}$.

Definition 29. *Let ∇ be a connection on the module $\mathcal{E}$ with curvature F. The* topological action *is given by*

$$\mathrm{Top}(\mathcal{E}) = (F, *_\theta F)_2 = \int\!\!\!\!\!\!- *_\theta \operatorname{tr}(F^2),$$

where the trace is taken over internal indices, and in the second equality we have used the identity $*_\theta \circ *_\theta = \mathrm{id}$ on S^4_θ.

Let us show that $\mathrm{Top}(\mathcal{E})$ does not depend on the choice of a connection on $\mathcal{E}$. Since two connections differ by an element α in $\mathrm{Hom}_{C^\infty(S^4_\theta)}(\mathcal{E}, \mathcal{E} \otimes \Omega^1(S^4_\theta))$, we have to establish that $(F', *_\theta F')_2 = (F, *_\theta F)_2$ where $F' = F + t[\nabla, \alpha] + \mathcal{O}(t^2)$ is the curvature of $\nabla' := \nabla + t\alpha$, $t \in \mathbb{R}$. By definition of the inner product $(\cdot,\cdot)_2$ we then have

$$\begin{aligned}(F', *_\theta F')_2 - (F, *_\theta F)_2 &= t(F, *_\theta[\nabla, \alpha])_2 + t([\nabla, \alpha], *_\theta F)_2 + \mathcal{O}(t^2)\\ &= t(F, [\nabla^*, *_\theta \alpha])_2 + t([\nabla^*, *_\theta\alpha], F)_2 + \mathcal{O}(t^2),\end{aligned}$$

which vanishes due to the Bianchi identity $[\nabla, F] = 0$.

The Hodge star operator $*_\theta$ splits $\Omega^2(S^4_\theta)$ into a self-dual and anti-self-dual space,

$$\Omega^2(S^4_\theta) = \Omega^2_+(S^4_\theta) \oplus \Omega^2_-(S^4_\theta). \tag{5.6}$$

In fact, $\Omega^2_\pm(S^4_\theta) = L_\theta\left(\Omega^2_\pm(S^4)\right)$. This decomposition is orthogonal with respect to the inner product $(\cdot,\cdot)_2$ so that we can write the Yang–Mills action functional as

$$\mathrm{YM}(\nabla) = \big(F_+, F_+\big)_2 + \big(F_-, F_-\big)_2. \tag{5.7}$$

Comparing this with the topological action,

$$\mathrm{Top}(\mathcal{E}) = \big(F_+, F_+\big)_2 - \big(F_-, F_-\big)_2, \tag{5.8}$$

we see that $\mathrm{YM}(\nabla) \geq |\mathrm{Top}(\mathcal{E})|$, with equality holding iff

$$*_\theta F = \pm F. \tag{5.9}$$

Solutions of these equations are called instantons. We conclude that instantons are absolute minima of the Yang–Mills action functional.

5.2. *On a generic four-dimensional M_θ*

We shall briefly describe how the just constructed Yang–Mills theory on S^4_θ can be generalized to any four-dimensional toric noncommutative manifold M_θ.

With G a compact semisimple Lie group, let $P \to M$ be a principal G bundle on M. We take M to be a compact four-dimensional Riemannian manifold equipped with an isometric action σ of the torus $\mathbb{T}^2$. For the construction to work, we assume that this action can be lifted to an action $\widetilde{\sigma}$ of a cover $\widetilde{\mathbb{T}}^2$ on P, while it commutes with the action of G. As in Sec. 3, we define the noncommutative algebras $C^\infty(P_\theta)$ and $C^\infty(M_\theta)$ as the vector spaces $C^\infty(P)$ and $C^\infty(M)$ with star products defined as in (3.2) with respect to the action of $\widetilde{\mathbb{T}}^2$ and $\mathbb{T}^2$ respectively; or, equivalently as the images of $C^\infty(P)$ and $C^\infty(M)$ under the corresponding quantization map L_θ. Since the action of $\widetilde{\mathbb{T}}^2$ commutes with the action of G on P, the corresponding action α of G on the algebra $C^\infty(P)$ by

$$\alpha_g(f)(p) = f(g^{-1} \cdot p), \tag{5.10}$$

induces an action of G by automorphisms on the algebra $C^\infty(P_\theta)$. This means that also the inclusion $C^\infty(M) \subset C^\infty(P)$ as G-invariant elements in $C^\infty(P)$ extends to an inclusion $C^\infty(M_\theta) \subset C^\infty(P_\theta)$ of G-invariant elements in $C^\infty(P_\theta)$. Clearly, the action of G translates trivially into a coaction of the Hopf algebra $C^\infty(G)$ on $C^\infty(P_\theta)$.

Proposition 30. *The inclusion $C^\infty(M_\theta) \hookrightarrow C^\infty(P_\theta)$ is a (principal) Hopf–Galois $C^\infty(G)$ extension.*

Proof. As in [47], it is enough to establish surjectivity of the canonical map

$$\begin{aligned} \chi : C^\infty(P_\theta) \otimes_{C^\infty(M_\theta)} C^\infty(P_\theta) &\to C^\infty(P_\theta) \otimes C^\infty(G), \\ f' \otimes_{C^\infty(M_\theta)} f &\mapsto f'\Delta_R(f) = f' f_{(0)} \otimes f_{(1)}; \end{aligned}$$

all additional nice properties would then follow from the cosemisimplicity of the Hopf algebra $C^\infty(G)$. Now, for the undeformed case, the bijectivity of the canonical map $\chi^{(0)} : C^\infty(P) \otimes_{C^\infty(M)} C^\infty(P) \to C^\infty(P) \otimes C^\infty(G)$ follows by the very definition of a principal bundle. Furthermore, there is an isomorphism of vector spaces:

$$T : C^\infty(P_\theta) \otimes_{C^\infty(M_\theta)} C^\infty(P_\theta) \to C^\infty(P) \otimes_{C^\infty(M_\theta)} C^\infty(P_\theta),$$
$$f' \otimes_{C^\infty(M_\theta)} f \mapsto \sum f'_r \otimes_{C^\infty(M)} \widetilde{\sigma}_{r\theta}(f),$$

where $f' = \sum_r f'_r$ is the homogeneous decomposition of f' under the action of $\widetilde{\mathbb{T}}^2$. We claim that the canonical map is given as the composition $\chi = \chi^{(0)} \circ T$; hence, it is bijective. Indeed,

$$\begin{aligned}\chi^{(0)} \circ T\big(f' \otimes_{C^\infty(M_\theta)} f\big) &= \sum_r f'_r \widetilde{\sigma}_{r\theta}(f_{(0)}) \otimes f_{(1)} \\ &= f' \times_\theta f_{(0)} \otimes f_{(1)} = \chi(f' \otimes_{C^\infty(M_\theta)} f),\end{aligned}$$

since the action of $\widetilde{\mathbb{T}}^2$ on $C^\infty(P_\theta)$ commutes with the coaction of $C^\infty(G)$. $\square$

Noncommutative associated bundles are defined as in (4.27) by setting

$$\mathcal{E} = C^\infty(P_\theta) \boxtimes_\rho W := \big\{f \in C^\infty(P_\theta) \otimes W | (\alpha_g \otimes \mathrm{id})(f) = (\mathrm{id} \otimes \rho(g)^{-1})(f)\big\}$$

for a representation ρ of G on W. These $C^\infty(M_\theta)$ bimodules are finite projective since they are of the form of the modules defined in Sec. 3.3 (cf. Proposition 18). Moreover, Proposition 20 generalizes and reads $\mathrm{End}(\mathcal{E}) \simeq C^\infty(P_\theta) \boxtimes_{\mathrm{ad}} L(W)$, where ad is the adjoint representation of G on $L(W)$. Also, one identifies the adjoint bundle as the module coming from the adjoint representation of G on $\mathfrak{g} \subset L(W)$, namely $\Gamma^\infty(\mathrm{ad}(P_\theta)) := C^\infty(P_\theta) \boxtimes_{\mathrm{ad}} \mathfrak{g}$.

For a (right) finite projective $C^\infty(M_\theta)$-module $\mathcal{E}$ we define an inner product $(\cdot,\cdot)_2$ on $\mathrm{Hom}_{C^\infty(M_\theta)}(\mathcal{E}, \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega(M_\theta))$ as in Sec. 5. The Yang–Mills action functional on the space $C(\mathcal{E})$ of compatible connections ∇ on $\mathcal{E}$ is then given in terms of the corresponding curvatures F as before by

$$\mathrm{YM}(\nabla) = (F, F)_2, \tag{5.11}$$

and is a gauge invariant, positive and quartic functional. The derivation of the Yang–Mills eqs. (5.5) on S^4_θ does not rely on the specific properties of S^4_θ and continues to hold on M_θ. The same is true for the topological action, and $\mathrm{YM}(\nabla) \geq |\mathrm{Top}(\mathcal{E})|$ with equality if and only if $*_\theta F = \pm F$. In other words, instanton connections are absolute minima of the Yang–Mills action.

6. Let Us Twist Symmetries

The noncommutative sphere S^4_θ can be realized as a quantum homogeneous space of the quantum orthogonal group $\mathrm{SO}_\theta(5)$ [64, 22]. In other words, $\mathcal{A}(S^4_\theta)$ can be obtained as the subalgebra of $\mathcal{A}(\mathrm{SO}_\theta(5))$ made of elements that are coinvariant under the natural coaction of $\mathrm{SO}_\theta(4)$ on $\mathrm{SO}_\theta(5)$. For our purposes, it turns out to be more convenient to take a dual point of view and consider an *action* instead of a coaction. One obtains a twisted action of the Lie algebra $so(5)$ on S^4_θ and elements of $so(5)$ act as twisted derivations on the algebra $\mathcal{A}(S^4_\theta)$. Similar considerations hold for any noncommutative sphere S^N_Θ.

When lifted to $S^7_{\theta'}$, the twisted rotational symmetry leaves invariant the basic instanton ∇_0 described above. In [48] we used a twisted conformal symmetry to construct instantons on S^4_θ, a construction that we shall review later on.

In fact, what we are really describing are Hopf algebras $\mathrm{U}_\theta(so(5))$ and $\mathrm{U}_\theta(so(5,1))$ which are obtained from the undeformed Hopf algebras $\mathrm{U}(so(5))$ and $\mathrm{U}(so(5,1))$ via a twist of Drinfel′d type. Twisting of algebras and coalgebras has been known for some time [32, 33, 36]. The twists relevant for toric noncommutative manifolds are associated to the Cartan subalgebra of a Lie algebra and were already introduced in [58]. Their use to implement symmetries of toric noncommutative manifolds was made explicit in [63].

The geometry of multi-parameter quantum groups and quantum enveloping algebras coming from twists has been studied in [2, 3]. Interesting consequences, e.g. for the nonassociativity of differential calculi were studied in [12]. For symmetries of the usual noncommutative planes and their use for quantum field theories on it, one has the approach of [65, 57, 66]. More recently [16, 67], a twist was used to implement Poincarè symmetry on the Moyal plane while conformal transformations are twisted in [54]. Twisting of infinitesimal diffeomorphisms and their use for gravity theories are in [4, 5] and infinite dimensional (infinitesimal) conformal symmetries on a two-dimensional Moyal plane are twisted in [50]. There are also studies of spin and statistics and their relations in the context of these twisted symmetries of the Moyal plane [9]. Finally, a deformation of nonrelativistic Schrödinger symmetry is in [10] while extensions to superspace, including super-Poincaré and superconformal symmetries, were treated in [11].

6.1. *Twisting Hopf algebras and their actions*

We review the known algebraic construction of twisting a Hopf algebra and its actions, for which we refer, for instance, to [52] or [17] for details.

The Hopf algebra that is relevant in the present paper is just the universal enveloping algera $H = \mathrm{U}(\mathfrak{g})$ with $\mathfrak{g}$ a Lie algebra. On elements $X \in \mathfrak{g}$, we have coproduct:

$$\Delta : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}), \qquad X \mapsto \Delta(X) = X \otimes \mathbb{I} + \mathbb{I} \otimes X, \tag{6.1}$$

counit:

$$\varepsilon : \mathrm{U}(\mathfrak{g}) \to \mathbb{C}, \qquad X \mapsto \varepsilon(X) = 0, \tag{6.2}$$

and antipode:

$$S : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}), \qquad X \mapsto S(X) = -X. \tag{6.3}$$

When $\mathfrak{g}$ is realized as a Lie algebra of vector fields acting on an algebra of functions $A = C^\infty(M)$, the coproduct (6.1) is just the implementation of the Leibniz rule for any $X \in \mathfrak{g}$,

$$X(ab) := \Delta X(a \otimes b) = X(a)b + aX(b), \tag{6.4}$$

saying that X is a derivation of A. Then, suitably twisting the Hopf algebra H goes together with twisting the product in A to a noncommutative algebra that carries an action of the twisted Hopf algebra.

Let us start with a Hopf algebra $H = (H, \mu, \mathbb{I}, \Delta, \varepsilon, S)$ over $\mathbb{C}$ (say), with multiplication $\mu : H \otimes H \to H$; comultiplication $\Delta : H \to H \otimes H$ (for which we use Sweedler notation, $\Delta(h) = h_{(1)} \otimes h_{(2)}$); unit $\mathbb{I} : \mathbb{C} \to H$ and counit $\varepsilon : H \to \mathbb{C}$; antipode $S : H \to H$. This structure is twisted by an invertible element $\mathcal{F} \in H \otimes H$ with properties,

$$(\mathcal{F} \otimes \mathbb{I})(\Delta \otimes \mathrm{id})\mathcal{F} = (\mathbb{I} \otimes \mathcal{F})(\mathrm{id} \otimes \Delta)\mathcal{F}, \tag{6.5}$$

$$(\varepsilon \otimes \mathrm{id})\mathcal{F} = \mathbb{I} = (\mathrm{id} \otimes \varepsilon)\mathcal{F}. \tag{6.6}$$

Then, the element of A

$$v = \mu(\mathrm{id} \otimes S)\mathcal{F}$$

is invertible with inverse given by

$$v^{-1} = \mu(S \otimes \mathrm{id})\mathcal{F}^{-1}.$$

The twisted Hopf algebra $H_{\mathcal{F}} = (H, \mu, \mathbb{I}, \Delta_{\mathcal{F}}, \varepsilon, S_{\mathcal{F}})$ has the same algebra and counit as H but twisted coproduct $\Delta_{\mathcal{F}} : H \to H \otimes H$ and antipode $S_{\mathcal{F}} : H \to H$,

$$\Delta_{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \qquad S_{\mathcal{F}}(h) = vS(h)v^{-1}. \tag{6.7}$$

The twist $\mathcal{F} \in H \otimes H$ is called a 2-cochain in general and the condition (6.5) is a cocycle condition that assures coassociativity of the twisted coproduct

$\Delta_{\mathcal{F}}$. By dropping condition (6.5), $\Delta_{\mathcal{F}}$ is however "almost coassociative" and this leads to the notion of a *quasi-Hopf algebra* [33].

The cocycle $\mathcal{F}$ can be used to twist the multiplication of any left H-module algebra A. This twisting yields a new algebra $A_{\mathcal{F}}$ which is naturally a left $H_{\mathcal{F}}$-module algebra.

Let us recall that a left H-module algebra A is first of all a left H-module, and this means that there is a map $\lambda : H \otimes A \to A$ such that the association $H \ni h \mapsto \lambda(h \otimes \cdot)$ is a homomorphism of algebras from H into the endomorphisms of A. In addition there is compatibility with respect to the algebra structure of A,

$$h \triangleright (ab) := \Delta h(a \otimes b) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \qquad h \triangleright 1 = \varepsilon(h)1, \tag{6.8}$$

for all $h \in H$, and $a, b \in A$; and we have used the notation $\lambda(h \otimes a) = h \triangleright a$.

Now, if $m : A \otimes A \to A, m(a \otimes b) = ab$ denotes the multiplication in A, the new algebra $A_{\mathcal{F}}$ is defined to be A with multiplication given by

$$m_{\mathcal{F}} = m \circ \mathcal{F}^{-1},$$

and associativity of this product is guaranteed by the cocycle condition (6.5). Suggestively, the new product can be indicated as

$$a \times_{\mathcal{F}} b = m\left((\mathcal{F}^{-1} \triangleright a \otimes b)\right). \tag{6.9}$$

As mentioned, the algebra $A_{\mathcal{F}}$ is a left $H_{\mathcal{F}}$-module algebra; in fact the action of any $h \in H_{\mathcal{F}}$ on any $a \in A_{\mathcal{F}}$ is just the old one $h \triangleright a$ but extended on products via the twisted comultiplication:

$$h \triangleright (a \times_{\mathcal{F}} b) = \Delta_{\mathcal{F}}(a \otimes b). \tag{6.10}$$

Dually, the cocycle $\mathcal{F}$ could be used to twist the comultiplication of any right H-module coalgebra B to get a coalgebra $B_{\mathcal{F}}$ carrying a natural action of $H_{\mathcal{F}}$.

Remark 31. To be precise, the twists that we use in the present paper are in fact formal power series, that is $\mathcal{F} \in H \otimes H[[\lambda]]$, with $\lambda = e^{2\pi i\theta}$ the deformation parameter. We shall avoid the use of formal power series and of λ-adic topology by working in explicit representation spaces and with explicit operators (see also [48]).

6.2. *The rotational symmetry of S^4_θ*

Let us start with the construction of the twisted symmetry $\mathrm{U}_\theta(so(5))$. The eight roots of the Lie algebra $so(5)$ are two-component vectors $r = (r_1, r_2)$

of the form $r = (\pm 1, \pm 1)$ and $r = (0, \pm 1), r = (\pm 1, 0)$. There are corresponding generators E_r together with two mutually commuting generators H_1, H_2 of the Cartan subalgebra. The Lie brackets are

$$\begin{gathered}[H_1, H_2] = 0, \quad [H_j, E_r] = r_j E_r, \\ [E_{-r}, E_r] = r_1 H_1 + r_2 H_2, \quad [E_r, E_{r'}] = N_{r,r'} E_{r+r'},\end{gathered} \tag{6.11}$$

with $N_{r,r'} = 0$ if $r + r'$ is not a root. The universal enveloping algebra $\mathrm{U}(so(5))$ is the algebra generated by elements $\{H_j, E_r\}$ modulo relations given by the previous Lie brackets. It is a Hopf algebra with the undeformed structure as in (6.1)–(6.3).

The twisted universal enveloping algebra $\mathrm{U}_\theta(so(5))$ is generated as above (i.e. one does not change the algebra structure) but is endowed with a twisted coproduct,

$$\Delta_\theta : \mathrm{U}_\theta(so(5)) \to \mathrm{U}_\theta(so(5)) \otimes \mathrm{U}_\theta(so(5)), \qquad X \mapsto \Delta_\theta(X) = \mathcal{F} \Delta_0(X) \mathcal{F}^{-1}.$$

For the symmetries studied in the present paper the twist $\mathcal{F}$ is given explicitly by

$$\mathcal{F} = \lambda^{\frac{1}{2}(-H_1 \otimes H_2 + H_2 \otimes H_1)}. \tag{6.12}$$

On the generators E_r, H_j, the twisted coproduct reads

$$\begin{aligned}\Delta_\theta(E_r) &= E_r \otimes \lambda^{\frac{1}{2}(-r_1 H_2 + r_2 H_1)} + \lambda^{\frac{1}{2}(r_1 H_2 - r_2 H_1)} \otimes E_r, \\ \Delta_\theta(H_j) &= H_j \otimes \mathbb{I} + \mathbb{I} \otimes H_j.\end{aligned} \tag{6.13}$$

This coproduct allows one to represent $\mathrm{U}_\theta(so(5))$ as an algebra of twisted derivations on both S^4_θ and $S^7_{\theta'}$ as we shall see below. With counit and antipode given by

$$\begin{aligned}&\varepsilon(E_r) = \varepsilon(H_j) = 0, \\ &S(E_r) = -\lambda^{\frac{1}{2}(r_2 H_1 - r_1 H_2)} E_r \lambda^{\frac{1}{2}(r_1 H_2 - r_2 H_1)}, \quad S(H_j) = -H_j,\end{aligned} \tag{6.14}$$

the algebra $\mathrm{U}_\theta(so(5))$ becomes a Hopf algebra [17]. At the classical value of the deformation parameter, $\theta = 0$, one recovers the Hopf algebra structure of $\mathrm{U}(so(5))$.

Remark 32. The operators $\lambda^{\pm\frac{1}{2} r_i H_j}$ in (6.16) are understood as exponentials of diagonal matrices due to the fact that on generators the operators H_1 and H_2 can be written as finite dimensional matrices. This will be clear presently when acting on both S^4_θ and $S^7_{\theta'}$.

We are ready for the representation of $\mathrm{U}_\theta(so(5))$ on S^4_θ. For convenience, we introduce "partial derivatives" ∂_μ and ∂^*_μ with the usual action on the generators of the algebra $\mathcal{A}(S^4_\theta)$ i.e, $\partial_\mu(z_\nu) = \delta_{\mu\nu}$, $\partial_\mu(z^*_\nu) = 0$, and $\partial^*_\mu(z^*_\nu) = \delta_{\mu\nu}$, $\partial^*_\mu(z_\nu) = 0$. Then, the action of $\mathrm{U}_\theta(so(5))$ on $\mathcal{A}(S^4_\theta)$ is given by the following operators,

$$\begin{aligned} H_1 &= z_1\partial_1 - z_1^*\partial_1^*, & H_2 &= z_2\partial_2 - z_2^*\partial_2^* \\ E_{+1,+1} &= z_2\partial_1^* - z_1\partial_2^*, & E_{+1,-1} &= z_2^*\partial_1^* - z_1\partial_2, \\ E_{+1,0} &= \frac{1}{\sqrt{2}}(2z_0\partial_1^* - z_1\partial_0), & E_{0,+1} &= \frac{1}{\sqrt{2}}(2z_0\partial_2^* - z_2\partial_0), \end{aligned} \tag{6.15}$$

and $E_{-r} = (E_r)^*$, with the obvious meaning of the adjoint. A comparison with Eq. (4.15) shows that H_1 and H_2 in (6.15) are the infinitesimal generators of the action of $\mathbb{T}^2$ on S^4_θ. These operators (not the partial derivatives!) are extended to the whole of $\mathcal{A}(S^4_\theta)$ as twisted derivations via the coproduct (6.13),

$$\begin{aligned} E_r(ab) &= \Delta_\theta(E_r)(a\otimes b) = E_r(a)\lambda^{\frac{1}{2}(-r_1H_2+r_2H_1)}(b) \\ &\quad + \lambda^{\frac{1}{2}(r_1H_2-r_2H_1)}(a)E_r(b), \\ H_j(ab) &= \Delta_\theta(H_j)(a\otimes b) = H_j(a)b + aH_j(b), \end{aligned} \tag{6.16}$$

for any two elements $a, b \in \mathcal{A}(S^4_\theta)$. With these twisted rules, one readily checks compatibility with the commutation relations (4.1) of $\mathcal{A}(S^4_\theta)$.

We can write the twisted action of $\mathrm{U}_\theta(so(5))$ on $\mathcal{A}(S^4_\theta)$ by using the quantization map L_θ of Sec. 3. For $L_\theta(a) \in \mathcal{A}(S^4_\theta)$ and $t \in \mathrm{U}(so(5))$ a twisted action is defined by

$$T \cdot L_\theta(a) = L_\theta(t\cdot a), \tag{6.17}$$

where T is the "quantization" of t and $t\cdot a$ is the classical action of $\mathrm{U}(so(5))$ on $\mathcal{A}(S^4)$ (a better but heavier notation for the action $T\cdot$ would be $t\cdot_\theta$). One checks that these two definitions of the twisted action coincide. The latter definition allows one to define an action of $\mathrm{U}(so(5))$ on $C^\infty(S^4_\theta)$ by allowing a to be in $C^\infty(S^4)$ in Eq. (6.17). Furthermore, as operators on the Hilbert space $\mathcal{H}$ of spinors, one could identify $\lambda^{\frac{1}{2}(r_1H_2-r_2H_1)} = U(\frac{1}{2}r\cdot\theta)$, with $r = (r_1, r_2)$, θ the antisymmetric two by two matrix with $\theta_{12} = -\theta_{21} = \theta$ and $U(s)$ the representation of $\mathbb{T}^2$ on $\mathcal{H}$ as in Sec. 3.1.

The twisted action of the Hopf algebra $\mathrm{U}(so(5))$ on $\mathcal{A}(S^4_\theta)$ is extended to the differential calculus $(\Omega(S^4_\theta), \mathrm{d})$ by requiring it to commute with the exterior derivative,

$$T\cdot \mathrm{d}\omega := \mathrm{d}(T\cdot\omega)$$

for $T \in \mathrm{U}(so(5))$, $\omega \in \Omega(S^4_\theta)$. Then, we need to use the rule (6.16) on a generic form. For instance, on 1-forms we have

$$\begin{aligned} E_r\Big(\sum_k a_k \mathrm{d} b_k\Big) &= \sum_k \Big(E_r(a_k)\mathrm{d}\big(\lambda^{\frac{1}{2}(-r_1H_2+r_2H_1)}(b_k)\big) \\ &\qquad + \lambda^{\frac{1}{2}(r_1H_2-r_2H_1)}(a_k)\mathrm{d}\big(E_r(b_k)\big)\Big), \qquad (6.18) \\ H_j\Big(\sum_k a_k \mathrm{d} b_k\Big) &= \sum_k \Big(H_j(a_k)\mathrm{d} b_k + a_k \mathrm{d}\big(H_j(b_k)\big)\Big). \end{aligned}$$

The representation of $\mathrm{U}_\theta(so(5))$ on S^4_θ given in (6.15) is the fundamental vector representation. When lifted to $S^7_{\theta'}$ one gets the fundamental spinor representation: as we see from the quadratic relations among corresponding generators, as given in (4.10), the lifting amounts to taking the "square root" representation. The action of $\mathrm{U}_\theta(so(5))$ on $\mathcal{A}(S^7_{\theta'})$ is constructed by requiring twisted derivation properties via the coproduct (6.16) so as to reduce to the action (6.15) on $\mathcal{A}(S^4_\theta)$ when using the defining quadratic relations (4.10). The action on $\mathcal{A}(S^7_{\theta'})$ can be given as the action of the matrices Γ's on the ψ's,

$$\psi_a \mapsto \sum_b \Gamma_{ab}\psi_b, \qquad \psi_a^* \mapsto \sum_b \widetilde{\Gamma}_{ab}\psi_b^* \qquad (6.19)$$

with the matrices $\Gamma = \{H_j, E_r\}$ given explicitly by

$$\begin{aligned} H_1 &= \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, & H_2 &= \frac{1}{2}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \\ E_{+1,+1} &= \begin{pmatrix} 0 & 0 \\ 0 & \begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} \end{pmatrix}, & E_{+1,-1} &= \begin{pmatrix} \begin{smallmatrix} 0 & 0 \\ -\mu & 0 \end{smallmatrix} & 0 \\ 0 & 0 \end{pmatrix}, \qquad (6.20) \\ E_{+1,0} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix} \\ \begin{smallmatrix} \mu & 0 \\ 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}, & E_{0,+1} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \begin{smallmatrix} 0 & \bar{\mu} \\ 0 & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}, \end{aligned}$$

and

$$\widetilde{\Gamma} := \sigma\Gamma\sigma^{-1}, \qquad \sigma := \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}. \qquad (6.21)$$

Furthermore, $E_{-r} = (E_r)^*$. With the twisted rules (6.16) for the action on products, one checks compatibility of the above action with the commutation relations (4.4) of $\mathcal{A}(S^7_{\theta'})$. Again, the operators $\lambda^{\pm\frac{1}{2}r_iH_j}$ in (6.16) are exponentials of diagonal matrices H_1 and H_2 given in the representation (6.20) and as above, one could think of $\lambda^{\frac{1}{2}(r_1 H_2 - r_2 H_1)}$ as the operator $U(\frac{1}{2}r \cdot \theta)$.

Remark 33. Compare the form of the matrices H_1 and H_2 in the representation (6.20) above with the lifted action $\widetilde{\sigma}$ of $\widetilde{\mathbb{T}}^2$ on $S^7_{\theta'}$ as defined in (4.16). One checks that

$$\widetilde{\sigma}_s = e^{\pi i((s_1+s_2)H_1 + (-s_1+s_2)H_2)},$$

when acting on the spinor $(\psi_1, \ldots, \psi_4)$.

Notice that $\widetilde{\Gamma} = -\Gamma^t$ at $\theta = 0$. There is a beautiful correspondence between the matrices in the representation (6.20) and the twisted Dirac matrices introduced in (4.13),

$$\begin{aligned} &\frac{1}{4}[\gamma_1^*, \gamma_1] = 2H_1, && \frac{1}{4}[\gamma_2^*, \gamma_2] = 2H_2, \\ &\frac{1}{4}[\gamma_1, \gamma_2] = (\mu + \overline{\mu})E_{+1,+1}, && \frac{1}{4}[\gamma_1, \gamma_2^*] = (\mu + \overline{\mu})E_{+1,-1}, \\ &\frac{1}{4}[\gamma_1, \gamma_0] = \sqrt{2}E_{+1,0}, && \frac{1}{4}[\gamma_2, \gamma_0] = \sqrt{2}\overline{\mu}E_{0,+1}. \end{aligned} \tag{6.22}$$

Also, the twisted Dirac matrices satisfy the following relations under conjugation by σ:

$$(\sigma\gamma_0\sigma^{-1})^t = \gamma_0, \qquad (\sigma\gamma_1\sigma^{-1})^t = \gamma_1\lambda^{H_2}, \qquad (\sigma\gamma_2\sigma^{-1})^t = \gamma_2\lambda^{H_1}. \tag{6.23}$$

As for S^4_θ, the twisted action of $so(5)$ on $\mathcal{A}(S^7_\theta)$ is straightforwardly extended to the differential calculus $(\Omega(S^7_\theta), \mathrm{d})$ by requiring that the action commutes with the exterior differential d and using the twisted rule when acting on products.

7. Instantons from Twisted Conformal Symmetries

Different instantons are obtained by a twisted symmetry action of $so(5,1)$. Classically, $so(5,1)$ is the conformal Lie algebra consisting of the infinitesimal diffeomorphisms leaving the conformal structure invariant. The Lie algebra $so(5,1)$ is given by adding 5 generators to $so(5)$. We explicitly describe its action on S^4_θ together with its lift to $S^7_{\theta'}$ as an algebra of twisted derivations. The induced action on forms leaves the conformal structure invariant and when acting on ∇_0 eventually results in a five-parameter family of instantons.

The Hopf algebra $U_\theta(so(5))$ described in the previous section is made of twisted infinitesimal symmetries under which a basic instanton – associated canonically with the noncommutative instanton bundle constructed previously – is invariant. We construct a collection of (infinitesimal) gauge-nonequivalent instantons, by acting with a twisted conformal symmetry $U_\theta(so(5,1))$ on the basic one. A completeness argument on this collection is provided using an index theoretical argument, similar to [8]. The dimension of the "tangent" to the moduli space can be computed as the index of a twisted Dirac operator and it turns out to be equal to its classical value, which is five.

Here, one has to be careful with the notion of tangent space to the moduli space. As will be discussed elsewhere [46] one can construct a noncommutative family of instantons, that is, instantons parametrized by the quantum quotient space of the deformed conformal group $SL_\theta(2,\mathbb{H})$ by the deformed gauge group $Sp_\theta(2)$. The basic instanton of [23] is a "classical point" in the moduli space of instantons. We perturb this connection ∇_0 linearly by sending $\nabla_0 \mapsto \nabla_0 + t\alpha$ where $t \in \mathbb{R}$ and $\alpha \in \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \Omega^1(S^4_\theta))$. In order for this new connection still to be an instanton we have to impose the self-duality equation on its curvature. After differentiating this equation with respect to t, at $t = 0$, we obtain the linearized self-duality equation to be fulfilled by α. It is in this sense that we are considering the tangent space to the moduli space of instantons at the origin ∇_0.

7.1. *The basic instanton*

We start with a technical lemma that simplifies the discussion. Let $\mathcal{E} = C^\infty(S^7_{\theta'}) \boxtimes_\rho W$ be the module of sections associated to a finite dimensional representation of SU(2), as defined in Eq. (4.27).

Lemma 34. *There is the following isomorphism of right $C^\infty(S^4_\theta)$-modules,*

$$\mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega(S^4_\theta) \simeq \Omega(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \mathcal{E}.$$

Consequently, $\mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega(S^4_\theta)) \simeq \Omega(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \mathrm{End}(\mathcal{E})$.

Proof. Recall from Proposition 18 that the right $C^\infty(S^4_\theta)$-module $\mathcal{E}$ has a homogeneous module-basis $\{e_\alpha, \alpha = 1, \ldots, N\}$ for some N and each e_α of degree r_α. A generic element in $\mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega(S^4_\theta)$ can be written as a sum $\sum_\alpha e_\alpha \otimes_{C^\infty(S^4_\theta)} \omega^\alpha$, with ω^α an element in $\Omega(S^4_\theta)$. Now, for every $\omega \in \Omega(S^4_\theta)$ there is an element $\widetilde{\omega} \in \Omega(S^4_\theta)$ – given explicitly by $\widetilde{\omega} = \sigma_{r_\alpha \cdot \theta}(\omega)$ – such that $e_\alpha \omega = \widetilde{\omega} e_\alpha$, where the latter equality holds inside the algebra $\Omega(S^7_{\theta'})$

(recall that $\Omega(S^4_\theta) \subset \Omega(S^7_{\theta'})$). We can thus define a map

$$T : \mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega(S^4_\theta) \xrightarrow{\simeq} \Omega(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \mathcal{E},$$

by $T(e_\alpha \otimes_{C^\infty(S^4_\theta)} \omega^\alpha) = \widetilde{\omega}^\alpha \otimes e_\alpha$; it is a right $C^\infty(S^4)$-module map:

$$T\left(e_\alpha \otimes_{C^\infty(S^4_\theta)} (\omega^\alpha \times_\theta f)\right) = (\widetilde{\omega}^\alpha \times_\theta \widetilde{f}) \otimes_{C^\infty(S^4_\theta)} e_\alpha = T(e_\alpha \otimes_{C^\infty(S^4_\theta)} \omega^\alpha) \times_\theta f.$$

Since an inverse map T^{-1} is easily constructed, T gives the desired isomorphism. $\square$

Thus, we can unambiguously use the notation $\Omega(S^4_\theta, \mathcal{E})$ for the above right $C^\infty(S^4_\theta)$-module $\mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega(S^4_\theta)$.

We let $\nabla_0 = p \circ \mathrm{d}$ be the canonical (Grassmann) connection on the projective module $\mathcal{E} = C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2 \simeq p(C^\infty(S^4_\theta))^4$, with the projection $p = \Psi^\dagger \Psi$ of (4.19) where Ψ is the matrix (4.17). When acting on equivariant maps, we can write ∇_0 as

$$\nabla_0 : \mathcal{E} \to \mathcal{E} \otimes_{C^\infty(S^4_\theta)} \Omega^1(S^4_\theta), \qquad (\nabla_0 f)_i = \mathrm{d}f_i + \omega_{ij} \times_\theta f_j, \tag{7.1}$$

where ω – called the gauge potential – is given in terms of the matrix Ψ by

$$\omega = \Psi^\dagger \mathrm{d}\Psi, \tag{7.2}$$

The above is a 2×2-matrix with entries in $\Omega^1(S^7_{\theta'})$ satisfying $\overline{\omega_{ij}} = -\omega_{ji}$ and $\sum_i \omega_{ii} = 0$. Note here that the entries ω_{ij} commute with all elements in $C^\infty(S^7_{\theta'})$. Indeed, from (4.17) we see that the elements in ω_{ij} are $\mathbb{T}^2$-invariant and hence central (as one-forms) in $\Omega(S^7_{\theta'})$. In other words $L_\theta(\omega) = \omega$, which shows that for an element $f \in \mathcal{E}$ as above, we have $\nabla_0(f)_i = \mathrm{d}f_i + \omega_{ij} \times_\theta f_j = \mathrm{d}f_i + \omega_{ij} f_j$ which coincides with the action of the classical connection $\mathrm{d}+\omega$ on f. The curvature $F_0 = \nabla_0^2 = \mathrm{d}\omega + \omega^2$ of ∇_0 is an element of $\mathrm{End}(\mathcal{E}) \otimes_{C^\infty(S^4_\theta)} \Omega^2(S^4_\theta)$ that satisfies [1, 22] the self-duality equation,

$$*_\theta F_0 = F_0; \tag{7.3}$$

hence this connection is an instanton. At the classical value of the deformation parameter, $\theta = 0$, the connection (7.2) is nothing but the SU(2) instanton of [13].

Its "topological charge", i.e. the values of $\mathrm{Top}(\mathcal{E})$ in Definition (29), was already computed in [23]. Clearly it depends only on the class $[p]$ of the bundle and can be evaluated as the index

$$\mathrm{Top}([p]) = \mathrm{index}(D_p) = ⨍ \gamma_5 \pi_D(\mathrm{ch}_2(p)) \tag{7.4}$$

of the twisted Dirac operator

$$D_p = p(D \otimes \mathbb{I}_4)p.$$

The last equality in (7.4) follows from the vanishing of the class $\mathrm{ch}_1(p)$ of the bundle. The Chern character classes and their realization as operators are in Sec. 3.5. On the other hand, one finds

$$\pi_D\big(\mathrm{ch}_2(p)\big) = 3\gamma_5,$$

which, together with the fact that

$$\int\!\!\!\!\!\!- \; 1 = \mathrm{Tr}_\omega(|D|^{-4}) = \frac{1}{3},$$

on S^4 (see for instance [37, 42]), gives the value $\mathrm{Top}([p]) = 1$.

We aim at constructing all connections ∇ on $\mathcal{E}$ whose curvature satisfies this self-duality equation and of topological charge equal to 1. We can write any such connection in terms of the canonical connection as in Eq. (2.9), i.e. $\nabla = \nabla_0 + \alpha$ with α a one-form valued endomorphism of $\mathcal{E}$. Clearly, this will not change the value of the topological charge. We are interested in SU(2)-instantons, so we impose that α is traceless and skew-Hermitian, with the trace taken in the second factor of $\mathrm{End}(\mathcal{E}) \simeq P \boxtimes_{\mathrm{ad}} M_2(\mathbb{C})$. When complexified, this yields an element $\alpha \in \Omega^1(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \Gamma^\infty(\mathrm{ad}(S^7_{\theta'})) =: \Omega^1(\mathrm{ad}(S^4_\theta))$ (cf. Example 22).

As usual, we impose an irreducibility condition on the instanton connections, a connection on $\mathcal{E}$ being *irreducible* if it cannot be written as the sum of two other connections on $\mathcal{E}$. We are interested only in the irreducible instanton connections on the module $\mathcal{E}$.

Remark 35. In Sec. 4.5, we constructed projections $p_{(n)}$ for all modules $C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^n$ over $C^\infty(S^4_\theta)$ associated to the irreducible representations $\mathbb{C}^n$ of SU(2). The induced Grassmann connections $\nabla_0^{(n)} := p_{(n)}\mathrm{d}$, when acting on $C^\infty(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^n$, were written as $\mathrm{d} + \omega_{(n)}$, with $\omega_{(n)}$ an $n \times n$ matrix with entries in $\Omega^1(S^7_{\theta'})$. A similar argument as above then shows that all $\omega_{(n)}$ have entries that are central (as one-forms) in $\Omega(S^7_{\theta'})$; again, this means that $L_\theta(\omega_{(n)}) = \omega_{(n)}$. In particular, this holds for the adjoint bundle associated to the adjoint representation on $su(2)_\mathbb{C} \simeq \mathbb{C}^3$ (as complex representation spaces), from which we conclude that $\nabla_0^{(2)}$ coincides with $[\nabla_0, \cdot]$ (since this is the case if $\theta = 0$).

The instanton potential in Eq. (7.2) is invariant under the action of $\mathrm{U}_\theta(so(5))$. Recall the latter's action on $S^7_{\theta'}$ given in eqs. (6.19)-(6.21) (and extended canonically to forms). Due to the form of $\widetilde{\Gamma}$ in (6.21) and the property $\Psi_{a2} = \sigma_{ab}\psi_b^*$ for the second column of the matrix Ψ in (4.17), the algebra $\mathrm{U}_\theta(so(5))$ acts on Ψ by left matrix multiplication by Γ, and by right matrix multiplication on Ψ^* by the matrix transpose $\widetilde{\Gamma}^t$ as follows:

$$\Psi_{ai} \mapsto \sum_b \Gamma_{ab}\Psi_{bi}, \qquad \Psi^*_{ia} \mapsto \sum_a \Psi^*_{ib}\widetilde{\Gamma}_{ab}. \tag{7.5}$$

These are used in the following

Proposition 36. *The instanton gauge potential ω is invariant under the twisted action of* $\mathrm{U}_\theta(so(5))$.

Proof. From the above observations, the gauge potential transforms as:

$$\omega = \Psi^*\mathrm{d}\Psi \mapsto \Psi^*\big(\widetilde{\Gamma}^t\lambda^{-r_1H_2} + \lambda^{r_2H_1}\Gamma\big)\mathrm{d}\Psi,$$

where $\lambda^{-r_iH_j}$ is understood in its representation (6.20) on $\mathcal{A}(S^7_{\theta'})$. Direct computation for $\Gamma = \{H_j, E_r\}$ shows that $\widetilde{\Gamma}^t\lambda^{-r_1H_2} + \lambda^{r_2H_1}\Gamma = 0$, which finishes the proof. □

7.2. *The infinitesimal conformal symmetry*

Different instantons are obtained by a twisted symmetry action of $\mathrm{U}_\theta(so(5,1))$. Classically, $so(5,1)$ is the conformal Lie algebra consisting of the infinitesimal diffeomorphisms leaving the conformal structure invariant. We construct the Hopf algebra $\mathrm{U}_\theta(so(5,1))$ by adding 5 generators to $\mathrm{U}_\theta(so(5))$ and describe its action on S^4_θ together with its lift to $S^7_{\theta'}$. The induced action of $\mathrm{U}_\theta(so(5,1))$ on forms leaves the conformal structure invariant. The action of $\mathrm{U}_\theta(so(5,1))$ on ∇_0 eventually results in a five-parameter family of (infinitesimal) instantons.

The conformal Lie algebra $so(5,1)$ consists of the generators of $so(5)$ together with the dilation and the so-called special conformal transformations. On $\mathbb{R}^4$ with coordinates $\{x_\mu, \mu = 1,\dots,4\}$ they are given by the operators $H_0 = \sum_\mu x_\mu\partial/\partial x_\mu$ and $G_\mu = 2x_\mu\sum_\nu x_\nu\partial/\partial x_\nu - \sum_\nu x_\nu^2(\partial/\partial x_\nu)$, respectively [49]. In the definition of the enveloping algebra $\mathrm{U}_\theta(so(5,1))$ we do not change the algebra structure, i.e. we take the relations of $\mathrm{U}(so(5,1))$, as we did for $\mathrm{U}_\theta(so(5))$. We thus define $\mathrm{U}_\theta(so(5,1))$ as the algebra $\mathrm{U}_\theta(so(5))$ with five extra generators adjoined, H_0, G_r, $r = (\pm1,0),(0,\pm1)$, subject to the relations of $\mathrm{U}_\theta(so(5))$ of Eq. (6.11) together with the (undeformed) relations,

$$\begin{aligned} &[H_0, H_i] = 0, &\qquad &[H_j, G_r] = r_jG_r, \\ &[H_0, G_r] = \sqrt{2}E_r, &\qquad &[H_0, E_r] = (\sqrt{2})^{-1}G_r, \end{aligned} \tag{7.6}$$

whenever $r = (\pm 1, 0), (0, \pm 1)$, and

$$\begin{aligned} &[G_{-r}, G_r] = 2r_1 H_1 + 2r_2 H_2, && [G_r, G_{r'}] = N_{r,r'} E_{r+r'}, \\ &[E_r, G_{r'}] = \widetilde{N}_{r,r'} G_{r+r'}, && [E_{-r}, G_r] = \sqrt{2} H_0, \end{aligned} \tag{7.7}$$

with as before, the constant $N_{r,r'} = 0$ if $r + r'$ is not a root of $so(5)$ and the constant $\widetilde{N}_{r,r'} = 0$ if $r + r' \notin \{(\pm 1, 0), (0, \pm 1)\}$. Although the algebra structure is unchanged, again the Hopf algebra structure of $\mathrm{U}_\theta(so(5,1))$ gets twisted. The twisted structures are given by eqs. (6.13) and (6.14) together with

$$\begin{aligned} &\Delta_\theta(G_r) = G_r \otimes \lambda^{\frac{1}{2}(-r_1 H_2 + r_2 H_1)} && \Delta_\theta(H_0) = H_0 \otimes 1 + 1 \otimes H_0, \\ &\qquad + \lambda^{\frac{1}{2}(r_1 H_2 - r_2 H_1)} \otimes G_r, && S(H_0) = -H_0, \\ &S(G_r) = -\lambda^{\frac{1}{2}(r_2 H_1 - r_1 H_2)} G_r \lambda^{\frac{1}{2}(r_1 H_2 - r_2 H_1)}, && \varepsilon(H_0) = 0 \\ &\varepsilon(G_r) = 0, && \end{aligned} \tag{7.8}$$

making $\mathrm{U}_\theta(so(5,1))$ a Hopf algebra.

The action of $\mathrm{U}_\theta(so(5,1))$ on $\mathcal{A}(S^4_\theta)$ is given by the operators (6.15) together with

$$\begin{aligned} H_0 &= \partial_0 - z_0(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 + z_2^*\partial_2^*), \\ G_{1,0} &= 2\partial_1^* - z_1(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + \overline{\lambda} z_2\partial_2 + \lambda z_2^*\partial_2^*), \\ G_{0,1} &= 2\partial_2^* - z_2(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 + z_2^*\partial_2^*), \end{aligned} \tag{7.9}$$

and $G_{-r} = (G_r)^*$. The introduction of the extra λ's in $G_{1,0}$ (and $G_{-1,0}$) is necessary for the algebra structure of $\mathrm{U}_\theta(so(5,1))$, as dictated by the Lie brackets in (7.6), (7.7) to be preserved. Since the operators H_0 and G_r are quadratic in the z's, one has to be careful when checking the Lie brackets and use the twisted rules (6.16). For instance, on the generator z_2, we have

$$\begin{aligned} [E_{-1,-1}, G_{1,0}](z_2) &= E_{-1,-1}(-\overline{\lambda} z_1 z_2) + G_{1,0}(z_1^*) \\ &= -\overline{\lambda}(E_{-1,-1}(z_1)\lambda^{H_2}(z_2) + \lambda^{H_1}(z_1)E_{-1,-1}(z_2)) \\ &\quad + G_{1,0}(z_1^*) \\ &= -z_2^* z_2 + z_1 z_1^* + 2 - z_1 z_1^* = G_{0,-1}(z_2). \end{aligned}$$

The operators in (6.15) and (7.9) give a well defined action of $\mathrm{U}_\theta(so(5,1))$ on the algebra $\mathcal{A}(S^4_\theta)$ provided one extends them to the whole of $\mathcal{A}(S^4_\theta)$ as twisted derivations via the rules (6.16) together with

$$\begin{aligned} G_r(ab) &= G_r(a)\lambda^{\frac{1}{2}(-r_1H_2+r_2H_1)}(b) + \lambda^{\frac{1}{2}(r_1H_2-r_2H_1)}(a)G_r(b), \\ H_0(ab) &= H_0(a)b + aH_0(b), \end{aligned} \tag{7.10}$$

for any two elements $a, b \in \mathcal{A}(S^4_\theta)$.

Equivalently, the Hopf algebra $\mathrm{U}_\theta(so(5,1))$ could be defined to act on $\mathcal{A}(S^4_\theta)$ by

$$T \cdot L_\theta(a) = L_\theta(t \cdot a), \tag{7.11}$$

for $T \in \mathrm{U}_\theta(so(5,1))$ deforming $t \in \mathrm{U}(so(5,1))$ and $L_\theta(a) \in \mathcal{A}(S^4_\theta)$ deforming $a \in \mathcal{A}(S^4)$. Again, equation (7.11) makes sense for $a \in C^\infty(S^4)$, which defines an action of $so(5,1)$ on $C^\infty(S^4_\theta)$. As before, the action on the differential calculus $(\Omega(S^4_\theta), \mathrm{d})$ is obtained by requiring commutation with the exterior derivative: $T \cdot \mathrm{d}\omega = \mathrm{d}(T \cdot \omega)$, for $T \in so(5,1)$ and $\omega \in \Omega(S^4_\theta)$. On products we shall have formulae like the one in (6.18),

$$\begin{aligned} G_r\left(\sum_k a_k \mathrm{d}b_k\right) &= \sum_k \Big(G_r(a_k)\mathrm{d}\big(\lambda^{\frac{1}{2}(-r_1H_2+r_2H_1)}(b_k)\big) \\ &\quad + \lambda^{\frac{1}{2}(r_1H_2-r_2H_1)}(a_k)\mathrm{d}\big(G_r(b_k)\big)\Big), \\ H_0\left(\sum_k a_k \mathrm{d}b_k\right) &= \sum_k \Big(H_0(a_k)\mathrm{d}b_k + a_k\mathrm{d}\big(H_0(b_k)\big)\Big). \end{aligned} \tag{7.12}$$

What we are dealing with are "infinitesimal" twisted conformal transformations:

Lemma 37. *The Hodge $*_\theta$-structure of $\Omega(S^4_\theta)$ is invariant under the twisted action of $\mathrm{U}_\theta(so(5,1))$,*

$$T \cdot (*_\theta \omega) = *_\theta(T \cdot \omega),$$

Proof. Recall that $T(L_\theta(a)) = L_\theta(t \cdot a)$ for $a \in \mathcal{A}(S^4)$ and T is the "quantization" of $t \in \mathrm{U}(so(5,1))$. Then, since $so(5,1)$ leaves the Hodge $*$-structure of $\Omega(S^4)$ invariant and the differential d commutes with the action of $\mathrm{U}_\theta(so(5,1))$, it follows that the latter algebra leaves the Hodge $*_\theta$-structure of $\Omega(S^4_\theta)$ invariant as well. □

7.3. *The construction of instantons*

Again, the action of $so(5,1)$ on S^4_θ can be lifted to an action on $S^7_{\theta'}$. And the latter action can be written as in (6.19) in terms of matrices Γ's acting on the ψ's,

$$\psi_a \mapsto \sum_b \Gamma_{ab}\psi_b, \qquad \psi^*_a \mapsto \sum_b \widetilde{\Gamma}_{ab}\psi^*_b, \tag{7.13}$$

where in addition to (6.20) we have also the matrices $\Gamma = \{H_0, G_r\}$, given explicitly by

$$\begin{aligned} H_0 &= \frac{1}{2}(-z_0\mathbb{I}_4 + \gamma_0), \\ G_{1,0} &= \frac{1}{2}(-z_1\lambda^{-H_2} + \gamma_1), \\ G_{0,1} &= \frac{1}{2}(-z_2 + \lambda^{-H_1}\gamma_2), \end{aligned} \tag{7.14}$$

with $G_{-r} = (G_r)^*$ and $\widetilde{\Gamma} = \sigma\Gamma\sigma^{-1}$. Notice the reappearance of the twisted Dirac matrices γ_μ, γ^*_μ of (4.13) in the above expressions. In the above expressions, the operators λ^{-H_j} are 4×4 matrices obtained from the spin representation (6.20) of H_1 and H_2 and are given explicitily by

$$\lambda^{-H_1} = \begin{pmatrix} \bar{\mu} & & & \\ & \mu & & \\ & & \mu & \\ & & & \bar{\mu} \end{pmatrix}, \qquad \lambda^{-H_2} = \begin{pmatrix} \mu & & & \\ & \bar{\mu} & & \\ & & \mu & \\ & & & \bar{\mu} \end{pmatrix}, \qquad \mu = \sqrt{\lambda}. \tag{7.15}$$

As for $so(5)$, the action of $so(5,1)$ on the matrix Ψ is found to be by left matrix multiplication by Γ and on Ψ^* by $\widetilde{\Gamma}$,

$$\Psi_{ai} \mapsto \sum\nolimits_b \Gamma_{ab}\Psi_{bi}, \qquad \Psi^*_{ia} \mapsto \sum\nolimits_b \widetilde{\Gamma}_{ab}\Psi^*_{ib}. \tag{7.16}$$

Here we have to be careful with the ordering between $\widetilde{\Gamma}$ and Ψ^* in the second term since the $\widetilde{\Gamma}$'s involve the (non-central) z's. There are the following useful commutation relations between the z_μ's and Ψ:

$$\begin{aligned} z_1\Psi_{ai} &= (\lambda^{-H_2})_{ab}\Psi_{bi}z_1, \qquad & z_2\Psi_{ai} &= (\lambda^{-H_1})_{ab}\Psi_{bi}z_2, \\ z_1\Psi^*_{ia} &= \Psi^*_{ib}(\lambda^{-H_2})_{ba}z_1, \qquad & z_2\Psi^*_{ia} &= \Psi^*_{ib}(\lambda^{-H_1})_{ba}z_2, \end{aligned} \tag{7.17}$$

with λ^{-H_j} understood as the explicit matrices (7.15).

Proposition 38. *The instanton gauge potential* $\omega = \Psi^* \mathrm{d}\Psi$ *transforms under the action of the Hopf algebra* $\mathrm{U}_\theta(so(5,1))$ *as* $\omega \mapsto \omega + \delta\omega_i$, *where*

$$\delta\omega_0 := H_0(\omega) = -z_0\omega - \frac{1}{2} dz_0 \mathbb{I}_2 + \Psi^* \ \gamma_0 \ \mathrm{d}\Psi,$$

$$\delta\omega_1 := G_{+1,0}(\omega) = -z_1\omega - \frac{1}{2} dz_1 \ \mathbb{I}_2 + \Psi^* \ \gamma_1 \ \mathrm{d}\Psi,$$

$$\delta\omega_2 := G_{0,+1}(\omega) = -z_2\omega - \frac{1}{2} dz_2 \ \mathbb{I}_2 + \Psi^* \ \gamma_2 \ \mathrm{d}\Psi,$$

$$\delta\omega_3 := G_{-1,0}(\omega) = -\omega\overline{z}_1 - \frac{1}{2} d\overline{z}_1 \ \mathbb{I}_2 + \Psi^* \ \gamma_1^* \ \mathrm{d}\Psi,$$

$$\delta\omega_4 := G_{0,-1}(\omega) = -\omega\overline{z}_2 - \frac{1}{2} d\overline{z}_2 \ \mathbb{I}_2 + \Psi^* \ \gamma_2^* \ \mathrm{d}\Psi,$$

with γ_μ, γ_μ^* *the twisted* 4×4 *Dirac matrices defined in* (4.13).

Proof. The action of H_0 on the instanton gauge potential $\omega = \Psi^* \mathrm{d}\Psi$ takes the form

$$H_0(\omega) = H_0(\Psi^*)\mathrm{d}\Psi + \Psi^* \mathrm{d}(H_0(\Psi)) = \Psi^*(-z_0\mathbb{I}_4 + \gamma_0)\mathrm{d}\Psi - \frac{1}{2} dz_0 \Psi^*\Psi,$$

since z_0 is central. Direct computation results in the above expression for $\delta\omega_0$. Instead, the twisted action of G_r on ω takes the form,

$$G_r : \omega_{ij} \mapsto \sum_{a,b,c} \widetilde{\Gamma}_{ab} \Psi_{ib}^* (\lambda^{-r_1 H_2})_{ac} \mathrm{d}\Psi_{cj} + (\lambda^{r_2 H_1})_{ab} \Psi_{ib}^* \Gamma_{ac} \mathrm{d}\Psi_{cj}$$
$$+ (\lambda^{r_2 H_1})_{ab} \Psi_{ib}^* (\mathrm{d}\Gamma_{ac}) \Psi_{cj},$$

where we used the fact that $\widetilde{H}_j = \sigma H_j \sigma^{-1} = -H_j$. Let us consider the case $r = (+1, 0)$. Firstly, note that the complex numbers $(\lambda^{-H_2})_{ac}$ commute with Ψ_{ib}^* so that from the definition of Γ and $\widetilde{\Gamma}$, and using (7.17), we obtain for the first two terms,

$$-z_1(\Psi^* \mathrm{d}\Psi)_{ij} + \frac{1}{2} \Psi_{ib}^* (\sigma\gamma_1\sigma^{-1})_{cb} (\lambda^{-H_2})_{cd} \mathrm{d}\Psi_{dj} + \frac{1}{2} \Psi_{ib}^* (\gamma_1)_{bc} \mathrm{d}\Psi_{cj}.$$

The first term forms the matrix $-z_1\omega$ whereas the second two terms combine to give $\Psi^*\gamma_1 \mathrm{d}\Psi$, due to relation (6.23). Finally, using Eq. (7.17) the term $\Psi_{ib}^*(\mathrm{d}\Gamma_{ac})\Psi_{cj}$ reduces to $-\frac{1}{2}\mathrm{d}z_1 \Psi_{ib}^* \Psi_{bj} = -\frac{1}{2}\mathrm{d}z_1 \mathbb{I}_2$. The formulae for $r = (-1, 0)$ and $r = (0, \pm 1)$ are established in a similar manner. □

The transformations in Proposition 38 of the gauge potential ω under the twisted symmetry $\mathrm{U}_\theta(so(5,1))$ induce natural transformations of the canonical connection ∇_0 in (7.1) to $\nabla_{t,i} := \nabla_0 + t\delta\omega_i + \mathcal{O}(t^2)$. We shall presently see explicitly that these new connections are (infinitesimal) instantons, i.e. their curvatures are self-dual. In fact, this also follows from Lemma 37 which states that $\mathrm{U}_\theta(so(5,1))$ acts by conformal transformation, thus leaving the self-duality equation $*_\theta F_0 = F_0$ for the basic instanton ∇_0 invariant.

We start by writing $\nabla_{t,i}$ in terms of the canonical connection on $\mathcal{E} \simeq p\big(\mathcal{A}(S^4_\theta)\big)^4$. Using the explicit isomorphism of Proposition 17, between this module and the module of equivariant maps $\mathcal{A}(S^7_{\theta'}) \boxtimes_\rho \mathbb{C}^2$, we find that $\nabla_{t,i} = p\mathrm{d} + t\delta\alpha_i + \mathcal{O}(t^2)$, with explicit expressions

$$\begin{aligned}
&\delta\alpha_0 = p\gamma_0(\mathrm{d}p)p - \frac{1}{2}\Psi\mathrm{d}z_0\Psi^*, \\
&\delta\alpha_1 = p\gamma_1(\mathrm{d}p)p - \frac{1}{2}\Psi\mathrm{d}z_1\Psi^*, \qquad \delta\alpha_3 = p\gamma_1^*(\mathrm{d}p)p - \frac{1}{2}\Psi\mathrm{d}z_1^*\Psi^*, \\
&\delta\alpha_2 = p\gamma_2(\mathrm{d}p)p - \frac{1}{2}\Psi\mathrm{d}z_2\Psi^*, \qquad \delta\alpha_4 = p\gamma_2^*(\mathrm{d}p)p - \frac{1}{2}\Psi\mathrm{d}z_2^*\Psi^*.
\end{aligned} \tag{7.18}$$

The $\delta\alpha_i' s$ are 4×4 matrices with entries in the one-forms $\Omega^1(S^4_\theta)$ and satisfying the relations $p\delta\alpha_i = \delta\alpha_i p = p\delta\alpha_i p = \delta\alpha_i$, as expected from the general theory of connections on modules in Sec. 2.2). Indeed, using relations (7.17) one can move the $\mathrm{d}z$'s to the left of Ψ at the cost of some μ's, so getting expressions like $\mathrm{d}z_i\, p \in M_4(\Omega^1(S^4_\theta))$.

From Eq. (2.10), the curvature $F_{t,i}$ of the connection $\nabla_{t,i}$ is given by

$$F_{t,i} = F_0 + tp\mathrm{d}(\delta\alpha_i) + \mathcal{O}(t^2). \tag{7.19}$$

In order to check self-duality (modulo t^2) of this curvature, we express it in terms of the projection p and consider $F_{t,i}$ as a two-form valued endomorphism on $\mathcal{E} \simeq p\big(\mathcal{A}(S^4_\theta)\big)^4$.

Proposition 39. *The curvatures $F_{t,i}$ of the connections $\nabla_{t,i}$, $i = 0,\ldots,4$, are given by $F_{t,i} = F_0 + t\delta F_i + \mathcal{O}(t^2)$, where $F_0 = p\mathrm{d}p\mathrm{d}p$ and*

$$\begin{aligned}
&\delta F_0 = -2z_0F_0, \\
&\delta F_1 = -2z_1\lambda^{H_2}F_0, \qquad \delta F_3 = -2z_1^*\lambda^{-H_2}F_0, \\
&\delta F_2 = -2z_2\lambda^{H_1}F_0, \qquad \delta F_4 = -2z_2^*\lambda^{-H_1}F_0.
\end{aligned} \tag{7.20}$$

Proof. A small computation yields for $\delta F_i = p\mathrm{d}(\delta\alpha_i)$, thought of as an $\Omega^2(S^4_\theta)$-valued endomorphism on $\mathcal{E}$, the expression $\delta F_i = p(\mathrm{d}p)\gamma_i(\mathrm{d}p)p - p\gamma_i(\mathrm{d}p)(\mathrm{d}p)p$, with the notation $\gamma_3 = \gamma_1^*$ and $\gamma_4 = \gamma_2^*$, and using

$p(\mathrm{d}p)p = 0$. Then, the identity $p((\mathrm{d}p)\gamma_i + \gamma_i \mathrm{d}p)(\mathrm{d}p)p = 0$ for $i = 0, \ldots, 4$, yields $\delta F_i = -2p\gamma_i(\mathrm{d}p)(\mathrm{d}p)p$. This is expressed as $\delta F_i = -2p\gamma_i p(\mathrm{d}p)(\mathrm{d}p)$ by using $\mathrm{d}p = (\mathrm{d}p)p + p\mathrm{d}p$. Finally, $p\gamma_i p = \Psi(\Psi^*\gamma_i\Psi)\Psi^*$, so that the result follows from the definition of the z's in terms of the Dirac matrices given in Eq. (4.12), together with the commutation relations between them and the matrix Ψ in Eq. (7.17). □

Proposition 40. *The connections $\nabla_{t,i}$ are (infinitesimal) instantons, i.e.*

$$*_\theta F_{t,i} = F_{t,i} \bmod t^2. \tag{7.21}$$

Moreover, the connections $\nabla_{t,i}$ are not gauge equivalent to ∇_0.

Proof. The first point follows directly from the above expressions for $\delta_i F$ and the self-duality of F_0. To establish the gauge inequivalence of the connections $\nabla_{t,i}$ with ∇_0, we recall that an infinitesimal gauge transformation is given by $\nabla_0 \mapsto \nabla_0 + t[\nabla_0, X]$ for $X \in \Gamma^\infty(\mathrm{ad}(S^7_{\theta'}))$. We need to show that $\delta_i\omega$ is orthogonal to $[\nabla_0, X]$ for any such X, i.e.

$$([\nabla_0, X], \delta_i\omega)_2 = 0,$$

with the natural inner product on $\Omega^1(\mathrm{ad}(S^7_{\theta'})) := \Omega^1(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \Gamma^\infty(\mathrm{ad}(S^7_{\theta'}))$. From Remark 35, it follows that

$$(\nabla_0^{(2)}(X), \delta_i\omega)_2 = \left(X, \left(\nabla_0^{(2)}\right)^* (\delta_i\omega)\right)_2,$$

which then should vanish for all X. From equation (7.11), we see that $\delta_i\omega = T_i(\omega)$ coincides with $L_\theta(t_i \cdot \omega^{(0)})$ with t_i and $\omega^{(0)}$ the classical counterparts of T_i and ω, respectively. In the undeformed case, the infinitesimal gauge potentials generated by acting with elements in $so(5,1) - so(5)$ on the basic instanton gauge potential $\omega^{(0)}$ satisfy $(\nabla_0^{(2)})^*(\delta_i\omega^{(0)}) = 0$, as shown in [8]. The result then follows from the observation that $\nabla_0^{(2)}$ commutes with the quantization map L_θ (cf. Remark 35). □

7.4. *Instantons on $\mathbb{R}^4_\theta$*

In this section, we obtain "local expressions" for the instantons on S^4_θ constructed in the previous section; that is we map them to a noncommutative $\mathbb{R}^4_\theta$ obtained by "removing a point" from S^4_θ. With $\lambda = e^{2\pi i\theta}$ as above, the algebra $\mathcal{A}(\mathbb{R}^4_\theta)$ of polynomial functions on the 4-plane $\mathbb{R}^4_\theta$ is defined to be the $*$-algebra generated by ζ_1, ζ_2 satisfying

$$\zeta_1\zeta_2 = \lambda\zeta_2\zeta_1, \qquad \zeta_1\zeta_2^* = \overline{\lambda}\zeta_2^*\zeta_1. \tag{7.22}$$

At $\theta = 0$ one recovers the $*$-algebra of polynomial functions on the usual 4-plane $\mathbb{R}^4$.

The algebra $\mathcal{A}(\mathbb{R}^4_\theta)$ can also be defined as the vector space $\mathcal{A}(\mathbb{R}^4)$ equipped with a deformed product $\times_\theta$ as in equation (3.2). Indeed, the torus $\mathbb{T}^2$ acts naturally on the two complex coordinates of $\mathbb{R}^4 \simeq \mathbb{C}^2$. This also allows us to define the smooth algebra $C^\infty_b(\mathbb{R}^4_\theta)$ as the vector space $C^\infty_b(\mathbb{R}^4)$ of bounded smooth functions on $\mathbb{R}^4$ equipped with a deformed product $\times_\theta$. However, for our purposes it suffices to consider the polynomial algebra $\mathcal{A}(\mathbb{R}^4_\theta)$ with one self-adjoint central generator ρ added together with relations $\rho^2(1+|\zeta|^2) = (1+|\zeta|^2)\rho^2 = 1$ where $|\zeta|^2 := \zeta_1^*\zeta_1 + \zeta_2^*\zeta_2$ (this enlargement was already done in [22]). In the following, we will denote the enlarged algebra by $\widetilde{\mathcal{A}}(\mathbb{R}^4_\theta)$ and will also use the notation

$$\rho^2 = (1+|\zeta|^2)^{-1} = \frac{1}{1+|\zeta|^2}. \tag{7.23}$$

Note that ρ^2 is an element in $C^\infty_b(\mathbb{R}^4_\theta)$. One defines elements $\widetilde{z}_\mu, \mu = 0,1,2$ in $\widetilde{\mathcal{A}}(\mathbb{R}^4_\theta)$ by

$$\widetilde{z}_0 = (1-|\zeta|^2)(1+|\zeta|^2)^{-1}, \qquad \widetilde{z}_j = 2\zeta_j(1+|\zeta|^2)^{-1} \quad j = 1,2, \tag{7.24}$$

and checks that they satisfy the same relations as in (4.1) of the generators z_μ of $\mathcal{A}(S^4_\theta)$. The difference is that the classical point $z_0 = -1, z_j = z_j^* = 0$ of S^4_θ is not in the spectrum of $\widetilde{z}_\mu$. We interpret the noncommutative plane $\mathbb{R}^4_\theta$ as a "chart" of the noncommutative 4-sphere S^4_θ and Eq. (7.24) as the (inverse) stereographic projection from S^4_θ to $\mathbb{R}^4_\theta$. In fact, one can cover S^4_θ by two such charts with domain $\mathbb{R}^4_\theta$, and transition functions on $\mathbb{R}^4_\theta\backslash\{0\}$, where $\{0\}$ is the classical point $\zeta_j = \zeta_j^* = 0$ of $\mathbb{R}^4_\theta$ (cf. [22] for more details).

A differential calculus $(\Omega(\mathbb{R}^4_\theta), \mathrm{d})$ on $\mathbb{R}^4_\theta$ is obtained from the general procedure described in Sec. 3. Explicitly, $\Omega(\mathbb{R}^4_\theta)$ is the graded $*$-algebra generated by the elements ζ_μ of degree 0 and $d\zeta_\mu$ of degree 1 with relations,

$$\begin{aligned} d\zeta_\mu d\zeta_\nu + \lambda_{\mu\nu} d\zeta_\nu d\zeta_\mu = 0, &\qquad d\zeta_\mu^* d\zeta_\nu + \lambda_{\nu\mu} d\zeta_\nu d\zeta_\mu^* = 0, \\ \zeta_\mu d\zeta_\nu = \lambda_{\mu\nu} d\zeta_\nu \zeta_\mu, &\qquad \zeta_\mu^* d\zeta_\nu = \lambda_{\nu\mu} d\zeta_\nu \zeta_\mu^*, \end{aligned} \tag{7.25}$$

and $\lambda_{12} = \overline{\lambda}_{21} =: \lambda$. There is a unique differential d on $\Omega(\mathbb{R}^4_\theta)$ such that $\mathrm{d} : \zeta_\mu \mapsto d\zeta_\mu$ and a Hodge star operator $*_\theta : \Omega^p(\mathbb{R}^4_\theta) \to \Omega^{4-p}(\mathbb{R}^4_\theta)$, obtained from the classical Hodge star operator. In terms of the standard Riemannian metric on $\mathbb{R}^4$, on two-forms we have,

$$*_\theta d\zeta_1 d\zeta_2 = -d\zeta_1 d\zeta_2, \qquad *_\theta d\zeta_1 d\zeta_1^* = -d\zeta_2 d\zeta_2^*, \qquad *_\theta d\zeta_1 d\zeta_2^* = d\zeta_1 d\zeta_2^*, \tag{7.26}$$

and $*_\theta^2 = \mathrm{id}$. These are the same formulae as the ones for the undeformed Hodge $*$ on $\mathbb{R}^4$ – since the metric is not changed in an isospectral deformation.

Again, we slightly enlarge the differential calculus $\Omega(\mathbb{R}^4_\theta)$ by adding the self-adjoint central generator ρ. The differential d on ρ is derived from the Leibniz rule for d applied to its defining relation,

$$(\mathrm{d}\rho^2)(1+|\zeta|^2) + \rho^2 \mathrm{d}(1+|\zeta|^2) = \mathrm{d}\left(\rho^2(1+|\zeta|^2)\right) = 0,$$

so that $\rho \mathrm{d}\rho = \frac{1}{2}\mathrm{d}\rho^2 = -\frac{1}{2}\rho^4 \mathrm{d}(1+|\zeta|^2) = -\frac{1}{2}\rho^4 \sum_\mu (\zeta_\mu d\zeta_\mu^* + \zeta_\mu^* d\zeta_\mu)$. The enlarged differential calculus will be denoted by $\widetilde{\Omega}(\mathbb{R}^4_\theta)$.

The stereographic projection from S^4 onto $\mathbb{R}^4$ is a conformal map commuting with the action of $\mathbb{T}^2$; thus it makes sense to investigate the form of the instanton connections on S^4_θ obtained in Proposition 38 on the local chart $\mathbb{R}^4_\theta$. As in [44], we first introduce a "local section" of the principal bundle $S^7_{\theta'} \to S^4_\theta$ on the local chart of S^4_θ defined in (7.24). Let $u = (u_1, u_2)$ be a complex spinor of modulus one, $u_1^* u_1 + u_2^* u_2 = 1$, and define

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \rho \begin{pmatrix}u_1\\ u_2\end{pmatrix}, \qquad \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} = \rho \begin{pmatrix}\zeta_1^* & \zeta_2^*\\ -\mu\zeta_2 & \overline{\mu}\zeta_1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}. \tag{7.27}$$

Here ρ is a central element in $C^\infty(\mathbb{R}^4_\theta)$ such that $\rho^2 = (1+|\zeta|^2)^{-1}$ and the commutation rules of the u_j's with the ζ_k's are dictated by those of the ψ_j,

$$u_1\zeta_j = \mu\zeta_j u_1 \quad u_2\zeta_j = \overline{\mu}\zeta_j u_2\,, \quad j = 1,2. \tag{7.28}$$

The right action of SU(2) rotates the vector u while mapping to the "same point" of S^4_θ, which, using the Definition (4.10), from the choice in (7.27) is found to be

$$2(\psi_1\psi_3^* + \psi_2^*\psi_4) = \widetilde{z}_1, \quad 2(-\psi_1^*\psi_4 + \psi_2\psi_3^*) = \widetilde{z}_2, \quad 2(\psi_1^*\psi_1 + \psi_2^*\psi_2) - 1 = \widetilde{z}_0, \tag{7.29}$$

and which is in the local chart (7.24), as expected.

Remark 41. Strictly speaking, the symbols ψ_a here denotes elements in the algebra $\mathcal{A}(S^7_{\theta'})$ enlarged by an extra generator which is the inverse of $1 + z_0 = 2(1 + \psi_1^*\psi_1 + \psi_2^*\psi_2)$. Intuitively, this corresponds to "removing" the fiber S^3 in the sphere $S^7_{\theta'}$ above the classical point $z_0 = -1, z_j = z_j^* = 0$ of the base space S^4_θ.

By writing the unit vector u as an SU(2) matrix, $u = \begin{pmatrix}u_1 & -u_2^*\\ u_2 & u_1^*\end{pmatrix}$, we have

$$\Psi = \rho \begin{pmatrix}\mathbb{I}_2 & 0\\ 0 & \mathcal{Z}\end{pmatrix}\begin{pmatrix}u\\ u\end{pmatrix}, \qquad \text{with } \mathcal{Z} = \begin{pmatrix}\zeta_1^* & \zeta_2^*\\ -\mu\zeta_2 & \overline{\mu}\zeta_1\end{pmatrix}. \tag{7.30}$$

Then, by direct computation the gauge potential $\omega = \Psi^* d\Psi$ takes the form

$$
\begin{aligned}
u\omega u^* &= \rho^{-1}\mathrm{d}\rho + \rho^2 \mathcal{Z}^* \mathrm{d}\mathcal{Z} + (\mathrm{d}u)u^* \\
&= \frac{1}{(1+|\zeta|^2)} \begin{pmatrix} \sum_i \zeta_i d\zeta_i^* - d\zeta_i \zeta_i^* & 2(\zeta_1 d\zeta_2^* - d\zeta_1 \zeta_2^*) \\ 2(\zeta_2 d\zeta_1^* - d\zeta_2 \zeta_1^*) & \sum_i \zeta_i^* d\zeta_i - d\zeta_i^* \zeta_i \end{pmatrix} + (\mathrm{d}u)u^*,
\end{aligned}
\tag{7.31}
$$

while its curvature $F = \mathrm{d}\omega + \omega^2$ is,

$$
uFu^* = \rho^4 \mathrm{d}\mathcal{Z}^* \mathrm{d}\mathcal{Z} = \frac{1}{(1+|\zeta|^2)^2} \begin{pmatrix} d\zeta_1 d\zeta_1^* - d\zeta_2 d\zeta_2^* & 2d\zeta_1 d\zeta_2^* \\ 2d\zeta_2 d\zeta_1^* & -d\zeta_1 d\zeta_1^* - d\zeta_2 d\zeta_2^* \end{pmatrix}.
\tag{7.32}
$$

From (7.26) one checks that this curvature is self-dual: $*_\theta(uFu^*) = uFu^*$, as expected.

The explicit local expressions for the transformed – under infinitesimal conformal transformations – gauge potentials and their curvature can be obtained in a similar manner. As an example, let us work out the local expression for $\delta_0\omega$, which is the most transparent one. Given the expression for $\delta_0\omega$ in Proposition 38, a direct computation shows that its local counterpart is

$$
u\delta_0\omega u^* = -2\rho\mathrm{d}\rho - 2\rho^4 \mathcal{Z}^* \mathrm{d}\mathcal{Z},
\tag{7.33}
$$

giving for the transformed curvature,

$$
uF_{t,0}u^* = F_0 + 2t(1-2\rho^2)F_0 + \mathcal{O}(t^2).
\tag{7.34}
$$

It is clear that this rescaled curvature still satisfies the self-duality equation; this is also in concordance with Proposition 39, being $\widetilde{z}_0 = 2\rho^2 - 1$.

7.5. *Moduli space of instantons*

We will closely follow the infinitesimal construction in [8] of instantons for the undeformed case. This will eventually result in the computation of the dimension of the "tangent space" to the moduli space of instantons on S^4_θ by index methods. It will turn out that the five-parameter family of instantons constructed in the previous section is indeed the complete set of infinitesimal instantons on S^4_θ.

Let us start by considering the following family of connections on S^4_θ,

$$
\nabla_t = \nabla_0 + t\alpha
\tag{7.35}
$$

where $\alpha \in \Omega^1(\mathrm{ad}(S^7_{\theta'})) \equiv \Omega^1(S^4_\theta) \otimes_{C^\infty(S^4_\theta)} \Gamma^\infty(\mathrm{ad}(S^7_{\theta'}))$ and after Example 22 we have denoted $\Gamma^\infty(\mathrm{ad}(S^7_{\theta'})) = C^\infty(S^7_{\theta'}) \otimes_{\mathrm{ad}} su(2)$. For ∇_t to be an instanton, we have to impose the self-duality equation $*_\theta F_t = F_t$ on the curvature $F_t = F_0^2 + t[\nabla, \alpha] + \mathcal{O}(t^2)$ of ∇_t. This leads, when differentiated with respect to t, at $t = 0$, to *linearized self-duality equations*

$$P_-[\nabla_0, \alpha] = 0, \tag{7.36}$$

with $P_- := \frac{1}{2}(1 - *_\theta)$ the projection onto the anti-self-dual 2-forms. Here $[\nabla_0, \alpha]$ is an element in $\Omega^2(\Gamma^\infty(\mathrm{ad}(S^7_{\theta'})))$,

$$[\nabla_0, \alpha]_{ij} = \mathrm{d}\alpha_{ij} + \omega_{ik}\alpha_{kj} - \alpha_{ik}\omega_{kj}, \tag{7.37}$$

and has vanishing trace, due to the fact that $\omega_{ik}\alpha_{kj} = \alpha_{kj}\omega_{ik}$ (cf. Eqs. (7.1), (7.2) and the related discussion).

If the family were obtained from an infinitesimal gauge transformation, we would have had $\alpha = [\nabla_0, X]$, for some $X \in \Gamma^\infty(\mathrm{ad}(S^7_{\theta'}))$. Indeed, $[\nabla_0, X]$ is an element in $\Omega^1(\mathrm{ad}(S^7_{\theta'}))$ and $P_-[\nabla_0, [\nabla_0, X]] = [P_- F_0, X] = 0$, since F_0 is self-dual. Hence, we have defined an element in the first cohomology group H^1 of the so-called *self-dual complex*:

$$0 \to \Omega^0(\mathrm{ad}(S^7_{\theta'})) \xrightarrow{\mathrm{d}_0} \Omega^1(\mathrm{ad}(S^7_{\theta'})) \xrightarrow{\mathrm{d}_1} \Omega^2_-(\mathrm{ad}(S^7_{\theta'})) \to 0, \tag{7.38}$$

with $\Omega^0(\mathrm{ad}(S^7_{\theta'})) = \Gamma^\infty(\mathrm{ad}(S^7_{\theta'}))$ and $\mathrm{d}_0 = [\nabla_0, \cdot\,]$, $\mathrm{d}_1 := P_-[\nabla_0, \cdot\,]$. Note that these operators are Fredholm operators, so that the cohomology groups of the complex are finite dimensional. As usual, the complex can be replaced by a single Fredholm operator

$$\mathrm{d}_0^* + \mathrm{d}_1 : \Omega^1(\mathrm{ad}(S^7_{\theta'})) \to \Omega^0(\mathrm{ad}(S^7_{\theta'})) \oplus \Omega^2_-(\mathrm{ad}(S^7_{\theta'})), \tag{7.39}$$

where d_0^* is the adjoint of d_0 with respect to the inner product (3.22).

Our goal is to compute $h^1 = \dim H^1$ – the number of "true" instantons. This is achieved by calculating the alternating sum $-h^0 + h^1 - h^2$ of Betti numbers from the index of the above Fredholm operator,

$$\mathrm{index}(d_0^* + d_1) = -h^0 + h^1 - h^2, \tag{7.40}$$

while showing that $h^0 = h^2 = 0$. By definition, H^0 consists of the covariantly constant elements in $\Gamma^\infty(\mathrm{ad}(S^7_{\theta'}))$. Since $[\nabla_0, \cdot]$ commutes with the action of $\mathbb{T}^2$ and coincides with $\nabla_0^{(2)}$ on $\Gamma^\infty(\mathrm{ad}(S^7_{\theta'}))$ (cf. Remark 35), being covariantly constant means

$$[\nabla_0, X] = \nabla_0^{(2)}(X) = 0. \tag{7.41}$$

If we write once more $X = L_\theta(X^{(0)})$ in terms of its classical counterpart, we find that this condition entails

$$\nabla_0^{(2)}(L_\theta(X^{(0)})) = L_\theta(\nabla_0^{(2)}(X^{(0)})) = 0, \tag{7.42}$$

since $\nabla_0^{(2)}$ commutes with L_θ (cf. Remark 35). Given that for the undeformed case there are no covariantly constant elements in $\Gamma^\infty(\mathrm{ad}(S^7))$ for an irreducible self-dual connection on $\mathcal{E}$, we conclude that $h^0 = 0$. A completely analogous argument for the kernel of the operator d_1^* shows that also $h^2 = 0$.

7.6. *Dirac operator associated to the complex*

The next step consists of computing the index of the Fredholm operator $\mathrm{d}_0^* + \mathrm{d}_1$ defined in (7.39). Firstly, this operator can be replaced by a Dirac operator on the spinor bundle $\mathcal{S}$ with coefficients in the adjoint bundle. For this, we need the following lemma, which is a straightforward modification of its classical analogue [8]. Recall that the $\mathbb{Z}^2$-grading γ_5 induces a decomposition of the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. Note also that $\mathcal{S}^-$ coincides classically with the charge -1 anti-instanton bundle. Indeed, the Levi–Civita connection – when lifted to the spinor bundle and restricted to negative chirality spinors – has anti-self-dual curvature. Similarly, $\mathcal{S}^+$ coincides with the charge $+1$ instanton bundle. Then Remark 18 implies that the $C^\infty(S^4)$-modules $\Gamma^\infty(S^4, \mathcal{S}^\pm)$ have a module-basis that is homogeneous under the action of $\widetilde{\mathbb{T}}^2$. We conclude from $\widetilde{\mathbb{T}}^2$-equivariance that $\Gamma^\infty(S^4_\theta, \mathcal{S}^-)$ is isomorphic to the charge -1 anti-instanton bundle $\Gamma^\infty(S^7_{\theta'} \times_{\mathrm{SU}(2)} \mathbb{C}^2)$ on S^4_θ. Similarly $\Gamma^\infty(S^4_\theta, \mathcal{S}^+)$ is isomorphic to the charge $+1$ instanton bundle.

Lemma 42. *There are the following isomorphisms of right $C^\infty(S^4_\theta)$-modules,*

$$\Omega^1(S^4_\theta) \simeq \Gamma^\infty(S^4_\theta, \mathcal{S}^+ \otimes \mathcal{S}^-) \simeq \Gamma^\infty(S^4_\theta, \mathcal{S}^+) \otimes_{C^\infty(S^4_\theta)} \Gamma^\infty(S^4_\theta, \mathcal{S}^-),$$
$$\Omega^0(S^4_\theta) \oplus \Omega^2_-(S^4_\theta) \simeq \Gamma^\infty(S^4_\theta, \mathcal{S}^- \otimes \mathcal{S}^-) \simeq \Gamma^\infty(S^4_\theta, \mathcal{S}^-) \otimes_{C^\infty(S^4_\theta)} \Gamma^\infty(S^4_\theta, \mathcal{S}^-).$$

Proof. Since classically $\Omega^1(S^4) \simeq \Gamma^\infty(S^4, \mathcal{S}^+ \otimes \mathcal{S}^-)$ as σ-equivariant $C^\infty(S^4)$-bimodules, Lemma 10 shows that $\Omega^1(S^4_\theta) \simeq \Gamma^\infty(S^4_\theta, \mathcal{S}^+ \otimes \mathcal{S}^-)$ as $C^\infty(S^4_\theta)$-bimodules. The observations above the Lemma indicate that $\mathcal{S}^\pm \simeq S^7 \times_{\rho^\pm} \mathbb{C}^2$ for the spinor representation $\rho^+ \oplus \rho^-$ of $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{C}^4$, so that, using Proposition 19,

$$\begin{aligned}\Gamma^\infty(S^4_\theta, \mathcal{S}^+ \otimes \mathcal{S}^-) &\simeq C^\infty(S^7_{\theta'}) \boxtimes_{\rho^+ \otimes \rho^-} (\mathbb{C}^2 \otimes \mathbb{C}^2) \\ &\simeq \left(C^\infty(S^7_{\theta'}) \boxtimes_{\rho^+} \mathbb{C}^2\right) \otimes_{C^\infty(S^4_\theta)} \left(C^\infty(S^7_{\theta'}) \boxtimes_{\rho^-} \mathbb{C}^2\right).\end{aligned}$$

This proves our claim. An analogous statement holds for the second isomorphism. □

Let us forget for the moment the adjoint bundle $\mathrm{ad}(S^7_{\theta'})$. Since $\Omega(S^4_\theta) \simeq \Omega(S^4)$ as vector spaces and both d and the Hodge $*$ commute with the action of $\mathbb{T}^2$, the operator $\mathrm{d}^* + P_-\mathrm{d}$ can be understood as a map from $\Omega^1(S^4) \to \Omega^0(S^4) \oplus \Omega^2_-(S^4)$ (see Sec. 3.4). Under the isomorphisms of the above Lemma, this operator is replaced [8] by a Dirac operator with coefficients in $\mathcal{S}^-$,

$$D' : \Gamma^\infty(S^4_\theta, \mathcal{S}^+ \otimes \mathcal{S}^-) \to \Gamma^\infty(S^4_\theta, \mathcal{S}^- \otimes \mathcal{S}^-). \tag{7.43}$$

Twisting by the adjoint bundle $\mathrm{ad}(S^7_{\theta'})$ merely results into a composition with the projection $p_{(2)}$ defining the bundle $\mathrm{ad}(S^7_{\theta'})$. Hence, eventually the operator $\mathrm{d}_0^* + \mathrm{d}_1$ is replaced by the Dirac operator with coefficients in the vector bundle $\mathcal{S}^- \otimes \mathrm{ad}(S^7_{\theta'})$ on S^4_θ:

$$\mathcal{D} : \Gamma^\infty(S^4_\theta, \mathcal{S}^+ \otimes \mathcal{S}^- \otimes \mathrm{ad}(S^7_{\theta'})) \to \Gamma^\infty(S^4_\theta, \mathcal{S}^- \otimes \mathcal{S}^- \otimes \mathrm{ad}(S^7_{\theta'})). \tag{7.44}$$

We have finally arrived at the computation of the index of this Dirac operator, which we do by means of the Connes–Moscovoci local index formula. It is given by the pairing

$$\mathrm{index}(\mathcal{D}) = \langle \phi, \mathrm{ch}(\mathcal{S}^- \otimes \mathrm{ad}(S^7_{\theta'}))\rangle = \langle \phi, \mathrm{ch}(\mathcal{S}^-) \cdot \mathrm{ch}(\mathrm{ad}(S^7_{\theta'}))\rangle. \tag{7.45}$$

Both the cyclic cocycle ϕ^* and the Chern characters, as well as their realization as operators $\pi_D(\mathrm{ch}(\mathcal{E})$ on the the Hilbert space $\mathcal{H}$, are recalled in Sec. 3.5. In [47] we computed these operators for all modules associated to the principal bundle $S^7_{\theta'} \to S^4_\theta$. In particular, for the adjoint bundle we found that

$$\pi_D\big(\mathrm{ch}_0(\mathrm{ad}(S^7_{\theta'}))\big) = 3, \quad \pi_D\big(\mathrm{ch}_1(\mathrm{ad}(S^7_{\theta'}))\big) = 0, \quad \pi_D\big(\mathrm{ch}_2(\mathrm{ad}(S^7_{\theta'}))\big) = 4(3\gamma_5).$$

To compute the Chern character of the spinor bundle $\mathcal{S}^-$ we use its mentioned identification with the charge -1 instanton bundle $\Gamma^\infty(S^7_{\theta'} \times_{\mathrm{SU}(2)} \mathbb{C}^2)$ on S^4_θ. It then follows from [23] (cf. also [47]) that

$$\pi_D\big(\mathrm{ch}_0(\mathcal{S}^-)\big) = 2, \qquad \pi_D\big(\mathrm{ch}_1(\mathcal{S}^-)\big) = 0, \qquad \pi_D\big(\mathrm{ch}_2(\mathcal{S}^-)\big) = -3\gamma_5.$$

Combining the two Chern characters and using the local index formula on S^4_θ, we have

$$\mathrm{index}(\mathcal{D}) = 6 \operatorname*{Res}_{z=0} z^{-1} \operatorname{tr}(\gamma_5 |D|^{-2z}) + 0 + \frac{1}{2}(2 \cdot 4 - 3 \cdot 1) \operatorname*{Res}_{z=0} \operatorname{tr}(3\gamma_5^2 |D|^{-4-2z}), \tag{7.46}$$

with D identified with the classical Dirac operator $\not{D}$ on S^4 (recall that we do not change it in the isospectral deformation). Now, the first term vanishes due to $\mathrm{index}(D) = 0$ for this classical operator. On the other hand $\gamma_5^2 = \mathbb{I}_4$, and

$$3\operatorname*{Res}_{z=0} \mathrm{tr}(|D|^{-4-2z}) = 6\,\mathrm{Tr}_\omega(|D|^{-4}) = 2,$$

since the Dixmier trace of $|D|^{-m}$ on the m-sphere equals $8/m!$ (cf. for instance [37, 42]). We conclude that $\mathrm{index}(\mathcal{D}) = 5$ and for the moduli space of instantons on S^4_θ, we have:

Theorem 43. *The tangent space at the base point ∇_0 to the moduli space of* (*irreducible*) $\mathrm{SU}(2)$*-instantons on S^4_θ is five-dimensional.*

For the global geometry of the moduli space, it appears that one can construct a noncommutative family of instantons, that is, instantons parametrized by the quantum quotient space of the deformed conformal group $\mathrm{SL}_\theta(2, \mathbb{H})$ by the deformed gauge group $\mathrm{Sp}_\theta(2)$; this will be reported elsewhere [46].

8. Final Remarks

A different quantum version of the SU(2) Hopf bundle $S^7 \to S^4$ was constructed in [45]. The quantum sphere S^7_q arises from the symplectic group $\mathrm{Sp}_q(2)$ and a quantum 4-sphere S^4_q is obtained via a suitable self-adjoint idempotent p whose entries generate the algebra $A(S^4_q)$ of polynomial functions over it. This projection determines a deformation of an instanton bundle over the classical sphere S^4.

One starts with the symplectic quantum groups $A(\mathrm{Sp}_q(2))$, i.e. the Hopf algebras generated by matrix elements T^j_i's with commutation rules coming from the R matrix of the C-series [34]. The symplectic quantum 7-sphere $A(S^7_q)$ is generated by the matrix elements of the first and the last column of T. The algebra $A(S^7_q)$ is the quantum version of the homogeneous space $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ and the injection $A(S^7_q) \hookrightarrow A(\mathrm{Sp}_q(2))$ is a quantum principal bundle with "structure Hopf algebra" $A(\mathrm{Sp}_q(1))$, an example of the general construction of [15]. In turn, the sphere S^7_q is the total space of a quantum $\mathrm{SU}_q(2)$ principal bundle over a quantum 4-sphere S^4_q. As mentioned, the algebra $A(S^4_q)$ is constructed as the subalgebra of $A(S^7_q)$ generated by the matrix elements of a self-adjoint projection p which generalizes (at $q \neq 1$) the instanton of charge -1 (at $q = 1$). Unlike the construction for $A(S^7_q)$ out of $A(\mathrm{Sp}_q(2))$, one does not have a quantum homogeneous structure. Still, there is a natural coaction of $\mathrm{SU}_q(2)$ on $A(S^7_q)$ with coinvariant algebra $A(S^4_q)$ and the injection $A(S^4_q) \hookrightarrow A(S^7_q)$ turns out to be a faithfully flat $A(\mathrm{SU}_q(2))$ noncommutative principal bundle (a Hopf–Galois extension).

To compute the charge of the projection and to prove the non-triviality of this principal bundle, one follows the general strategy of noncommutative index theorems [20]. One constructs the representations of the algebra $A(S^4_q)$ and the corresponding K-homology. The analogue of the fundamental class of S^4 is given by a non-trivial Fredholm module μ. The natural coupling between μ and the projection p is computed via the pairing of the corresponding Chern characters $\mathrm{ch}^*(\mu) \in HC^*[A(S^4_q)]$ and $\mathrm{ch}_*(p) \in HC_*[A(S^4_q)]$ in cyclic cohomology and homology respectively. As expected, the result of this pairing, which is an integer being the index of a Fredholm operator, is actually -1 and therefore the bundle is non-trivial.

Clearly the next step would be to repeat the analysis of the toric four-sphere and define a Yang–Mills action functional and self-duality equations. To this end one needs a "metric structure" on the bundle; for this, the recently found [28] isospectral noncommutative geometry for $\mathrm{SU}_q(2)$ promises to be useful.

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References

[1] P. Aschieri and F. Bonechi, On the noncommutative geometry of twisted spheres, *Lett. Math. Phys.* **59** (2002) 133–156.

[2] P. Aschieri and L. Castellani, Bicovariant calculus on twisted $ISO(N)$, quantum Poincaré group and quantum Minkowski space, *Int. J. Mod. Phys. A* **11** (1996) 4513–4549.

[3] P. Aschieri and L. Castellani, Universal enveloping algebra and differential calculi on inhomogeneous orthogonal q-groups, *J. Geom. Phys.* **26** (1998) 247–271.

[4] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, A gravity theory on noncommutative spaces, *Class. Quant. Grav.* **22** (2005) 3511–3532.

[5] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, Noncommutative geometry and gravity, *Class. Quant. Grav.* **23** (2006) 1883–1912.

[6] M. F. Atiyah, *The Geometry of Yang–Mills Fields*, Fermi Lectures. Scuola Normale, Pisa, 1979.

[7] M. F. Atiyah, N. J. Hitchin, V. G. Drinfel′d and Y. I. Manin, Construction of instantons, *Phys. Lett. A* **65** (1978) 185–187.

[8] M. F. Atiyah, N. J. Hitchin and I. M. Singer, Self-duality in four-dimensional Riemannian geometry, *Proc. Roy. Soc. Lond. A* **362** (1978) 425–461.
[9] A. P. Balachandran, G. Mangano, A. Pinzul and S. Vaidya, Spin and statistics on the Groenewold–Moyal plane: Pauli-forbidden levels and transitions, *Int. J. Mod. Phys. A* **21** (2006) 3111–3126.
[10] R. Banerjee, Deformed Schrödinger symmetry on noncommutative space, *Eur. Phys. J. C* **47** (2006) 541–545.
[11] R. Banerjee, C. Lee and S. Siwach, Deformed conformal and super-Poincaré symmetries in the non-(anti)commutative spaces, *Eur. Phys. J. C* **48** (2006) 305–313.
[12] E. J. Beggs and S. Majid, Quantization by cochain twists and nonassociative differentials, math.QA/0506450.
[13] A. Belavin, A. Polyakov, A. Schwarz and Y. S. Tyupkin, Pseudoparticle solutions of the Yang–Mills equations, *Phys. Lett.* **59B** (1975) 85–87.
[14] T. Brzeziński and P. M. Hajac, The Chern–Galois character, *C.R. Acad. Sci. Paris Ser. I* **338** (2004) 113–116.
[15] T. Brzeziński and S. Majid, Quantum group gauge theory on quantum spaces, *Commun. Math. Physics* **157** (1993) 591–638.
[16] M. Chaichian, P. Kulish, K. Nishijima and A. Tureanu, On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT, *Phys. Lett. B* **604** (2004) 98–102.
[17] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994.
[18] S. Chern and X. Hu, Equivariant Chern character for the invariant Dirac operator, *Michigan Math. J.* **44** (1997) 451–473.
[19] A. Connes, Noncommutative differential geometry, *Publ. Math. IHES* **39** (1985) 257–360.
[20] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.
[21] A. Connes, Gravity coupled with matter and the foundation of noncommutative geometry, *Commun. Math. Phys.* **182** (1996) 155–176.
[22] A. Connes and M. Dubois-Violette, Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples, *Commun. Math. Phys.* **230** (2002) 539–579.
[23] A. Connes and G. Landi, Noncommutative manifolds: The instanton algebra and isospectral deformations, *Commun. Math. Phys.* **221** (2001) 141–159.
[24] A. Connes and M. Marcolli, A walk in the noncommutative garden, math.QA/0601054; this volume.
[25] A. Connes and H. Moscovici, The local index formula in noncommutative geometry, *Geom. Funct. Anal.* **5** (1995) 174–243.
[26] J. Cuntz and D. Quillen, Algebra extension and nonsingularity, *J. Amer. Math. Soc.* **8** (1995) 251–289.
[27] L. Dąbrowski, H. Grosse and P. M. Hajac, Strong connections and Chern-Connes pairing in the Hopf–Galois theory, *Commun. Math. Phys.* **220** (2001) 301–331.
[28] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J. C. Várilly, The Dirac operator on $\mathrm{SU}_q(2)$, *Commun. Math. Phys.* **259** (2005) 729–759.

[29] J. Dixmier, Existence de traces non normales, *C.R. Acad. Sci. Paris Sér A-B* **262** (1966) A1107–A1108.
[30] S. K. Donaldson, Polynomial invariants for smooth manifolds, *Topology* **29** (1990) 257–315.
[31] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Science Publications, Oxford, 1990.
[32] V. G. Drinfel′d, Constant quasiclassical solutions of the Yang–Baxter quantum equation, *Sov. Math. Dokl.* **28** (1983) 667–671.
[33] V. G. Drinfel′d, Quasi-Hopf algebras, *Leningrad Math. J.* **1** (1990) 1419–1457.
[34] L. D. Faddeev, N. Y. Reshetikhin and L. A. Takhtajan, Quantization of Lie groups and Lie algebras, *Leningrad Math. J.* **1** (1990) 193–225.
[35] V. Gayral, B. Iochum and J. C. Várilly, Dixmier traces on noncompact isospectral deformations, *J. Funct. Anal.* **237** (2006) 507–539.
[36] A. Giaquinto and J. J. Zhang, Bialgebra actions, twists, and universal deformation formulas, *J. Pure Appl. Alg.* **128** (1998) 133–151.
[37] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, Boston, 2001.
[38] P. M. Hajac, Strong connections on quantum principal bundles, *Commun. Math. Phys.* **182** (1996) 579–617.
[39] P. M. Hajac and S. Majid, Projective module description of the q-monopole, *Commun. Math. Phys.* **206** (1999) 247–264.
[40] P. Julg, K-théorie équivariante et produits croisés, *C.R. Acad. Sci. Paris.* **292** (1981) 629–632.
[41] H. F. Kreimer and M. Takeuchi, Hopf algebras and Galois extensions of an algebra, *Indiana Univ. Math. J.* **30** (1981) 675–692.
[42] G. Landi, *An Introduction to Noncommutative Spaces and their Geometry*, Springer-Verlag, Berlin, 1997.
[43] G. Landi, Deconstructing monopoles and instantons, *Rev. Math. Phys.* **12** (2000) 1367–1390.
[44] G. Landi, Spin-Hall effect with quantum group symmetry, *Lett. Math. Phys.* **75** (2006) 187–200.
[45] G. Landi, C. Pagani and C. Reina, A Hopf bundle over a quantum four-sphere from the symplectic group, *Commun. Math. Phys.* **263** (2006) 65–88.
[46] G. Landi, C. Pagani, C. Reina and W. van Suijlekom, work in progress.
[47] G. Landi and W. van Suijlekom, Principal fibrations from noncommutative spheres, *Commun. Math. Phys.* **260** (2005) 203–225.
[48] G. Landi and W. van Suijlekom, Noncommutative instantons from twisted conformal symmetries, *Commun. Math. Phys.* **271** (2007) 591–634.
[49] S. Lie, *Theorie der Transformationsgruppen*, Chelsea, New York, 1970.
[50] F. Lizzi, S. Vaidya and P. Vitale, Twisted conformal symmetry in noncommutative two-dimensional quantum field theory, hep-th/0601056.
[51] J.-L. Loday, *Cyclic Homology*, Springer-Verlag, Berlin, 1992.
[52] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, 1995.

[53] S. Majid, Quantum and braided group Riemannian geometry, *J. Geom. Phys.* **30** (1999) 113–146.
[54] P. Matlock, Non-commutative geometry and twisted conformal symmetry, *Phys. Rev. D* **71** (2005) 126007.
[55] S. Montgomery, *Hopf Algebras and their Actions on Rings*, Amer. Math. Soc. Providence, 1993.
[56] N. Nekrasov and A. Schwarz, Instantons on noncommutative $\mathbb{R}^4$, and $(2,0)$ superconformal six-dimensional theory, *Commun. Math. Phys.* **198** (1998) 689–703.
[57] R. Oeckl, Untwisting noncommutative $\mathbb{R}^g$ and the equivalence of quantum field theories, *Nucl. Phys. B* **581** (2000) 559–574.
[58] N. Reshetikhin, Multiparameter quantum groups and twisted quasitriangular Hopf algebras, *Lett. Math. Phys.* **20** (1990) 331–335.
[59] M. A. Rieffel, Deformation quantization for actions of $\mathbb{R}^d$, *Mem. Amer. Math. Soc.* 506 (1993).
[60] M. A. Rieffel, K-groups of C^*-algebras deformed by actions of $\mathbb{R}^d$, *J. Funct. Anal.* **116** (1993) 199–214.
[61] P. Schauenburg, Bi-Galois objects over Taft algebras, *Israel J. Math.* **115** (2000) 101–123.
[62] H.-J. Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, *Israel J. Math.* **72** (1990) 167–195.
[63] A. Sitarz, Twists and spectral triples for isospectral deformations, *Lett. Math. Phys.* **58** (2001) 69–79.
[64] J. C. Várilly, Quantum symmetry groups of noncommutative spheres, *Commun. Math. Phys.* **221** (2001) 511–523.
[65] P. Watts, Noncommutative string theory, the R-matrix, and Hopf algebras, *Phys. Lett. B* **474** (2000) 295–302.
[66] P. Watts, Derivative and the role of the Drinfel′d twist in noncommutative string theory, hep-th/0003234.
[67] J. Wess, Deformed coordinate spaces: derivatives, hep-th/0408080.
[68] E. Witten, Topological quantum field theory, *Commun. Math. Phys.* **117** (1988) 353–386.

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An Invitation to Noncommutative Geometry

LECTURE NOTES ON NONCOMMUTATIVE ALGEBRAIC GEOMETRY AND NONCOMMUTATIVE TORI

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Contents

0. Introduction

I would like to thank all the organizers, namely, *M. Khalkhali, M. Marcolli, M. Shahshahani* and *M. M. Sheikh-Jabbari*, of the *International Workshop on Noncommutative Geometry, 2005* for giving me the opportunity to speak.

In section 1 we shall browse through some interesting definitions and constructions which will be referred to later on. In section 2 we shall discuss noncommutative projective geometry as initiated by *Artin* and *Zhang* in [AZ94]. This section is rather long and the readers can easily skip over some details. In section 3 we shall provide a brief overview of the algebraic aspects of noncommutative tori, which comprise the most widely studied class of noncommutative differentiable manifolds. These notes are not entirely self-contained and should be read in tandem with those of *B. Noohi* for background material and of *J. Plazas* for a better understanding of the topological and differential aspects of noncommutative tori.

1. Some Preliminaries

In a paper entitled *Some Algebras Associated to Automorphisms of Elliptic Curves* Artin, Tate and Van den Bergh [TVdB90] gave a nice description of noncommutative algebras which should, in principle, be algebras of functions of some nonsingular "noncommutative schemes". In the commutative case, nonsingularity is reflected in the regularity of the ring. However, this notion is insufficient for noncommutative purposes. So Artin and Schelter gave a stronger regularity condition which we call the *Artin–Schelter* (AS)-*regularity* condition. The main result of the above-mentioned paper says that AS-regular algebras of dimension 3 (global dimension) can be described neatly as some algebras associated to automorphisms of projective schemes, mainly elliptic curves. Also such algebras are both left and right Noetherian. This subsection is entirely based on the contents of [TVdB90].

To begin with, we fix an algebraically closed field k of characteristic 0. We shall mostly be concerned with $\mathbb{N}$-graded k-algebras $A = \oplus_{i\geq 0} A_i$, that are finitely generated in degree 1, with A_0 finite dimensional as a k-vector space. Such algebras are called *finitely graded* for short, though the term could be a bit misleading at first sight. A finitely graded ring is called *connected graded* if $A_0 = k$. A_+ stands for the two-sided *augmentation ideal* $\oplus_{i>0} A_i$.

Definition 1.1. (AS-regular algebra) *A connected graded ring A is called Artin–Schelter* (*AS*) *regular of dimension d if it satisfies the following conditions*:

(a) *A has global dimension d.*

(b) $\mathrm{GKdim}(A) < \infty$.
(c) *A is AS-Gorenstein.*

It is worthwhile to say a few words about *Gelfand–Kirillov dimension* (GKdim) and the *AS-Gorenstein* condition of algebras.

Take any connected graded k-algebra A and choose a finite dimensional k-vector space V such that $k[V] = A$. Now set $F^n A = k + \sum_{i=1}^{n} V^i$ for $n \geq 1$. This defines a filtration of A. Then the $\mathrm{GKdim}(A)$ is defined to be

$$\mathrm{GKdim}(A) = \lim\sup_{n} \frac{\ln(\dim_k F^n A)}{\ln(n)}\,.$$

Of course, one has to check that the definition does not depend on the choice of V.

Remark 1.2. *Bergman* [KL85] has shown that GKdim can be any real number $\alpha \geq 2$. However, if GKdim ≤ 2, then it is either 0 or 1.

There are some equivalent formulations of the AS-Gorenstein condition available in literature. We shall be content by saying the following:

Definition 1.3. (AS-Gorenstein condition) *A connected graded k-algebra A of global dimension $d < \infty$ is AS-Gorenstein if*

$$\mathrm{Ext}^i_A(k, A) = 0 \; for \; i \neq d \; and \; \mathrm{Ext}^d_A(k, A) \simeq k\,.$$

All regular commutative rings are AS-Gorenstein, which supports our conviction that the AS-Gorenstein hypothesis is desirable for noncommutative analogues of regular commutative rings. Further, note that the usual *Gorenstein* condition (for commutative rings) requires that they be Noetherian of finite injective dimension as modules over themselves.

Now we take up the task of describing the minimal projective resolution $(\star)$ of an AS-regular algebra of dimension $d = 3$. As a fact, let us also mention that the global dimension of a graded algebra is equal to the projective dimension of the left module ${}_A k$. Let

$$0 \longrightarrow P^d \longrightarrow \cdots \xrightarrow{f_2} P^1 \xrightarrow{f_1} P^0 \longrightarrow {}_A k \longrightarrow 0 \tag{1}$$

be a minimal projective resolution of the left module ${}_A k$. P^0 turns out to be A; P^1 and P^2 need an investigation into the structure of A for their descriptions. Suppose $A = T/I$, where $T = k\{x_1, \ldots, x_n\}$ is a free associative algebra generated by homogeneous elements x_i with degrees l_{1j} (also assume that $\{x_1, \ldots, x_n\}$ is a minimal set of generators). Then

$$P^1 \approx \bigoplus_{j=1}^{n} A(-l_{1j})\,. \tag{2}$$

The map $P^1 \longrightarrow P^0$, denoted x, is given by right multiplication with the column vector $(x_1, \ldots, x_n)^t$.

Coming to P^2, let $\{f_j\}$ be a minimal set of homogeneous generators for the ideal I such that $\deg f_j = l_{2j}$. In T, write each f_j as

$$f_j = \sum_j m_{ij} x_j \,, \tag{3}$$

where $m_{ij} \in T_{l_{2i}-l_{1j}}$. Let M be the image in A of the matrix (m_{ij}). Then

$$P^2 \approx \bigoplus_j A(-l_{2j}) \tag{4}$$

and the map $P^2 \longrightarrow P^1$, denoted M, is just right multiplication by the matrix M.

In general, it is not so easy to interpret all the terms of the resolution (1). However, for a regular algebra of dimension 3, the resolution looks like

$$0 \longrightarrow A(-s-1) \xrightarrow{x^t} A(-s)^r \xrightarrow{M} A(-1)^r \xrightarrow{x} A \longrightarrow {}_A k \longrightarrow 0\,, \tag{5}$$

where $(r,s) = (3,2)$ or $(2,3)$. Thus such an algebra has r generators and r relations each of degree s, $r+s=5$. Set $g = (x^t)M$; then

$$g^t = ((x^t)M)^t = QMx = Qf \tag{6}$$

for some $Q \in GL_r(k)$.

Now, with some foresight, we introduce a new definition, that of a *standard algebra*, in which we extract all the essential properties of AS-regular algebras of dimension 3.

Definition 1.4. *An algebra A is called* standard *if it can be presented by r generators x_j of degree* 1 *and r relations f_i of degree s, such that, with M defined by* (3), $(r,s) = (2,3)$ *or* $(3,2)$ *as above, and there is an element $Q \in GL_r(k)$ such that* (6) *holds.*

Remark 1.5. For a standard algebra A, (5) is just a complex and if it is a resolution, then A is a regular algebra of dimension 3.

1.1. *Twisted homogeneous coordinate rings*

Here we sketch a very general recipe for manufacturing interesting noncommutative rings out of a completely "commutative geometric" piece of datum, called an *abstract triple*, which turns out to be an isomorphism invariant for "AS-regular algebras".

Definition 1.6. *An abstract triple $\mathcal{T} = (X, \sigma, \mathscr{L})$ is a triple consisting of a projective scheme X, an automorphism σ of X and an invertible sheaf $\mathscr{L}$ on X.*

It is time to construct the *Twisted Homogeneous Coordinate Ring* $B(\mathcal{T})$ out of an abstract triple. For each integer $n \geq 1$ set

$$\mathscr{L}_n = \mathscr{L} \otimes \mathscr{L}^{\sigma} \otimes \cdots \otimes \mathscr{L}^{\sigma^{n-1}}, \tag{7}$$

where $\mathscr{L}^{\sigma} := \sigma^* \mathscr{L}$. The tensor products are taken over $\mathcal{O}_X$ and we set $\mathscr{L}_0 = \mathcal{O}_X$. As a graded vector space, $B(\mathcal{T})$ is defined as

$$B(\mathcal{T}) = \bigoplus_{n \geq 0} H^0(X, \mathscr{L}_n).$$

For every pair of integers $m, n \geq 0$, there is a canonical isomorphism

$$\mathscr{L}_m \otimes_k \mathscr{L}_n^{\sigma^m} \longrightarrow \mathscr{L}_{m+n}$$

and hence it defines a multiplication on $B(\mathcal{T})$

$$H^0(X, \mathscr{L}_m) \otimes_k H^0(X, \mathscr{L}_n) \longrightarrow H^0(X, \mathscr{L}_{m+n}).$$

Example 1. Let us compute (more precisely, allude to the computation of) the twisted homogeneous coordinate ring in a very simple case. Let $\mathcal{T} = (\mathbb{P}^1, \mathcal{O}(1), \sigma)$, where $\sigma(a_0, a_1) = (qa_0, a_1)$ for some $q \in k^*$. It is based on our understanding of Example 3.4 of [SvdB01].

We may choose a parameter u for $\mathbb{P}^1$, so that the standard affine open cover consisting of $U = \mathbb{P}^1 \setminus \{\infty\}$ and $V = \mathbb{P}^1 \setminus \{0\}$ has rings of regular functions $\mathcal{O}(U) = k[u]$ and $\mathcal{O}(V) = k[u^{-1}]$ respectively. Now we can identify $\mathcal{O}(1)$ with the sheaf of functions on $\mathbb{P}^1$ which have at most a simple pole at infinity; in other words, it is the subsheaf of $k(u) = k(\mathbb{P}^1)$ generated by $\{1, u\}$. It can be checked that $H^0(X, \mathcal{O}(n))$ is spanned by $\{1, u, \ldots, u^n\}$ and that, as a graded vector space $B(\mathbb{P}^1, \mathrm{id}, \mathcal{O}(1)) = k\{x, y\}$ (the free algebra over k generated by x and y and not the usual polynomial ring), where $x = 1$ and $y = u$, thought of as elements of $B_1 = H^0(X, \mathcal{O}(1))$. It should be mentioned that σ acts on the rational functions on the right as $f^{\sigma}(p) = f(\sigma(p))$ for any $f \in k(\mathbb{P}^1)$ and $p \in \mathbb{P}^1$. From the presentation of the algebra it is evident that $\mathcal{O}(1)^{\sigma} \cong \mathcal{O}(1)$. So as a graded vector space $B(\mathcal{T}) \cong B(\mathbb{P}^1, \mathrm{id}, \mathcal{O}(1))$. The multiplication is somewhat different though.

$$y \cdot x = y \otimes x^{\sigma} = u \otimes 1^{\sigma} = u \otimes 1 = u \in H^0(\mathbb{P}^1, \mathcal{O}(2)).$$

On the other hand,

$$x \cdot y = x \otimes y^{\sigma} = 1 \otimes u^{\sigma} = 1 \otimes qu = qu \in H^0(\mathbb{P}^1, \mathcal{O}(2)).$$

So we find a relation between x and y, namely, $x \cdot y - qy \cdot x = 0$ and a little bit more work shows that this is the only relation. So the twisted homogeneous coordinate ring associated to $\mathcal{T}$ is

$$B(\mathcal{T}) = k\{x, y\}/(x \cdot y - qy \cdot x)\,.$$

A good reference for a better understanding of these rings is [AVdB90].

1.2. *A cursory glance at Grothendieck categories*

For the convenience of the reader let us say a few words about a *generator* of a category. An object G of a category $\mathcal{C}$ is called a *generator* if, given a pair of morphisms $f, g : A \longrightarrow B$ in $\mathcal{C}$ with $f \neq g$, there exists an $h : G \longrightarrow A$ with $fh \neq gh$ (more briefly, $\mathrm{Hom}(G, \_) : \mathcal{C} \longrightarrow Set$ is a faithful functor). A family of objects $\{G_i\}_{i \in I}$ is called a *generating set* if, given a pair of morphisms $f, g : A \rightrightarrows B$ with $f \neq g$, there exists an $h_i : G_i \longrightarrow A$ for some $i \in I$ with $fh_i \neq gh_i$. Strictly speaking, this is a misnomer. In a cocomplete category (i.e. closed under all coproducts), a family of objects $\{G_i\}_{i \in I}$ forms a generating set if and only if the coproduct of the family forms a generator.

Remark 1.7. Let $\mathcal{C}$ be a cocomplete abelian category. Then an object G is a generator if and only if, for any object $A \in Ob(\mathcal{C})$, there exists an epimorphism

$$G^{\oplus I} \longrightarrow A$$

for some indexing set I.

Definition 1.8. (Grothendieck category) *It is a* (*locally small*) *cocomplete abelian category with a generator and satisfying, for every family of short exact sequences indexed by a filtered category I* [*i.e. I is nonempty and, if $i, j \in Ob(I)$ then $\exists\ k \in Ob(I)$ and arrows $i \longrightarrow k$ and $j \longrightarrow k$, and for any two arrows $i \underset{v}{\overset{u}{\rightrightarrows}} j\ \exists\ k \in Ob(I)$ and an arrow $w : j \longrightarrow k$ such that $wu = wv$* (*think of a categorical formulation of a directed set*)].

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$

the following short sequence is also exact

$$0 \longrightarrow \varinjlim_{i \in I} A_i \longrightarrow \varinjlim_{i \in I} B_i \longrightarrow \varinjlim_{i \in I} C_i \longrightarrow 0\,,$$

i.e. passing on to filtered colimits preserves exactness. This is equivalent to the sup condition of the famous AB5 Property.

Remark 1.9. The original AB5 Property requires the so-called *sup* condition, besides cocompleteness. An abelian category satisfies *sup* if for any ascending chain Ω of subobjects of an object M, the supremum of Ω exists; and for any subobject N of M, the canonical morphism

$$sup\,\{L \cap N | L \in \Omega\} \xrightarrow{\sim} (sup\,\Omega) \cap N$$

is an isomorphism. Hence, another definition of a Grothendieck category could be a cocomplete abelian category, having a generator and satisfying the sup condition, *i.e.* an *AB5* category with a generator.

Example 2. (Grothendieck Categories) 1. Mod(R), where R is an associative ring with unity.
2. The category of sheaves of R-modules on an arbitrary topological space.
3. In the same vein, the category of abelian pre-sheaves on a *site* $\mathcal{T}$. Actually this is just $Funct(\mathcal{T}^{op}, Ab)$.
4. QCoh(X), where X is a quasi-compact and quasi-separated scheme. (a morphism of schemes $f : X \longrightarrow Y$ is called quasi-compact if, for any open quasi-compact $U \subseteq Y$, $f^{-1}(U)$ is quasi-compact in X and it becomes quasi-separated if the canonical morphism $\delta_f : X \longrightarrow X \times_Y X$ is quasi-compact. A scheme X is called quasi-compact (resp. quasi-separated) if the canonical unique morphism $X \longrightarrow Spec(\mathbb{Z})$, $Spec(\mathbb{Z})$ being the final object, is quasi-compact (resp. quasi-separated)).

Grothendieck categories have some remarkable properties which make them amenable to homological arguments.

1. Grothendieck categories are complete, i.e. they are closed under products.
2. In a Grothendieck category every object has an injective envelope, in particular there are enough injectives.

The deepest result about Grothendieck categories is given by the following theorem.

Theorem 1.10. (Gabriel, Popescu) *Let $\mathcal{C}$ be a Grothendieck category and let $\mathcal{G}$ be a generator of $\mathcal{C}$. Put $S = \mathrm{End}(\mathcal{G})$. Then the functor*

$$\mathrm{Hom}(\mathcal{G}, -) : \mathcal{C} \longrightarrow \mathrm{Mod}(S^{op})$$

is fully faithful (and has an exact left adjoint).

1.3. *Justification for bringing in Grothendieck categories*

We begin by directly quoting *Manin* [Man88] — "... Grothendieck taught us, to do geometry you really don't need a space, all you need is a category of sheaves on this would-be space." This idea gets a boost from the following reconstruction theorem.

Theorem 1.11. (Gabriel, Rosenberg [Ros98]) *Any scheme can be reconstructed uniquely up to isomorphism from the category of quasi-coherent sheaves on it.*

Remark 1.12. When it is known in advance that the scheme to be reconstructed is an affine one, we can just take the center of the category, which is the endomorphism ring of the identity functor of the category. More precisely, let $X = Spec\, A$ be an affine scheme and let $\mathcal{A}$ be the category of quasi-coherent sheaves on X, which is the same as $\mathrm{Mod}(A)$. Then the centre of $\mathcal{A}$, denoted $\mathrm{End}(Id_{\mathcal{A}})$, is canonically isomorphic to A. (The centre of an abelian category is manifestly commutative and, in general, it gives us only the centre of the ring, that is, $\mathcal{Z}(A)$. But here we are talking about honest schemes and hence $\mathcal{Z}(A) = A$.)

We can also get a derived analogue of the above result, which is, however, considerably weaker. Also it is claimed to be an easy consequence of the above theorem in [BO01].

Theorem 1.13. (Bondal, Orlov [BO01]) *Let X be a smooth irreducible projective variety with ample canonical or anti-canonical sheaf. If $\mathcal{D} = D^b\mathrm{Coh}(X)$ is equivalent as a graded category to $D^b\mathrm{Coh}(X')$ for some other smooth algebraic variety X', then X is isomorphic to X'.*

Remark 1.14. Notice that for elliptic curves or, in general, abelian varieties the theorem above is not applicable.

Finally, consider a pre-additive category with a single object, say $*$. Then, being a pre-additive category, $\mathrm{Hom}(*,*)$ is endowed with an abelian group structure. If we define a product on it by composition, then it is easy to verify that the two operations satisfy the ring axioms. So $\mathrm{Hom}(*,*)$ or simply $\mathrm{End}(*)$ is a ring and that is all we need to know about the pre-additive category. Extrapolating this line of thought, we say that pre-additive categories generalise the concept of rings and since schemes are concocted from commutative rings, it is reasonable to believe that a pre-additive category with some geometric properties should give rise to a "non-commutative scheme". The geometric properties desirable in an abelian category were written down by *Grothendieck* as the famous *AB Properties* in [Gro57].

1.4. *A brief discussion on construction of quotient categories*

This discussion is based on *Gabriel's* article [Gab62] (p. 365) and curious readers are encouraged to go through the details from there.

Recall that we call a full subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ *thick* if the following condition is satisfied:

for all short exact sequences in $\mathcal{A}$ of the form $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, we have

$$M \in \mathcal{C} \iff \text{both } M', M'' \in \mathcal{C}\,.$$

Now we construct the quotient of $\mathcal{A}$ by a thick subcategory $\mathcal{C}$, denoted $\mathcal{A}/\mathcal{C}$, as follows:

$$Ob(\mathcal{A}/\mathcal{C}) = \text{objects of } \mathcal{A}\,.$$

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M,N) = \varinjlim_{\substack{\forall\, M',N' \text{ subobj.} \\ s.t.\ M/M',N' \in \mathcal{C}}} \mathrm{Hom}_{\mathcal{A}}(M', N/N')\,.$$

One needs to check that as M' and N' run through all subobjects of M and N respectively, such that M/M' and N' are in $Ob(\mathcal{C})$ (take intersection and sum respectively), the abelian groups $\mathrm{Hom}_{\mathcal{C}}(M', N/N')$ form a directed system. It satisfies the obvious universal properties which the readers are invited to formulate. It comes equipped with a canonical quotient functor $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$.

Proposition 1.15. (Gabriel) *Let $\mathcal{C}$ be a thick subcategory of an abelian category $\mathcal{A}$. Then the category $\mathcal{A}/\mathcal{C}$ is abelian and the canonical functor π is exact.*

The essence of this quotient construction is that the objects of $\mathcal{C}$ become isomorphic to zero.

Example 3. Let $\mathcal{A} = \mathrm{Mod}(\mathbb{Z})$ and $\mathcal{C} =$ Torsion *groups*. Then one can show that $\mathcal{A}/\mathcal{C} \simeq \mathrm{Mod}(\mathbb{Q})$.

Let us define a functor from $\mathcal{A}/\mathcal{C}$ to $\mathrm{Mod}(\mathbb{Q})$ by tensoring with $\mathbb{Q}$. We simplify the Hom sets of $\mathcal{A}/\mathcal{C}$. Using the structure theorem, write every abelian group as a direct sum of its torsion part and torsion-free part. If one of the variables is torsion, it can be shown that in the limit Hom becomes 0. So we may assume that both variables are torsion-free and for simplicity let us consider both of them to be $\mathbb{Z}$. Then,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(\mathbb{Z},\mathbb{Z}) &= \varinjlim_{n} \mathrm{Hom}_{\mathcal{A}}(n\mathbb{Z},\mathbb{Z}) \\
&= \bigcup_n \frac{1}{n}\mathbb{Z} \\
&= \mathbb{Q} = \mathrm{Hom}(\mathbb{Q},\mathbb{Q})
\end{aligned}$$

This shows that the functor is fully faithful and essential surjectivity can be verified easily.

Now we are ready to discuss a model of noncommutative projective geometry after *Artin* and *Zhang* [AZ94]. We would also like to bring to the notice of readers the works of *Verevkin* (see [Ver92]). But before that let us go through one nice result in the affine case. Let X be an affine scheme and put $A = \Gamma(X, \mathcal{O}_X)$. Then it is well-known that $\mathrm{QCoh}(X)$ is equivalent to Mod(A). This fact encourages us to ask: which Grothendieck categories can be written as Mod(A) for some possibly noncommutative ring A?

The answer to this question is given by the theorem below.

Theorem 1.16. ([Ste75]) *Let $\mathcal{C}$ be a Grothendieck category with a projective generator G and assume that G is small* [*i.e.* $\mathrm{Hom}(G, -)$ *commutes with all direct sums*]. *Then $\mathcal{C} \simeq \mathrm{Mod}(A^{op})$, for $A = \mathrm{End}(G)$.*

Note that the Gabriel–Popescu Theorem 1.10 gave just a fully faithful embedding with an exact left adjoint and not an equivalence.

2. Noncommutative Projective Geometry

Fix an algebraically closed field k; then we shall mostly be dealing with categories which are *k-linear* abelian categories [i.e. the bifunctor Hom ends up in Mod(k)]. Since in commutative algebraic geometry one mostly deals with finitely generated k-algebras, which are Noetherian, here we assume that our k-algebras are at least *right Noetherian.* Later on we shall need to relax this Noetherian condition, but for now we stick to it. Let R be a graded algebra. Then we introduce some categories here:

QCoh(X) := category of quasi-coherent sheaves on a scheme X.

Coh(X) := category of coherent sheaves on X.

Mod(A) := category of right A-modules, where A is a k-algebra.

Gr(R) := category of $\mathbb{Z}$-graded right R-modules, with degree 0 morphisms.

Tor(R) := full subcategory of Gr(R) generated by torsion modules (i.e. M such that $\forall\ x \in M$, $xR_{\geq s} = 0$ for some s), which is thick.

QGr(R) := the quotient category Gr(R)/Tor(R) (refer to the quotient construction before).

Notice that QCoh(X) is not obtained from Coh(X) by a quotient construction as QGr(R) is from Gr(R). In fact, when X is Noetherian, Coh(X) is the subcategory of QCoh(X) generated by all Noetherian objects in it.

Remark 2.1. (Standard Convention) If $XYuvw(\dots)$ denotes an abelian category, then we shall denote by $xyuvw(\dots)$ the full subcategory consisting of Noetherian objects and if $A, B, \dots, M, N, \dots$ denote objects in Gr(R) then we shall denote by $\mathcal{A}, \mathcal{B}, \dots, \mathcal{M}, \mathcal{N}, \dots$ the corresponding objects in QGr(R).

Some people denote QGr(R) by $Tails(R)$, but we shall stick to our notation. We denote the quotient functor $\mathrm{Gr}(R) \longrightarrow \mathrm{QGr}(R)$ by π. It has a right adjoint functor $\omega : \mathrm{QGr}(R) \longrightarrow \mathrm{Gr}(R)$ and so, for all $M \in \mathrm{Gr}(R)$ and $\mathscr{F} \in \mathrm{QGr}(R)$ one obtains

$$\mathrm{Hom}_{\mathrm{QGr}(R)}(\pi M, \mathscr{F}) \cong \mathrm{Hom}_{\mathrm{Gr}(R)}(N, \omega\mathscr{F}) .$$

The Hom's of $\mathrm{QGr}(R)$ take a more intelligible form with the assumptions on R. It turns out that for any $N \in gr(R)$ and $M \in \mathrm{Gr}(R)$,

$$\mathrm{Hom}_{\mathrm{QGr}(R)}(\pi N, \pi M) \cong \varinjlim \mathrm{Hom}_{\mathrm{Gr}(R)}(N_{\geq n}, M) \tag{8}$$

For any functor F from a k-linear category $\mathcal{C}$ equipped with an autoequivalence s, we denote by $\underline{F}$ the graded analogue of F given by $\underline{F}(A) := \oplus_{n\in\mathbb{Z}} F(s^n A)$ for any $A \in Ob(\mathcal{C})$. Further, to simplify notation we shall sometimes denote $s^n A$ by $A[n]$ when there is no chance of confusion.

Keeping in mind the notations introduced above we have

Lemma 2.2. $\omega\pi M \cong \varinjlim \underline{\mathrm{Hom}}_R(R_{\geq n}, M)$

Sketch of proof.

$$\begin{aligned}
\omega\pi M &= \underline{\mathrm{Hom}}_R(R, \omega\pi M) \quad [\text{since } R \in gr(R)] \\
&= \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}(R)}(R, \omega\pi M[d]) \\
&= \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{QGr}(R)}(\pi R, \pi M[d]) \quad [\text{by adjointness of } \pi \text{ and } \omega] \\
&= \bigoplus_{d\in\mathbb{Z}} \varinjlim \mathrm{Hom}_{\mathrm{Gr}(R)}(R_{\geq n}, M[d]) \quad [\text{by (8)}] \\
&= \varinjlim \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}(R)}(R_{\geq n}, M[d]) \\
&= \varinjlim \underline{\mathrm{Hom}}_R(R_{\geq n}, M) .
\end{aligned}$$

□

The upshot of this lemma is that there is a natural equivalence of functors $\omega \simeq \underline{\mathrm{Hom}}(\pi R, \_)$.

2.1. Proj R

Let X be a projective scheme with a line bundle $\mathscr{L}$. Then the homogeneous coordinate ring B associated to $(X, \mathscr{L})$ is defined by the formula $B = \oplus_{n\in\mathbb{N}} \Gamma(X, \mathscr{L}^n)$ with the obvious multiplication. Similarly, if $\mathscr{M}$ is a quasi-coherent sheaf on X, $\Gamma_h(\mathscr{M}) = \oplus_{n\in\mathbb{N}} \Gamma(X, \mathscr{M} \otimes \mathscr{L}^n)$ defines a graded B-module. Thus, the compostion of Γ_h with the natural projection from $\mathrm{Gr}(B)$ to $\mathrm{QGr}(B)$ yields a functor $\bar{\Gamma}_h : \mathrm{QCoh}(X) \longrightarrow \mathrm{QGr}(B)$. This functor works particularly well when $\mathscr{L}$ is ample, as is evident from the following fundamental result due to *Serre*.

Theorem 2.3. [Ser55] 1. *Let $\mathscr{L}$ be an ample line bundle on a projective scheme X. Then the functor $\bar{\Gamma}_h(-)$ defines an equivalence of categories between* $\mathrm{QCoh}(X)$ *and* $\mathrm{QGr}(B)$.

2. *Conversely, if R is a commutative connected graded k-algebra, that is, $R_0 = k$ and it is generated by R_1 as an R_0-algebra, then there exists a line bundle $\mathscr{L}$ over $X = \mathrm{Proj}(R)$ such that $R = B(X, \mathscr{L})$, up to a finite dimensional vector space. Once again,* $\mathrm{QGr}(R) \simeq \mathrm{QCoh}(X)$.

In commutative algebraic geometry one defines the Proj of a graded ring to be the set of all homogeneous prime ideals which do not contain the augmentation ideal. This notion is not practicable over arbitrary algebras. However, *Serre's* theorem filters out the essential ingredients to define the Proj of an arbitrary algebra. The equivalence is controlled by the category $\mathrm{QCoh}(X)$, the structure sheaf $\mathcal{O}_X$ and the autoequivalence given by tensoring with $\mathscr{L}$, which depends on the polarization of X. Borrowing this idea we get to the definition of Proj. Actually one should have worked with a $\mathbb{Z}$-graded algebra R and defined its Proj but it has been shown in [AZ94] that, with the definition to be provided below, $\mathrm{Proj}\, R$ is the same as $\mathrm{Proj}\, R_{\geqslant 0}$. Hence, we assume that R is an $\mathbb{N}$-graded k-algebra. $\mathrm{Gr}(R)$ has a shift operator s such that $s(M) = M[1]$ and a special object, R_R. We can actually recover R from the triple $(\mathrm{Gr}(R), R_R, s)$ by

$$R = \bigoplus_{i \in \mathbb{N}} \mathrm{Hom}(R_R, s^i(R_R))$$

and the composition is given as follows: $a \in R_i$ and $b \in R_j$, then $ab = s^j(a) \circ b \in R_{i+j}$.

Let $\mathcal{R}$ denote the image of R in $\mathrm{QGr}(R)$ and we continue to denote by s the autoequivalence induced by s on $\mathrm{QGr}(R)$.

Definition 2.4. ($\mathrm{Proj}\, R$) *The triple* $(\mathrm{QGr}(R), \mathcal{R}, s)$ *is called the projective scheme of R and is denoted* $\mathrm{Proj}\, R$. *Keeping in mind our convention we denote* $(\mathrm{qgr}(R), \mathcal{R}, s)$ *by* $\mathrm{proj}\, R$. *This is just as good because there is a way to switch back and forth between* $\mathrm{QGr}(R)$ *and* $\mathrm{qgr}(R)$.

2.2. *Characterization of* $\mathrm{Proj}\, R$

We have simply transformed *Serre's* theorem into a definiton. It is time to address the most natural question: which triples $(\mathcal{C}, \mathcal{A}, s)$ are of the form $\mathrm{Proj}(R)$ for some graded algebra R? This problem of characterization has been dealt with comprehensively by *Artin* and *Zhang*. We will be content with just taking a quick look at the important points. Let us acquaint ourselves with morphisms of such triples. A morphism between $(\mathcal{C}, \mathcal{A}, s)$ and $(\mathcal{C}', \mathcal{A}', s')$ is given by a triple (f, θ, μ), where $f : \mathcal{C} \longrightarrow \mathcal{C}'$ is a k-linear functor, $\theta : f(\mathcal{A}) \longrightarrow \mathcal{A}'$ is an isomorphism in $\mathcal{C}'$ and μ is a natural isomorphism of functors $f \circ s \longrightarrow s' \circ f$. The question of characterization is easier to deal with when s is actually an automorphism of $\mathcal{C}$. To circumvent this problem, an elegant construction has been provided in [AZ94] whereby one can pass to a different triple, where s becomes necessarily an automorphism. If s is an automorphism one can take negative powers of s as well and it becomes easier to define the graded analogues of all functors (refer to [2.1]). Sweeping that discussion under the carpet, henceforth, we tacitly assume that s is an automorphism of $\mathcal{C}$ (even though we may write s as an autoequivalence).

The definition of Proj was conjured up from *Serre's* theorem where the triple was $(\mathrm{QCoh}(X), \mathcal{O}_X, \_ \otimes \mathscr{L})$. Of course, one can easily associate a graded k-algebra to any $(\mathcal{C}, \mathcal{A}, s)$.

$$\Gamma_h(\mathcal{C}, \mathcal{A}, s) = \bigoplus_{n \geq 0} \mathrm{Hom}(\mathcal{A}, s^n \mathcal{A}) \tag{9}$$

with multiplication $a \cdot b = s^n(a)b$ for $a \in \mathrm{Hom}(\mathcal{A}, s^m \mathcal{A})$ and $b \in \mathrm{Hom}(\mathcal{A}, s^n \mathcal{A})$.

Remark 2.5. Let X be a scheme, $\sigma \in Aut(X)$ and $\mathscr{L}$ be a line bundle on X. Then one obtains the *twisted homogeneous coordinate ring*, as discussed in section 1, as a special case of the above construction applied to the triple $(\mathrm{QCoh}(X), \mathcal{O}_X, \sigma_*(- \otimes \mathscr{L}))$. [Hint: to verify this, use the projection formula for sheaves.]

Notice that $\mathscr{L}$ needs to be ample for *Serre's* theorem to work. So we need a notion of ampleness in the categorical set-up.

Definition 2.6. (Ampleness) *Assume that $\mathcal{C}$ is locally Noetherian. Let $\mathcal{A} \in Ob(\mathcal{C})$ be a Noetherian object and let s be an autoequivalence of $\mathcal{C}$. Then the pair $(\mathcal{A}, s)$ is called ample if the following conditions hold*:

1. *For every Noetherian object $\mathcal{O} \in Ob(\mathcal{C})$ there are positive integers $l_1, \ldots, l_p$ and an epimorphism from $\bigoplus_{i=0}^{p} \mathcal{A}(-l_i)$ to $\mathcal{O}$.*

2. *For every epimorphism between Noetherian objects $\mathcal{P} \longrightarrow \mathcal{Q}$ the induced map* $\mathrm{Hom}(\mathcal{A}(-n), \mathcal{P}) \longrightarrow \mathrm{Hom}(\mathcal{A}(-n), \mathcal{Q})$ *is surjective for $n \gg 0$.*

Remark 2.7. The first part of this definition corresponds to the standard definition of an ample sheaf and the second part to the homological one.

Now we are in good shape to state one part of the theorem of *Artin* and *Zhang* which generalises that of *Serre*.

Theorem 2.8. *Let $(\mathcal{C}, \mathcal{A}, s)$ be a triple as above such that the following conditions hold*:

(H1) *$\mathcal{A}$ is Noetherian,*

(H2) $A := \mathrm{Hom}(\mathcal{A}, \mathcal{A})$ *is a right Noetherian ring and* $\mathrm{Hom}(\mathcal{A}, \mathcal{M})$ *is a finite A-module for all Noetherian $\mathcal{M}$, and*

(H3) $(\mathcal{A}, s)$ *is ample.*

Then $\mathcal{C} \simeq \mathrm{QGr}(B)$ *for* $B = \Gamma_h(\mathcal{C}, \mathcal{A}, s)$. *Besides, B is right Noetherian.*

The converse to this theorem requires an extra hypothesis, which is the so-called χ_1 condition. One could suspect, and rightly so, that there is a χ_n condition for every n. They are all some kind of condition on the graded Ext functor. However, they all look quite mysterious at first glance. Actually most naturally occurring algebras satisfy them but the reason behind their occurrence is not well understood. We shall discuss them in some cases later but we state a small proposition first.

Proposition 2.9. *Let* $M \in \mathrm{Gr}(B)$ *and fix* $i \geq 0$. *There is a right B-module structure on* $\underline{\mathrm{Ext}}^n_B(B/B_+, M)$ *coming from the right B-module structure of* B/B_+. *Then the following are equivalent*:

1. *for all* $j \leq i$, $\underline{\mathrm{Ext}}^j_B(B/B_+, M)$ *is a finite B-module*;
2. *for all* $j \leq i$, $\underline{\mathrm{Ext}}^j_B(B/B_{\geq n}, M)$ *is finite for all* n;
3. *for all* $j \leq i$ *and all* $N \in \mathrm{Gr}(B)$, $\underline{\mathrm{Ext}}^j_B(N/N_{\geq n}, M)$ *has a right bound independent of* n;
4. *for all* $j \leq i$ *and all* $N \in \mathrm{Gr}(B)$, $\varinjlim \underline{\mathrm{Ext}}^j_B(N/N_{\geq n}, M)$ *is right bounded.*

The proof is a matter of unwinding the definitions of the terms suitably and then playing with them. We shall do something smarter instead — make a definition out of it.

Definition 2.10. (χ conditions) *A graded algebra B satisfies χ_n if, for any finitely generated graded B-module M, one of the equivalent conditions of the above proposition is satisfied (after substituting $i = n$ in them). Moreover, we say that B satisfies χ if it satisfies χ_n for every n.*

Remark 2.11. Since B/B_+ is a finitely generated B_0-module ($B_0 = k$) we could have equally well required the finiteness of $\underline{\mathrm{Ext}}^n_B(B/B_+, M)$ over $B_0 = k$, i.e. $\dim_k \underline{\mathrm{Ext}}^n_B(B/B_+, M) < \infty$ for χ_n.

Let B be an $\mathbb{N}$-graded right Noetherian algebra and $\pi : \mathrm{Gr}(B) \longrightarrow \mathrm{QGr}(B)$.

Theorem 2.12. *If B satisfies χ_1 as well, then* (H1), (H2) *and* (H3) *hold for the triple* $(\mathrm{qgr}(B), \pi B, s)$. *Moreover, if* $A = \Gamma_h(\mathrm{QGr}(B), \pi B, s)$, *then* $\mathrm{Proj}\, B$ *is isomorphic to* $\mathrm{Proj}\, A$ *via a canonical homomorphism* $B \longrightarrow A$. [*We have a canonical map* $B_n = \mathrm{Hom}_B(B, B[n]) \longrightarrow \mathrm{Hom}(\pi B, \pi B[n]) = A_n$ *given by the functor* π.]

The proofs of these theorems are once again quite long and involved. So they are omitted. What we need now is a good cohomology theory for studying such noncommutative projective schemes.

2.3. *Cohomology of* $\operatorname{Proj} R$

The following rather edifying theorem due to *Serre* gives us some insight into the cohomology of projective (commutative) spaces.

Theorem 2.13. [Har77] *Let X be a projective scheme over a Noetherian ring A, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over $\operatorname{Spec} A$. Let $\mathscr{F}$ be a coherent sheaf on X. Then:*

1. *for each $i \geq 0$, $H^i(X, \mathscr{F})$ is a finitely generated A-module,*
2. *there is an integer n_0, depending on $\mathscr{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(X, \mathscr{F}(n)) = 0$.*

There is an analogue of the above result and we zero in on that. We have already come across the χ conditions, which have many desirable consequences. Actually the categorical notion of ampleness does not quite suffice. For the desired result to go through, we need the algebra to satisfy χ too. Without inundating our minds with all the details of χ we propose to get to the point i.e. cohomology. Set $\pi R = \mathcal{R}$. On a projective (commutative) scheme X one can define the sheaf cohomology of $\mathscr{F} \in \operatorname{Coh}(X)$ as the right derived functor of the global sections functor, i.e. Γ. But $\Gamma(X, \mathscr{F}) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathscr{F})$. Buoyed by this fact, the proposed definition of the cohomology for every $\mathcal{M} \in \operatorname{qgr}(R)$ is

$$H^n(\mathcal{M}) := \operatorname{Ext}^n_{\mathcal{R}}(\mathcal{R}, \mathcal{M})\,.$$

However, taking into consideration the graded nature of our objects we also define the following:

$$\underline{H}^n(\operatorname{Proj} R, \mathcal{M}) := \underline{\operatorname{Ext}}^n_{\mathcal{R}}(\mathcal{R}, \mathcal{M}) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Ext}^n_{\mathcal{R}}(\mathcal{R}, \mathcal{M}[i])\,.$$

The category $\operatorname{QGr}(R)$ has enough injectives and one can choose a nice "minimal" injective resolution of $\mathcal{M}$ to compute its cohomologies, the details of which are available in chapter 7 of [AZ94].

Let $M \in \operatorname{Gr}(R)$ and write $\mathcal{M} = \pi M$. Then one should observe that

$$\begin{aligned} \underline{H}^n(\mathcal{M}) &= \underline{\operatorname{Ext}}^n_{\operatorname{QGr}(R)}(\mathcal{R}, \mathcal{M}) \\ &\cong \varinjlim \underline{\operatorname{Ext}}^n_{\operatorname{Gr}(R)}(R_{\geq n}, M) \quad [\text{by } (8)]\,. \end{aligned} \tag{10}$$

As R-modules we have the following exact sequence,

$$0 \longrightarrow R_{\geq n} \longrightarrow R \longrightarrow R/R_{\geq n} \longrightarrow 0\,.$$

For any $M \in \operatorname{Gr}(R)$, the associated $\underline{\operatorname{Ext}}$ sequence in $\operatorname{Gr}(R)$ looks like

$$\ldots \underline{\operatorname{Ext}}^j(R/R_{\geq n}, M) \longrightarrow \underline{\operatorname{Ext}}^j(R, M) \longrightarrow \underline{\operatorname{Ext}}^j(R_{\geq n}, M) \longrightarrow \ldots$$

Since R is projective as an R module, $\underline{\mathrm{Ext}}^j(R, M) = 0$ for every $j \geq 1$. Thus, we get the following exact sequence:

$$0 \to \underline{\mathrm{Hom}}(R/R_{\geq n}, M) \to M \to \underline{\mathrm{Hom}}(R_{\geq n}, M) \to \underline{\mathrm{Ext}}^1(R/R_{\geq n}, M) \to 0 \tag{11}$$

and, for every $j \geq 1$, an isomorphism

$$\underline{\mathrm{Ext}}^j(R_{\geq n}, M) \cong \underline{\mathrm{Ext}}^{j+1}(R/R_{\geq n}, M) \tag{12}$$

The following theorem is an apt culmination of all our efforts.

Theorem 2.14. (Serre's finiteness theorem) *Let R be a right Noetherian $\mathbb{N}$-graded algebra satisfying χ, and let $\mathscr{F} \in \mathrm{qgr}(R)$. Then,*

(H4) *for every $j \geq 0$, $H^j(\mathscr{F})$ is a finite right R_0-module, and*

(H5) *for every $j \geq 1$, $\underline{H}^j(\mathscr{F})$ is right bounded; i.e. for $d \gg 0$, $H^j(\mathscr{F}[d]) = 0$.*

Sketch of proof. Write $\mathscr{F} = \pi M$ for some $M \in gr(R)$. Suppose that $j = 0$. Since $\chi_1(M)$ holds, $\underline{\mathrm{Ext}}^i_R(R/R_{\geq n}, M)$ is a finite R-module for each $i = 1, 2$ and together with (11) it implies that $\omega\mathscr{F} \cong \underline{H}^0(\mathscr{F})$ is finite (recall ω from the paragraph after Remark 2.1). Now taking the 0-graded part on both sides we get (H4) for $j = 0$.

Suppose that $j \geq 1$. Since R satisfies χ_{j+1}, invoking Proposition 2.9 we get

$$\underset{\to}{\mathrm{Lim}}\underline{\mathrm{Ext}}^{j+1}_R(R/R_{\geq n}, M)$$

is right bounded. Combining (10) and (12) this equals $\underline{H}^j(\mathscr{F})$. This immediately proves (H5) as $\underline{H}^j(\mathscr{F})_d = H^j(\mathscr{F}[d])$. We now need left boundedness and local finiteness of $\underline{H}^j(\mathscr{F})$ to finish the proof of (H4) for $j \geq 1$. These we have already observed (at least tacitly) but one can verify them by writing down a resolution of $R/R_{\geq n}$ involving finite sums of shifts of R, and then realising the cohomologies as subquotients of a complex of modules of the form $\underline{\mathrm{Hom}}_R(\oplus_{i=0}^p R[l_i], M)$. □

Our discussion does not quite look complete unless we investigate the question of the "dimension" of the objects that we have defined.

2.4. *Dimension of* Proj R

The *cohomological dimension* of Proj R, denoted by $cd(\mathrm{Proj}\, R)$, is defined to be

$$cd(\mathrm{Proj}\, R) := \begin{cases} sup\{i \mid H^i(\mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \mathrm{qgr}(R)\} & \text{if it is finite,} \\ \infty & \text{otherwise.} \end{cases}$$

Remark 2.15. As H^i commutes with direct limits one could have used $\mathrm{QGr}(R)$ in the definition of cohomological dimension.

The following proposition gives us what we expect from a Proj construction regarding dimension and also provides a useful way of calculating it.

Proposition 2.16. 1. *If* $cd(\mathrm{Proj}\, R)$ *is finite, then it is equal to* $sup\{i \mid \underline{H}^i(R) \neq 0\}$.

2. *If the left global dimension of* R *is* $d < \infty$, *then* $cd(\mathrm{Proj}\, R) \leq d - 1$.

Sketch of proof. 1. Let d be the cohomological dimension of $\mathrm{Proj}\, R$. It is obvious that $sup\{i \mid \underline{H}^i(R) \neq 0\} \leq d$. We need to prove the other inequality. So we choose an object for which the supremum is attained, i.e. $\mathcal{M} \in \mathrm{qgr}(R)$ such that $H^d(\mathcal{M}) \neq 0$ and, hence, $\underline{H}^d(\mathcal{M}) \neq 0$. By the ampleness condition we may write down the following exact sequence:

$$0 \longrightarrow \mathcal{N} \longrightarrow \bigoplus_{i=0}^{p} R[-l_i] \longrightarrow \mathcal{M} \longrightarrow 0$$

for some $\mathcal{N} \in \mathrm{qgr}(R)$. By the long exact sequence of derived functors $\underline{H}^i$ we have

$$\ldots \longrightarrow \bigoplus_{i=0}^{p} \underline{H}^d(R[-l_i]) \longrightarrow \underline{H}^d(\mathcal{M}) \longrightarrow \underline{H}^{d+1}(\mathcal{N}) = 0\,.$$

This says that $\underline{H}^d(R[-l_i]) \neq 0$ for some i and hence, $\underline{H}^d(R) \neq 0$.

2. It has already been observed that $\underline{H}^i(M) \cong \mathrm{Lim}_{n\to\infty}\underline{\mathrm{Ext}}^i(R_{\geq n}, M)$ for all $i \geq 0$. Now, if the left global dimension of R is d, then $\underline{\mathrm{Ext}}^j(N, M) = 0$ for all $j > d$ and all $N, M \in \mathrm{Gr}(R)$. Putting $N = R/R_{\geq n}$ and using (12) we get $\underline{H}^d(M) = 0$ for all $M \in \mathrm{Gr}(R)$. Therefore, $cd(\mathrm{Proj}\, R) \leq d - 1$. □

Remark 2.17. If R is a Noetherian *AS-regular* graded algebra, then the *Gorenstein* condition can be used to prove that $cd(\mathrm{Proj}\, R)$ is actually equal to $d - 1$.

2.5. *Some examples* (*mostly borrowed from* [AZ94])

Example 4. (Twisted graded rings) Let σ be an automorphism of a graded algebra A. Then define a new multiplication $*$ on the underlying graded k-module $A = \oplus_n A_n$ by

$$a * b = a\sigma^n(b)\,,$$

where a and b are homogeneous elements in A and $\deg(a) = n$. Then algebra is called the *twist* of A by σ and it is denoted by A^σ. By [TVdB91] and [Zha96] $gr(A) \simeq gr(A^\sigma)$ and hence, $\mathrm{proj}(A) \simeq \mathrm{proj}(A^\sigma)$.

For example, if $A = k[x, y]$ where $\deg(x) = \deg(y) = 1$, then any linear operator on the space A_1 defines an automorphism, and hence a twist of A. If k is an algebraically closed field then, after a suitable linear change of variables, a twist can be brought into one of the forms $k_q[x, y] := k\{x, y\}/(yx - qxy)$ for some $q \in k$, or $k_j[x, y] := k\{x, y\}/(x^2 + xy - yx)$. Hence, $\operatorname{proj} k[x, y] \simeq \operatorname{proj} k_q[x, y] \simeq \operatorname{proj} k_j[x, y]$. The projective scheme associated to any one of these algebras is the projective line $\mathbb{P}^1$.

Example 5. (Changing the structure sheaf) Though the structure sheaf is a part of the definition of Proj, one might ask, given a k-linear abelian category $\mathcal{C}$, which objects $\mathcal{A}$ could serve the purpose of the structure sheaf. In other words, for which $\mathcal{A}$ do the conditions (H1), (H2) and (H3) of Theorem [2.8] hold? Since (H3) involves both the structure sheaf and the polarization s, the answer may depend on s. We propose to illustrate the possibilities by the simple example in which $\mathcal{C} = \mathrm{Mod}(R)$ when $R = k_1 \oplus k_2$, where $k_i = k$ for $i = 1, 2$ and where s is the automorphism which interchanges the two factors. The objects of $\mathcal{C}$ have the form $V \simeq k_1^{n_1} \oplus k_2^{n_2}$, and the only requirement for (H1), (H2) and (H3) is that both r_1 and r_2 not be zero simultaneously.

We have $s^n(V) = k_1^{r_2} \oplus k_2^{r_1}$ if n is odd and $s^n(V) = V$ otherwise. Thus, if we set $\mathcal{A} = V$ and $A = \Gamma_h(\mathcal{C}, \mathcal{A}, s)$, then $A_n \simeq k_1^{r_1 \times r_1} \oplus k_2^{r_2 \times r_2}$ if n is even, and $A_n \simeq k_1^{r_1 \times r_2} \oplus k_2^{r_2 \times r_1}$ otherwise. For example, if $r_1 = 1$ and $r_2 = 0$, then $A \simeq k[y]$, where y is an element of degree 2. Both of the integers r_i would need to be positive if s were the identity functor.

Example 6. (Commutative Noetherian algebras satisfy χ condition) Let A be a commutative Noetherian k-algebra. Then the module structure on $\underline{\mathrm{Ext}}^n_A(A/A_+, M)$ can be obtained both from the right A-module structure of A/A_+ and that of M. Choose a free resolution of A/A_+, consisting of finitely generated free modules. The cohomology of this complex of finitely generated A-modules is given by the $\underline{\mathrm{Ext}}$'s, whence they are finite.

Example 7. (Noetherian AS-regular algebras satisfy χ condition) If A is a Noetherian connected $\mathbb{N}$-graded algebra having global dimension 1, then A is isomorphic to $k[x]$, where $\deg(x) = n$ for some $n > 0$, which satisfies the condition χ by virtue of the previous example. In higher dimensions we have the following proposition.

Proposition 2.18. *Let A be a Noetherian AS-regular graded algebra of dimension $d \geq 2$ over a field k. Then A satisfies the condition χ.*

Sketch of proof. A is Noetherian and locally finite (due to finite GKdim). For such an A it is easy to check that $\underline{\mathrm{Ext}}^j(N, M)$ is a locally finite k-module whenever N, M are finite. A_0 is finite and hence, $\underline{\mathrm{Ext}}^j(A_0, M)$ is locally finite for every finite M and every j. Since A is connected graded, $A_0 = k$.

For any n and any finite A-module M we first show that $\underline{\mathrm{Ext}}^n(A_0, M) = \underline{\mathrm{Ext}}^n(k, M)$ is bounded using induction on the projective dimension of M. If $pd(M) = 0$, then $M = \oplus_{i=0}^{p} A[-l_i]$. By the *Gorenstein* condition (see Definition 1.3) of an AS-regular algebra A, $\underline{\mathrm{Ext}}^n(k, A[-l_i])$ is bounded for each i. Therefore, so is $\underline{\mathrm{Ext}}^n(k, M)$. If $pd(M) > 0$, we choose an exact sequence

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0\,,$$

where P is projective. Then $pd(N) = pd(M) - 1$. By induction, $\underline{\mathrm{Ext}}^n(k, N)$ and $\underline{\mathrm{Ext}}^n(k, P)$ are bounded, hence, so is $\underline{\mathrm{Ext}}^n(k, M)$. Now A/A_+ is finite and we have just shown $\underline{\mathrm{Ext}}^n(k, M)$ is finite (since bounded together with locally finite implies finite); then $\mathrm{Hom}(A/A_+, \underline{\mathrm{Ext}}^n(k, M)) \cong \underline{\mathrm{Ext}}^n(A/A_+, M)$ is locally finite and clearly bounded. Therefore, $\underline{\mathrm{Ext}}^n(A/A_+, M)$ is finite for every n and every finite M. □

Most of these examples are taken directly from the original article by *Artin* and *Zhang* [AZ94]. There is a host of other examples on algebras satisfying χ up to varying degrees, for which we refer the interested readers to [AZ94], [Rog02] and [SZ94]. Also one should take a look at [TVdB90] where these ideas, in some sense, germinated. Finally, a comprehensive survey article by *J. T. Stafford* and *M. van den Bergh* [SvdB01] should be consulted for further curiosities in the current state of affairs in noncommutative algebraic geometry.

3. Algebraic Aspects of Noncommutative Tori

The section is mostly based on the article *Noncommutative two-tori with real multiplication as noncommutative projective spaces* by A. Polishchuk [Pol04b]. Noncommutative two-tori with real multiplication will be explained by Jorge Plazas and by now we know what we mean by noncommutative projective varieties. In the following paragraph the gist of the paper has been provided (objects within quotes will be explained either by Jorge Plazas or by B. Noohi or the reader is expected to look it up for himself/herself). Interested readers are also encouraged to take a look at chapter 4 of [Man06], where *Manin* gives a very insightful overview of this work. We merely fill in some details here for pedagogical reasons.

One considers the category of "holomorphic vector bundles" on a noncommutative torus $\mathbb{T}_\theta$, θ being a real parameter, whose algebra of smooth functions is denoted A_θ.[a] We always assume θ to be irrational. In keeping with the general philosophy, $\mathbb{T}_\theta$ and A_θ will be used interchangeably

[a]Usually A_θ is used to denote the algebra of continuous functions and $\mathcal{A}_\theta$ is used to denote the smooth ones. To ease LaTeX-ing, the algebra of smooth functions has been consistently denoted by A_θ.

for a noncommutative torus. There is a fully faithful functor [PS03] from the "derived category of holomorphic bundles on a noncommutative torus A_θ" to the derived category of coherent sheaves on a complex elliptic curve X, denoted by $D^b(X)$ (refer to Remark 3.2), sending holomorphic bundles to the "heart" of a "t-structure" depending on θ, denoted by $\mathcal{C}^\theta$. The elliptic curve X is determined by the choice of a "complex structure" on A_θ, depending on a complex parameter τ in the lower half plane i.e. $X = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. It follows from [Pol04a] that the category of holomorphic vector bundles on A_θ is actually equivalent to the heart $\mathcal{C}^\theta$ and the "standard bundles" end up being the so-called "stable" objects of $D^b(X)$. We also know that the heart has "cohomological dimension" 1 and is derived equivalent to $D^b(X)$. The "real multiplication" of A_θ gives rise to an auto-equivalence, say F, of $D^b(X)$, which preserves the heart up to a shift. One knows when it actually preserves the heart, viz. when the matrix inducing the real multiplication has positive real eigenvalues. Now by choosing a "stable" object, say $\mathscr{G}$, in $D^b(X)$, one can construct graded algebras from the triple $\mathcal{C}^\theta$, $\mathscr{G}$ and F as described before (see (9)). Some criteria for the graded algebras to be generated in degree 1, quadratic and "Koszul" are also known. For the details one may refer to e.g. [Pla].

Remark 3.1. It is known that two noncommutative tori, say A_θ and $A_{\theta'}$, are Morita equivalent if $\theta' = g\theta$ for some $g \in SL(2,\mathbb{Z})$ [Rie81].

The equivalence defined in [PS03] between the "derived category of holomorphic bundles on A_θ" and $D^b(X)$ actually sends the holomorphic bundles on A_θ to $\mathcal{C}^{-\theta^{-1}}$ up to some shift and the real multiplication on A_θ descends to an element $F \in Aut(D^b(X))$, which preserves $\mathcal{C}^{-\theta^{-1}}$ (up to some shift). This is not too bad, as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\theta = -\theta^{-1}$ (action by fractional linear transformation), which says that $A_{-\theta^{-1}}$ is Morita equivalent to A_θ. The generators of $SL(2,\mathbb{Z})$ are $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The matrix g acts by translation by 1 and so up to Morita equivalence θ may be brought within the interval $[0,1]$ and θ, being irrational, $\theta \in (0,1)$. The image of this interval under $x \longmapsto -x^{-1}$ is $(-\infty,-1)$. We label the noncommutative torus by $A_{-\theta^{-1}}$, $-\theta^{-1} \in (-\infty,-1)$, so that when we pass on to the heart $\mathcal{C}^\theta$, $\theta \in (0,1)$.

Remark 3.2. By the bounded derived category of coherent sheaves one should actually understand $D^b(\mathrm{Coh}(X))$. However, we may not be able to find injective resolutions in $\mathrm{Coh}(X)$. So the precise category we want is $D^b(X) := D^b_{\mathrm{Coh}(X)}(\mathrm{QCoh}(X))$, *i.e.* the bounded derived category of complexes of quasi-coherent sheaves on X with the cohomology objects in $\mathrm{Coh}(X)$. According to Lemma 2.3 of [ST01] one knows that the two categories under consideration are equivalent. It is known that for a smooth curve X, every object of $D^b(X)$ (which is a complex) is quasi-isomorphic to the direct sum of its cohomologies.[b] This is not true in higher dimensions.

Let X be an elliptic curve over the complex numbers. For $F \in \mathrm{Coh}(X)$, let $\mathrm{rk}(F)$ stand for the generic rank of F and $\chi(F)$ for the Euler characteristic of F. Since X has genus 1, by Riemann-Roch the degree of F is the same as the Euler characteristic of F, i.e. $\deg(F) = \chi(F) := \dim_{\mathbb{C}}\mathrm{Hom}(\mathcal{O}_X, F) - \dim_{\mathbb{C}}\mathrm{Ext}^1(\mathcal{O}_X, F)$. So the *slope* of a coherent sheaf F, denoted by $\mu(F)$, which is just the rational number $\frac{\deg(F)}{\mathrm{rk}(F)}$, also equals $\frac{\chi(F)}{\mathrm{rk}(F)}$. The latter fraction is more suitable for our purposes and, hence, we take that as the definition of slope. We define the rank (resp. the Euler characteristic) of a complex in the derived category of $\mathrm{Coh}(X)$ as the alternating sum of the ranks (resp. the Euler characteristics) of the individual terms of the corresponding cohomology complex. Then the same definition as above extends the notion of slope to the objects of the derived category. A coherent sheaf F is called *semistable* (resp. *stable*) if for any nontrivial exact sequence $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ one has $\mu(F') \leq \mu(F)$ (resp. $\mu(F') < \mu(F)$) or equivalently $\mu(F) \leq \mu(F'')$ (resp. $\mu(F) < \mu(F'')$).

It is well known that every coherent sheaf on X splits as a direct sum of its torsion and torsion-free parts. Since X is smooth, projective and of dimension 1, every torsion-free coherent sheaf is locally free and for any $F \in \mathrm{Coh}(X)$ there exists a unique filtration [HN75]:

$$F = F_0 \supset F_1 \supset \cdots \supset F_n \supset F_{n+1} = 0 \tag{13}$$

such that

- F_i/F_{i+1} for $0 \leq i \leq n$ are semistable and
- $\mu(F_0/F_1) < \mu(F_1/F_2) < \cdots < \mu(F_n)$.

The filtration above is called the *Harder–Narasimhan filtration* of F and the graded quotients F_i/F_{i+1} are called the *semistable factors* of F. We set $\mu_{\min}(F) = \mu(F_0/F_1)$ and $\mu_{\max}(F) = \mu(F_n)$.

One calls an object $F \in D^b(X)$ *stable* if $F = V[n]$, where V is either a stable vector bundle (stable as above) or a coherent sheaf supported at a point (the stalk is the residue field).

[b]By induction, it is enough to show for complexes of length 2. Let $F^\bullet \in D^b(X)$.

$F^\bullet = \ldots \longrightarrow 0 \longrightarrow F^{-1} \xrightarrow{f} F^0 \longrightarrow 0 \longrightarrow \ldots.$

Consider the triangle: *ker* $f[1] \xrightarrow{\theta} F^\bullet \longrightarrow$ *cone* $\theta \xrightarrow{\xi}$ *ker* $f[2]$. Check that in $D^b(X)$ *cone* θ is *coker* f. Now $\xi \in \mathrm{Hom}(\mathit{coker}\ f, \mathit{ker}\ f[2]) = \mathrm{Hom}^2(\mathit{coker}\ f, \mathit{ker}\ f) = 0$. The last equality is due to *Fact* 2 (it appears later on). So $F^\bullet[1] = \mathit{coker}\ f[1] \oplus \mathit{ker}\ f[2]$. Note that *coker* f and *ker* f are the cohomologies of $F^\bullet$.

3.1. *t-structures on $D^b(X)$ depending on θ*

One way to obtain t-structures is via "torsion theories". So let us define a torsion pair $(\mathrm{Coh}_{>\theta}, \mathrm{Coh}_{\leq\theta})$ in $\mathrm{Coh}(X)$.

$$\mathrm{Coh}_{>\theta} := \{F \in \mathrm{Coh}(X) \ : \ \mu_{\min}(F) > \theta\},$$
$$\mathrm{Coh}_{\leq\theta} := \{F \in \mathrm{Coh}(X) \ : \ \mu_{\max}(F) \leq \theta\}.$$

We consider the full subcategories generated by these objects inside $\mathrm{Coh}(X)$. Notice that *torsion sheaves*, having slope $= \infty$, belong to $\mathrm{Coh}_{>\theta}$.

To show that this is indeed a torsion pair we need to verify two conditions.

1. $\mathrm{Hom}(T, F) = 0$ *for all* $T \in \mathrm{Coh}_{>\theta}$ *and* $F \in \mathrm{Coh}_{\leq\theta}$.

Lemma 3.3. *Let F and F' be a pair of semistable bundles. Then $\mu(F) > \mu(F')$ implies that* $\mathrm{Hom}(F, F') = 0$.

Proof. Suppose $f : F \longrightarrow F'$ is a nonzero morphism. Let G be the image of f. Then G is a quotient of F and so one has $\mu(G) \geq \mu(F)$. On the other hand, G is a torsion-free subsheaf of a vector bundle on a smooth curve and so it is locally free. Thus, one has $\mu(G) \leq \mu(F')$, which implies $\mu(F) \leq \mu(G) \leq \mu(F')$. Take the contrapositive to obtain the desired result. $\square$

Let $T \in \mathrm{Coh}_{>\theta}$ and $F \in \mathrm{Coh}_{\leq\theta}$. Further, suppose $\sigma \in \mathrm{Hom}(T, F)$. Let us write down the *Harder–Narasimhan filtrations* of T and F respectively.

$$0 \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_{m-1} \longrightarrow T_m = T$$

$$0 \longrightarrow F_0 \longrightarrow \cdots \longrightarrow F_{n-1} \longrightarrow F_n = F$$

Restrict $\sigma : T \longrightarrow F$ to T_0 and compose it with the canonical projection onto F_n/F_{n-1}. Now T_0 is a semistable factor of T and F_n/F_{n-1} that of F. Since $T \in \mathrm{Coh}_{>\theta}$ and $F \in \mathrm{Coh}_{\leq\theta}$, by the lemma above this map is 0. So the image lies in F_{n-1}. Apply the same argument after replacing F_n/F_{n-1} by F_{n-1}/F_{n-2} to conclude that the image lies in F_{n-2}. Iterating this process we may conclude that σ restricted to T_0 is 0. So σ factors through T/T_0. The *Harder–Narasimhan filtration* of T/T_0 is

$$0 \longrightarrow T_1/T_0 \longrightarrow \cdots \longrightarrow T_{m-1}/T_0 \longrightarrow T_m/T_0 = T/T_0$$

This filtration has the same semistable factors as that of T and so they satisfy the conditions of the *Harder–Narasimhan filtration*. So by the uniqueness of the *Harder–Narasimhan filtration* this is that of T/T_0. Iterate the argument above after replacing T by T/T_0 and taking the induced map of σ between T/T_0 and F to conclude that σ vanishes on T_1/T_0. But σ also vanishes on T_0. So it must vanish on T_1. Repeating this argument finitely many times one may show that σ vanishes on the whole of T.

2. *For every* $F \in \mathrm{Coh}(X)$ *there should be an exact sequence* (*necessarily unique up to isomorphism*)

$$0 \longrightarrow t(F) \longrightarrow F \longrightarrow F/t(F) \longrightarrow 0$$

such that $t(F) \in \mathrm{Coh}_{>\theta}$ *and* $F/t(F) \in \mathrm{Coh}_{\leq\theta}$.

Proof. Let $0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$ be the Harder–Narasimhan filtration of F. Let i be the unique integer such that $\mu(F_i/F_{i-1}) > \theta$ and $\mu(F_{i+1}/F_i) \leq \theta$. Then set $t(F) = F_i$. It is easy to see that $F_i \in \mathrm{Coh}_{>\theta}$ and $F/F_i \in \mathrm{Coh}_{\leq\theta}$. By the way, if no such i exists, then F is already either an element of $\mathrm{Coh}_{>\theta}$ or $\mathrm{Coh}_{\leq\theta}$. $\square$

Fact 1. (see for instance [HRS96]) *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair on an abelian category* $\mathcal{A}$. *Let* $\mathcal{C}$ *be the heart of the associated t-structure. Then* $\mathcal{C}$ *is an abelian category, equipped with a torsion pair* $(\mathcal{F}[1], \mathcal{T})$.

Recall that the *cohomological dimension* (perhaps, global dimension is a more appropriate term) of an abelian category $\mathcal{A}$ is the minimum integer n such that $\mathrm{Ext}^i(A, B) = 0$ for all $A, B \in \mathcal{A}$ and for all $i > n$ and ∞ if no such n exists.

Fact 2. [Ser55] *If* X *is a smooth projective curve* (*i.e.* $\dim X = 1$), *then the cohomological dimension of* $\mathrm{Coh}(X)$ *is* 1.

Now we shall associate a t-structure to this torsion pair (see for instance [HRS96]) as follows:

$$D^{\theta,\leq 0} := \left\{K \in D^b(X) \ : \ H^{>0}(K) = 0, H^0(K) \in \mathrm{Coh}_{>\theta}\right\},$$

$$D^{\theta,\geq 1} := \left\{K \in D^b(X) \ : \ H^{<0}(K) = 0, H^0(K) \in \mathrm{Coh}_{\leq\theta}\right\}.$$

It is customary to denote $D^{\theta,\leq 0}[-n]$ by $D^{\theta,\leq n}$ and $D^{\theta,\geq 0}[-n]$ by $D^{\theta,\geq n}$. Let $\mathcal{C}^\theta := D^{\theta,\leq 0} \cap D^{\theta,\geq 0}$ be the heart of the t-structure, which is known to be an abelian category. An interesting thing is that $(\mathrm{Coh}_{\leq\theta}[1], \mathrm{Coh}_{>\theta})$ defines a torsion pair on $\mathcal{C}^\theta$ (refer to *Fact* 1 above). As a matter of convention, the family of t-structures is extended to $\theta = \infty$ by putting the standard t-structure on it, whose heart is just $\mathrm{Coh}(X)$.

Our next aim is to show that $\mathcal{C}^\theta$ has cohomological dimension 1.

3.2. *ASIDE on Serre duality*

Let X be a smooth projective scheme of dimension n. Then there is a *dualizing sheaf* ω such that one has natural isomorphisms

$$H^i(X,F) \cong \mathrm{Ext}^{n-i}(F,\omega)^*,$$

where F is any coherent sheaf on X.

Remark 3.4. The definition of a *dualizing sheaf* exists for all proper schemes. For nonsingular projective varieties it is known that the dualizing sheaf is isomorphic to the canonical sheaf.

Definition 3.5. ([BO01] Def. 1.2, Prop. 1.3 and Prop. 1.4) *Let $\mathcal{D}$ be a k-linear triangulated category with finite dimensional* Hom*'s. An autoequivalence $S : \mathcal{D} \longrightarrow \mathcal{D}$ is called a Serre functor if there are bi-functorial isomorphisms*

$$\mathrm{Hom}_{\mathcal{D}}(A,B) \cong \mathrm{Hom}_{\mathcal{D}}(B,SA)^*$$

which are natural for all $A, B \in \mathcal{D}$.

Bondal and *Kapranov* have shown that in a reasonable manner Serre Duality of a smooth projective scheme X can be reinterpreted as the existence of a Serre functor (if it exists it is unique up to a graded natural isomorphism) on $D^b(X)$ [BK89].

It is also known that for a smooth projective variety of dimension n the Serre functor is $\_ \otimes \omega_X[n]$. For an elliptic curve X the Serre functor will be just the translation functor [1] (since $\dim X = 1$ and the canonical sheaf ω_X of an elliptic curve is trivial).

Lemma 3.6. *Any $F \in \mathrm{Coh}(X)$ is isomorphic to the direct sum of its semistable factors.*

Proof. The proof is by induction on the length of the Harder–Narasimhan filtration and for simplicity we treat only the case of length two. The category $\mathrm{Coh}(X)$ has the so-called *Calabi–Yau property*, which says that $\mathrm{Ext}^1(F,G) \cong \mathrm{Hom}(G,F)^*$ for all $F, G \in \mathrm{Coh}(X)$ (this follows from Serre duality as discussed above). Let $F \in \mathrm{Coh}(X)$ and let

$$0 \subset F_1 \subset F_2 = F$$

be its Harder–Narasimhan filtration.

Then its semistable factors are F_1 and $F/F_1 =: G$. Thus we obtain an exact sequence

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow G \longrightarrow 0,$$

where F_1 and G are semistable. By the properties of the Harder–Narasimhan filtration we have $\mu(G) < \mu(F_1)$. Due to the Calabi–Yau property we know that $\mathrm{Ext}^1(G, F_1) \cong \mathrm{Hom}(F_1, G)^*$. From Lemma 3.3 it follows that $\mathrm{Hom}(F_1, G) = 0$ and hence $\mathrm{Ext}^1(G, F_1) = 0$. Therefore the short exact sequence above splits and we obtain $F \cong F_1 \oplus G$. $\square$

Proposition 3.7. *$\mathcal{C}^\theta$ has cohomological dimension* 1.

Proof. First of all, observe that it is enough to show $\mathrm{Hom}^{>1}_{D^b(X)}(A, B) = 0$ for all A, B belonging to $\mathrm{Coh}_{>\theta}$ and $\mathrm{Coh}_{\leq\theta}$ only. Now $\mathrm{Coh}(X)$ has cohomological dimension 1 (*Fact* 2) and $\mathrm{Coh}_{>\theta}$ and $\mathrm{Coh}_{\leq\theta}$ are full subcategories of $\mathrm{Coh}(X)$. So if A, B were both either in $\mathrm{Coh}_{>\theta}$ or in $\mathrm{Coh}_{\leq\theta}[1]$ then there would have been nothing to prove. Let $A \in \mathrm{Coh}_{>\theta}$ and $B \in \mathrm{Coh}_{\leq\theta}[1]$.

Then $\mathrm{Hom}^i_{D^b(X)}(A, B) = \mathrm{Hom}^{i+1}_{D^b(X)}(A, B[-1])$. But $B \in \mathrm{Coh}_{\leq\theta}$, which is a subcategory of $\mathrm{Coh}(X)$. So $\mathrm{Hom}^{i+1}_{D^b(X)}(A, B[-1]) = 0$ for all $i \geq 1$. On the other hand,

$$\begin{aligned}
\mathrm{Hom}^i_{D^b(X)}(B, A) &= \mathrm{Hom}^{i-1}_{D^b(X)}(B[-1], A) \\
&= \mathrm{Hom}_{D^b(X)}(B[-1], A[1-i]) \\
&\cong \mathrm{Hom}_{D^b(X)}(A[1-i], B)^* \quad \text{(use Serre functor} = [1] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as explained before)} \\
&= \mathrm{Hom}_{D^b(X)}(A[2-i], B[-1])^* \\
&= \mathrm{Hom}^{2-i}_{D^b(X)}(A, B[-1])^*
\end{aligned}$$

So, for $i > 2$, $\mathrm{Hom}^i_{D^b(X)}(B, A)$ is evidently 0. Due to the first axiom of a torsion pair, $\mathrm{Hom}_{\mathrm{Coh}(X)}(A, B[-1]) = 0$ when $i = 2$. $\square$

Proposition 3.8. *The categories $\mathcal{C}^\theta$ and* $\mathrm{Coh}(X)$ *are derived equivalent, i.e. $D^b(\mathcal{C}^\theta) \cong D^b(X)$.*

Proof. It is known that if a torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ is *cotilting*, i.e. *every object of $\mathcal{A}$ is a quotient of an object in $\mathcal{F}$*, then the heart of the t-structure induced by the torsion pair is derived equivalent to $\mathcal{A}$ (see for instance Proposition 5.4.3 and the remark thereafter in [BvdB03]). Thus, it is enough to check that the torsion pair $(\mathrm{Coh}_{>\theta}, \mathrm{Coh}_{\leq\theta})$ is cotilting.

Given any $F \in \mathrm{Coh}(X)$ we need to produce an object in $\mathrm{Coh}_{\leq\theta}$ which surjects onto F. Let L be an ample line bundle on X, i.e. $\deg(L) > 0$. By Serre's theorem one may twist F by a large enough power of L such that it becomes generated by global sections, i.e. the quotient of a free sheaf. In other words, there exists $N > 0$ large enough such that for all $n > N$ there is an epimorphism $\oplus_{i\in I}\mathcal{O}_X \longrightarrow F \otimes L^n$, I finite. One may twist it back to obtain an epimorphism $\oplus_{i\in I}\check{L}^n \longrightarrow F$, where $\check{L}$ is the dual line bundle.

This shows that there exists an epimorphism onto F from a finite direct sum of copies of $\check{L}^n$. Since $\deg(\check{L}^n) = -n.\deg(L) < 0$ it is possible to make the slope of $\check{L}^n$, which is equal to $\deg(\check{L}^n)$, less than θ by choosing a large enough n. Being a line bundle $\check{L}^n$ is clearly semistable and we observe that the direct sum of copies of $\check{L}^n$ lies in $\mathrm{Coh}_{\leq\theta}$. □

Remark 3.9. In fact, all "bounded" t-structures on $D^b(X)$ come from some cotilting torsion pair in $\mathrm{Coh}(X)$. All such cotilting torsion pairs (up to an action of $Aut(D^b(X))$) have been listed in [GKR04]. I am especially thankful to *S. A. Kuleshov* for explaining the above argument.

Since $\mathcal{C}^\theta$ has cohomological dimension 1, if it were equivalent to $\mathrm{Coh}(Y)$ for some Y, then Y had better be a smooth curve (cf., *Fact* 2). The problem of dealing with categories of holomorphic bundles on A_θ has been reduced to studying t-structures on $D^b(X)$.

We have already seen some technical conditions involving a categorical incarnation of "ampleness" 2.6 to verify when a given k-linear ($k = \mathbb{C}$ now) abelian category is of the form $\mathrm{Proj}\, R$ for some graded k-algebra R. One of the requirements of Theorem 2.8 is that the category be locally Noetherian (i.e. the category has a Noetherian set of generators). Unfortunately, this condition fails to be true in our situation.

Proposition 3.10. *θ irrational implies that every nonzero object in $\mathcal{C}^\theta$ is not Noetherian.*

We would still like to say that what we have seen so far was not entirely useless. Recall that in our discussion of $\mathrm{Proj}\, R$ after *Artin* and *Zhang* we had assumed our graded algebra to be right Noetherian. *Polishchuk* has shown that even if one dispenses with the Noetherian assumption there is a way to recover *Serre's Theorem* 2.8. He gives an analogue of an "ample sequence of objects" and proves that if a k-linear abelian category has an ample sequence of objects then it is equivalent to "$\mathrm{cohproj}\, R$", where R is a "coherent" $\mathbb{Z}$-algebra. Unfortunately the words in quotes in the previous sentence will not be explained anymore. Interested readers are encouraged to look them up from [Pol05]. Finally, as an apt culmination of all our efforts we have the following theorem [Pol04b].

Theorem 3.11. (Polishchuk) *For every quadratic irrationality $\theta \in \mathbb{R}$ there exists an auto-equivalence $F : D^b(X) \longrightarrow D^b(X)$ preserving $\mathcal{C}^\theta$ and a stable object $\mathscr{G} \in \mathcal{C}^\theta$ such that the sequence $(F^n\mathscr{G}, n \in \mathbb{Z})$ is ample (in the modified sense of Polishchuk). Hence, the corresponding algebra $A_{F,\mathscr{G}} := \Gamma_h(\mathcal{C}^\theta, \mathscr{G}, F)$ is right coherent and $\mathcal{C}^\theta \simeq \mathrm{cohproj}\, A_{F,\mathscr{G}}$.*

Remark 3.12. There is some anomaly in the choice of the algebra R, whose $\mathrm{cohproj}\, R$ should be equivalent to $\mathcal{C}^\theta$. However, even in the commutative case one can show that if S and S' are two graded commutative rings, such that $S_n \cong S'_n$ for all $n \gg 0$, then $\mathrm{Proj}\, S \simeq \mathrm{Proj}\, S'$ (commutative Proj construction).

Final Remark. "Another perspective for the future work is to try to connect our results with Manin's program in [Man06] to use noncommutative two-tori with real multiplication for the explicit construction of the maximal Abelian extensions of real quadratic fields". ——— *A. Polishchuk.*

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References

[AVdB90] M. Artin and M. Van den Bergh, Twisted homogeneous coordinate rings, *J. Alg.* **133**, 249–271, 1990.

[AZ94] M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. Math.* **109**, 228–287, 1994.

[BK89] A. I. Bondal and M. M. Kapranov, Representable functors, Serre functors and reconstructions, *Izv. Akad. Nauk SSSR Ser. Mat.* **53**, 1183–1205, 1337, 1989.

[BO01] A. I. Bondal and D. Orlov, Reconstruction of a variety from the derived category and groups of autoequivalences, *Compositio Math.* **125**, 327–344, 2001.

[BvdB03] A. Bondal and M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.* **3**, 1–36, 258, 2003.

[Gab62] P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90**, 323–448, 1962.

[GKR04] A. L. Gorodentsev, S. A. Kuleshov and A. N. Rudakov, t-stabilities and t-structures on triangulated categories, *Izv. Ross. Akad. Nauk Ser. Mat.* **68**, 117–150, 2004.

[Gro57] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tôhuku Math. J.* **9**, 119–221, 1957.

[Har77] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977.

[HN75] G. Harder and M. S. Narasimhan, On the cohomology groups of moduli spaces of vector bundles on curves, *Math. Ann.* **212**, 215–248, 1974/75.

[HRS96] D. Happel, I. Reiten and S. O. Smalø, Tilting in abelian categories and quasitilted algebras, *Mem. Amer. Math. Soc.* **120**(575):viii+ 88, 1996.

[KL85] G. R. Krause and T. H. Lenagan, Growth of algebras and Gelfand–Kirillov dimension, *Research Notes in Math.*, Vol. 116, Pitman, Boston, 1985.

[Man88] Yu. I. Manin, Quantum groups and noncommutative geometry, Univ. de Montréal, Centre de Recherches Math., Montreal, QC, 1988.

[Man06] Yu. I. Manin, The notion of dimension in geometry and algebra, *Bull. Amer. Math. Soc.* (*N.S.*) **43**, 139–161 (electronic), 2006.

[Pla] J. Plazas, Arithmetic structures on noncommutative tori with real multiplication, math.QA/0610127.

[Pol04a] A. Polishchuk, Classification of holomorphic vector bundles on noncommutative two-tori, *Doc. Math.* **9**, 163–181 (electronic), 2004.

[Pol04b] A. Polishchuk, Noncommutative two-tori with real multiplication as noncommutative projective varieties, *J. Geom. Phys.* **50**, 162–187, 2004.

[Pol05] A. Polishchuk, Noncommutative proj and coherent algebras, *Math. Res. Lett.* **12**, 63–74, 2005.

[PS03] A. Polishchuk and A. Schwarz, Categories of holomorphic vector bundles on noncommutative two-tori, *Comm. Math. Phys.* **236**, 135–159, 2003.

[Rie81] M. A. Rieffel, C^*-algebras associated with irrational rotations, *Pacific J. Math.* **93**, 415–429, 1981.

[Rog02] D. Rogalski, *Examples of generic noncommutative surfaces*, PhD thesis, University of Michigan, May 2002.

[Ros98] A. Rosenberg, The spectrum of abelian categories and reconstructions of schemes, *Rings, Hopf Algebras, and Brauer groups, Lectures Notes in Pure and Appl. Math.*, Vol. 197, Marcel Dekker, New York, 257–274, 1998.

[Ser55] J. P. Serre, Faisceaux algébriques cohérents, *Ann. Math.* **61**, 197–278, 1955.

[ST01] P. Seidel and R. Thomas, Braid group actions on derived categories of coherent sheaves, *Duke Math. J.* **108**, 37–108, 2001.

[Ste75] B. Stenström, Rings of quotients, *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*, Vol. 217, Springer Verlag, Berlin, 1975.

[SvdB01] J. T. Stafford and M. van den Bergh, Noncommutative curves and noncommutative surfaces, *Bull. Amer. Math. Soc.* (*N.S.*) **38**, 171–216 (electronic), 2001.

[SZ94] J. T. Stafford and J. J. Zhang, Examples in non-commutative projective geometry, *Math. Proc. Cambridge Phil. Soc.* **116**, 415–433, 1994.

[TVdB90] M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, *The Grothendieck Festschrift*, Vol. 1, Birkhäuser, Boston, 33–85, 1990.

[TVdB91] M. Artin, J. Tate and M. Van den Bergh, Modules over regular algebras of dimension 3, *Invent. Math.* **106**, 335–388, 1991.

[Ver92] A. B. Verevkin, On a non-commutative analogue of the category of coherent sheaves on a projective scheme, *Amer. Math. Soc. Transl.* **151**, 41–53, 1992.

[Zha96] J. J. Zhang, Twisted graded algebras and equivalences of graded categories, *Proc. London Math. Soc.* (*3*) **72**, 281–311, 1996.

An Invitation to Noncommutative Geometry

LECTURES ON DERIVED AND TRIANGULATED CATEGORIES

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Contents

These are the notes of three lectures given in the International Workshop on Noncommutative Geometry held in I.P.M., Tehran, Iran, September 11–22, 2005.

The first lecture is an introduction to the basic notions of *abelian category theory*, with a view toward their algebraic geometric incarnations (as categories of modules over rings or sheaves of modules over schemes).

In the second lecture, we motivate the importance of *chain complexes* and work out some of their basic properties. The emphasis here is on the notion of *cone* of a chain map, which will consequently lead to the notion of an exact triangle of chain complexes, a generalization of the cohomology long exact sequence. We then discuss the homotopy category and the *derived category* of an abelian category, and highlight their main properties.

As a way of formalizing the properties of the cone construction, we arrive at the notion of a *triangulated category.* This is the topic of the third lecture. After presenting the main examples of triangulated categories (i.e., various homotopy/derived categories associated to an abelian category), we discuss the problem of constructing abelian categories from a given triangulated category using t-structures.

A word on style. In writing these notes, we have tried to follow a *lecture style* rather than an *article style.* This means that, we have tried to be very concise, keeping the explanations to a minimum, but not less (hopefully). The reader may find here and there certain remarks written in small fonts; these are meant to be side notes that can be skipped without affecting the flow of the material. With a few insignificant exceptions, the topics are arranged in linear order.

A word on references. The references given in these lecture notes are mostly suggestions for further reading and are not meant for giving credit or to suggest originality.

Lecture 1: Abelian Categories

Overview. In this lecture, we introduce *additive* and *abelian* categories, and discuss their most basic properties. We then concentrate on the examples of abelian categories that we are interested in. The most fundamental example is the category of *modules* over a ring. The next main class of examples consists of various categories of *sheaves of modules* over a space; a special type of these examples, and a very important one, is the category of *quasi-coherent sheaves* on a scheme. The idea here is that one can often recover a space from an appropriate category of sheaves on it. For example, we can recover a scheme from the category of quasi-coherent sheaves on it. This point of view allows us to think of an abelian category as "a certain category of sheaves on a certain hypothetical space". One might also attempt to extract the "ring of functions" of this hypothetical space. Under some conditions this is possible, but the ring obtained is not unique. This leads to the notion of *Morita equivalence* of rings. The slogan is that the "ring of functions" on a noncommutative space is well defined only up to Morita equivalence.

1. **Products and Coproducts in Categories** ([HiSt], [We], [Fr], [Ma], [GeMa])

Let C be a category and $\{X_i\}_{i\in I}$ a set of objects in C.

The **product** $\prod_{i\in I} X_i$ is an object in C, together with a collection of morphisms $\pi_i\colon \prod_{i\in I} X_i \to X_i$, satisfying the following universal property:

Given any collection of morphisms $f_i\colon Y \to X_i$,

$$\begin{tikzcd} & \prod_{i\in I} X_i \arrow[d, "\pi_i"] \\ Y \arrow[ur, dotted, "\exists!\, f"] \arrow[r, "f_i"'] & X_i \end{tikzcd}$$

The **coproduct** $\coprod_{i\in I} X_i$ is an object in C, together with a collection of morphisms $\iota_i\colon X_i \to \coprod_{i\in I} X_i$, satisfying the following universal property:

Given any collection of morphisms $g_i\colon X_i \to Y$,

$$\begin{tikzcd} & \coprod_{i\in I} X_i \arrow[dl, dotted, "\exists!\, f"'] \\ Y & X_i \arrow[l, "g_i"] \arrow[u, "\iota_i"'] \end{tikzcd}$$

Remark 1.1. Products and coproducts may or may not exist, but if they do they are unique up to canonical isomorphism.

Example 1.2.

1. $\mathsf{C} = \textbf{Sets}$: $\coprod =$ disjoint union, $\prod =$ cartesian product.
2. $\mathsf{C} = \textbf{Groups}$: $\coprod =$ free product, $\prod =$ cartesian product.
3. $\mathsf{C} = \textbf{UnitalCommRings}$: $\coprod\limits_{\text{finite}} = \otimes_{\mathbb{Z}}$, $\prod =$ cartesian product.
4. $\mathsf{C} = \textbf{Fields}$: $\coprod =$ does not exist, $\prod =$ does not exist.
5. $\mathsf{C} = R\textbf{-Mod} =$ left R-modules, R a ring: $\coprod\limits_{\text{finite}} = \prod\limits_{\text{finite}} = \oplus$, the usual direct sum of modules.

Exercise. Show that in R-**Mod** there is a natural morphism $\coprod_{i\in I} X_i \to \prod_{i\in I} X_i$, and give an example where this is not an isomorphism.

2. Abelian Categories ([HiSt], [We], [Fr], [Ma], [GeMa])

We discuss three types of categories: *Ab-categories*, *additive categories*, and *abelian categories*. Each type of category has more structure/properties than the previous one.

An *Ab*-**category** is a category C with the following *extra structure*: each $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ is endowed with the structure of an abelian group. We require that composition is linear:

$$X \xrightarrow{u} Y \underset{g}{\overset{f}{\rightrightarrows}} Z \xrightarrow{v} T$$

$$u(f+g) = uf + ug, \quad (f+g)v = fv + gv.$$

An **additive functor** between *Ab*-categories is a functor that induces group homomorphisms on Hom-sets.

An **additive** category is a special type of *Ab*-category. More precisely, an additive category is an *Ab*-category with the following *properties*:

- ▶ There exists a *zero* object 0 such that

$$\forall X, \quad \mathrm{Hom}_{\mathsf{C}}(0, X) = \{0\} = \mathrm{Hom}_{\mathsf{C}}(X, 0).$$

- ▶ Finite sums exist. (Equivalently, finite products exist; see Proposition 2.1.)

Before defining abelian categories, we discuss some basic facts about *Ab*-categories.

Proposition 2.1. *Let* C *be an Ab-category with a zero object, and let* $\{X_i\}_{i \in I}$ *be a finite set of objects in* C*. Then* $\coprod X_i$ *exists if and only if* $\prod X_i$ *exists. In this case,* $\coprod X_i$ *and* $\prod X_i$ *are naturally isomorphic.*

Proof. Exercise. (Or see [HiSt].) □

Notation. Thanks to Proposition 2.1, it makes sense to use the same symbol $\oplus$ for both product and coproduct (of finitely many objects) in an additive category.

Exercise. In Example 1.2, which ones can be made into additive categories?

A morphism $f \colon B \to C$ in an additive category is called a **monomorphism** if

$$\boxed{\forall\ X \xrightarrow{g} B \xrightarrow{f} C, \quad f \circ g = 0 \quad \Rightarrow \quad g = 0.}$$

A **kernel** of a morphism $f \colon B \to C$ in an additive category is a morphism $i \colon A \to B$ such that $f \circ i = 0$, and that for every $g \colon X \to B$ with $f \circ g = 0$

$$\begin{tikzcd} A \arrow[r, "i"] & B \arrow[r, "f"] & C \\ & X \arrow[u, "g"] \arrow[ul, dotted, "\exists!"'] \arrow[ur, "0"'] & \end{tikzcd}$$

Proposition 2.2. *If* i *is a kernel for some morphism, then* i *is a monomorphism. The converse is not always true.*

We can also make the dual definitions.

A morphism $f \colon B \to C$ in an additive category is called an **epimorphism** if

$$\boxed{\forall\ B \xrightarrow{f} C \xrightarrow{h} D, \quad h \circ f = 0 \quad \Rightarrow \quad h = 0.}$$

A **cokernel** of a morphism $f \colon B \to C$ in an additive category is a morphism $p \colon C \to D$ such that $p \circ f = 0$, and such that for every $h \colon B \to Y$ with $h \circ f = 0$

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "0"'] & C \arrow[r, "p"] \arrow[d, "h"] & D \arrow[dl, dotted, "\exists!"] \\ & Y & \end{tikzcd}$$

Proposition 2.3. *If p is a cokernel for some morphism, then p is an epimorphism. The converse is not always true.*

Remark 2.4. Kernels and cokernels may or may not exist, but if they do they are unique up to a canonical isomorphism.

Example 2.5.

1. In $\mathsf{A} = R$-**Mod** kernels and cokernels always exist.
2. Let A be the category of finitely generated free $\mathbb{Z}$-modules. Then kernels and cokernels always exist. (**Exercise.** What is the cokernel of $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$?)
3. Let A be the category of $\mathbb{C}$-vector spaces of even dimension. Then kernels and cokernels do not always exist in A. (Give an example.) The same thing is true if A is the category of infinite dimensional vector spaces.

An **abelian** category is an additive category A with the following properties:

- Kernels and cokernels always exist in A.
- Every monomorphism is a kernel and every epimorphism is a cokernel.

Main example. For every ring R, the additive category R-**Mod** is abelian.

Remark 2.6. Note that an *Ab*-category is a category *with an extra structure*. However, an additive category is just an *Ab*-category which satisfies some property (but no additional structure). An abelian category is an additive category which satisfies some more properties.

Exercise. In Example 2.5 show that (**1**) is abelian, but (**2**) and (**3**) are not. In (**2**) the map $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ is an epimorphism that is not the cokernel of any morphism, because it is also a monomorphism!

Proposition 2.7. *Let $f: B \to C$ be a morphism in an abelian category. Let $i: \ker(f) \to B$ be its kernel and $p: C \to \operatorname{coker}(f)$ its cokernel. Then* *there is a natural isomorphism* $\operatorname{coker}(i) \xrightarrow{\sim} \ker(p)$ *fitting in the following commutative diagram*:

$$\begin{tikzcd}
\ker(f) \arrow[r, hook, "i"] & B \arrow[r, "f"] \arrow[d, two heads] & C \arrow[r, two heads, "p"] & \operatorname{coker}(f) \\
& \operatorname{coker}(i) \arrow[r, "\sim"] & \ker(p) \arrow[u, hook] &
\end{tikzcd}$$

Corollary 2.8. *Every morphism* $f\colon B \to C$ *in an abelian category has a unique* (*up to a unique isomorphism*) *factorization*

$$\begin{tikzcd}
B \arrow[rr, "f"] \arrow[dr, two heads, "f_{epi}"'] & & C \\
& I \arrow[ur, hook, "f_{mono}"'] &
\end{tikzcd}$$

Corollary 2.9. *In an abelian category, mono + epi* $\Leftrightarrow$ *iso.*

The object I (together with the two morphisms f_{epi} and f_{mono}) in the above corollary is called the **image** of f and is denoted by $\operatorname{im}(f)$. The morphism f_{mono} factors through every monomorphism into C through which f factors. Dually, f_{epi} factors through every epimorphism originating from B through which f factors. Either of these properties characterizes the image.

In an abelian category, a sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called **exact** if f is a monomorphism, g is an epimorphism, and $\operatorname{im}(f) = \ker(g)$. An additive functor between abelian categories is called exact if it takes exact sequences to exact sequences.

We saw that R-modules form an abelian category for every ring R. In fact, every small abelian category is contained in some R-**Mod**. (A category is called *small* if its objects form a set.)

Theorem 2.10. (Freyd–Mitchell Embedding Theorem [Fr, Mil]) *Let* A *be a small abelian category. Then there exists a unital ring* R *and an exact fully faithful functor* $\mathsf{A} \to R$-**Mod**.

3. Categories of Sheaves ([Ha1], [KaSch], [Iv], [GeMa])

The main classes of examples of abelian categories are categories of sheaves over spaces. We give a quick review of sheaves and describe the abelian category structure on them.

Let C be an arbitrary category (base) and A another category (values).

A **presheaf** on C with values in A is a functor $\mathcal{F}\colon \mathsf{C}^{op} \to \mathsf{A}$. A morphism $f\colon \mathcal{F} \to \mathcal{G}$ of presheaves is a natural transformation of functors. The category of presheaves is denoted by $\mathbf{PreSh}(\mathsf{C}, \mathsf{A})$.

Typical example. Let X be a topological space, and let $\mathsf{C} = \mathsf{Open}_X$ be the category whose objects are open sets of X and whose morphisms are inclusions. Let $\mathsf{A} = \mathsf{Ab}$, the category of abelian groups. A presheaf $\mathcal{F}$ of abelian groups on X consists of:

- ▷ A collection of abelian groups
$$\mathcal{F}(U), \quad \forall\, U \text{ open};$$
- ▷ "Restriction" homomorphisms
$$\mathcal{F}(U) \to \mathcal{F}(V), \quad \forall\, V \subseteq U.$$
Restriction homomorphisms should respect triple inclusions $W \subseteq V \subseteq U$ and be equal to the identity for $U \subseteq U$.

Example 3.1.

1. (*Pre*)*sheaf of continuous functions on a topological space* X. The assignment
$$U \mapsto \mathcal{O}_X^{cont}(U) = \{\text{continuous functions on } U\}$$
is a presheaf on X. The restriction maps are simply restriction of functions. In fact, this is a presheaf of *rings* because restriction maps are ring homomorphisms.

2. *Constant presheaf.* Let A be an abelian group. The assignment
$$U \mapsto A$$
is a presheaf of groups. The restriction maps are the identity maps.

Variations. There are many variations on these examples. For instance, in (**1**) one can take the (pre)sheaf of C^r functions on a C^r-manifold, or (pre)sheaf of holomorphic functions on a complex manifold, and so on. These are called *structure sheaves. Idea:* Structure sheaves encode all the information about the structure in question (e.g, C^r, analytic, holomorphic, etc.). So, for instance, a complex manifold X can be recovered from its *underlying topological space* X^{top} and the *sheaf* $\mathcal{O}_X^{holo}$. That is, we can think of the pair $(X^{top}, \mathcal{O}_X^{holo})$ as a complex manifold.

Exercise. Formulate the notion of a holomorphic map of complex manifolds purely in terms of the pair $(X^{top}, \mathcal{O}_X^{holo})$.

Proposition 3.2. *Let* C *be an arbitrary category, and* A *an abelian category. Then* $\mathbf{PreSh}(\mathsf{C}, \mathsf{A})$ *is an abelian category.*

The kernel and cokernel of a morphism $f\colon \mathcal{F} \to \mathcal{G}$ of presheaves are given by

$$\boxed{\begin{array}{rl} \ker(f)\colon & U \mapsto \ker\left(\mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U)\right) \\ \operatorname{coker}(f)\colon & U \mapsto \operatorname{coker}\left(\mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U)\right) \end{array}}$$

A presheaf $\mathcal{F}$, say of abelian groups, rings etc., on X is called a **sheaf** if for every open $U \subseteq X$ and every open cover $\{U_\alpha\}$ of U the sequence

$$\boxed{\begin{array}{l} \mathcal{F}(U) \xrightarrow{res} \prod_\alpha \mathcal{F}(U_\alpha) \longrightarrow \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta) \\ \qquad\qquad\qquad (f_\alpha) \mapsto (f_\alpha|_{U_\alpha \cap U_\beta} - f_\beta|_{U_\alpha \cap U_\beta}) \end{array}}$$

is exact.

Example 3.3. Structure sheaves (e.g., Example 3.1.1) are sheaves! More generally, every vector bundle $E \to X$ gives rise to a sheaf of vector spaces via the assignment $U \mapsto E(U)$, where $E(U)$ stands for the space of sections of E over U. Is a constant presheaf (Example 3.1.2) a sheaf?

For an abelian category A, we denote the full subcategory of $\mathbf{PreSh}(X, \mathsf{A})$ whose objects are sheaves by $\mathbf{Sh}(X, \mathsf{A})$.

Proposition 3.4. *The category* $\mathbf{Sh}(X, \mathsf{A})$ *is abelian.*

Remark 3.5.

1. Monomorphisms in $\mathbf{Sh}(X, \mathsf{A})$ are the same as the ones in $\mathbf{PreSh}(X, \mathsf{A})$, but epimorphisms are different: $f\colon \mathcal{F} \to \mathcal{G}$ is an epimorphism if for every open U and every $a \in \mathcal{G}(U)$,

$$\boxed{\begin{array}{l} \exists\ \{U_\alpha\},\ \text{open cover of } U,\ \text{such that:} \\ \forall\, \alpha,\ a|_{U_\alpha} \text{ is in the image of } f(U_\alpha)\colon \mathcal{F}(U_\alpha) \to \mathcal{G}(U_\alpha). \end{array}}$$

2. Kernels in $\mathbf{Sh}(X, \mathsf{A})$ are defined in the same way as kernels in $\mathbf{PreSh}(X, \mathsf{A})$, but cokernels are defined differently: if $f\colon \mathcal{F} \to \mathcal{G}$

is a morphism of sheaves, the cokernel of f is the

$$\boxed{\text{sheaf associated to the presheaf} \quad U \mapsto \operatorname{coker}\big(\mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U)\big).}$$

We remark that there is a general procedure for producing a sheaf $\mathcal{F}^{sh}$ out of a presheaf $\mathcal{F}$. This is called *sheafification*. The sheaf $\mathcal{F}^{sh}$ is called the *sheaf associated* to $\mathcal{F}$ and it comes with a natural morphism of presheaves $i\colon \mathcal{F} \to \mathcal{F}^{sh}$ which is universal among morphisms $\mathcal{F} \to \mathcal{G}$ to sheaves $\mathcal{G}$. More details on this can be found in [Ha1].

4. Abelian Category of Quasi-Coherent Sheaves on a Scheme ([Ha1], [GeMa])

We give a super quick review of schemes. We then look at the category of quasi-coherent sheaves on a scheme.

The **affine scheme** $(\operatorname{Spec} R, \mathcal{O}_R)$ associated to a commutative unital ring R is a topological space $\operatorname{Spec} R$, the space of prime ideals in R, together with a sheaf of rings $\mathcal{O}_R$, the *structure sheaf*, on it. Recall that, in the topology of $\operatorname{Spec} R$ (called the *Zariski topology*) a closed set is the set $V(I)$ of all prime ideals containing a given ideal I. The sheaf $\mathcal{O}_R$ is uniquely determined by the fact that, for every $f \in R$, the ring of sections of $\mathcal{O}_R$ over the open set $U_f := \operatorname{Spec} R - V(f)$ is the localization $R_{(f)}$, that is, $\mathcal{O}_R(U_f) = R_{(f)}$. In particular, $\mathcal{O}_R(\operatorname{Spec} R) = R$. (Note: the open sets U_f form a basis for the Zariski topology on $\operatorname{Spec} R$.)

A **scheme** X consists of a pair $(X^{Zar}, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X^{Zar}. We require that X can be covered by open sets U such that each $(U, \mathcal{O}_U)$ is isomorphic to an affine scheme. (Here, $\mathcal{O}_U$ is the restriction of $\mathcal{O}_X$ to U.)

Remark 4.1. This definition is modeled on Example 3.1.1; in particular, read the paragraph after the example.

A **sheaf of modules** over a scheme X is a sheaf $\mathcal{F}$ of abelian groups on X^{Zar} such that, for every open $U \subseteq X^{Zar}$, $\mathcal{F}(U)$ is endowed with an $\mathcal{O}_X(U)$-module structure (and restriction maps respect the module structure).

A sheaf of modules $\mathcal{F}$ over X is called **quasi-coherent** if for every inclusion of the form $\operatorname{Spec} S = V \subseteq U = \operatorname{Spec} R$ of open sets in X^{Zar} we have

$$\boxed{\mathcal{F}(V) \cong S \otimes_R \mathcal{F}(U).}$$

Proposition 4.2. *The category* $\mathcal{O}_X$-**Mod** *of* $\mathcal{O}_X$*-modules on a scheme* X *is an abelian category. The full subcategory* **Quasi-Coh**$_X$ *is also an abelian category.*

Example 4.3. To an R-module M there is associated a quasi-coherent sheaf $\widetilde{M}$ on $X = \operatorname{Spec} R$ which is characterized by the property that $\widetilde{M}(U_f) = M_{(f)}$, for every $f \in R$. In fact, every quasi-coherent sheaf on $\operatorname{Spec} R$ is of this form. More precisely, we have an equivalence of categories ([Ha1], Corollary II.5.5)

$$\boxed{\mathbf{Quasi\text{-}Coh}_{\operatorname{Spec} R} \cong R\text{-}\mathbf{Mod}.}$$

The category **Quasi-Coh**$_X$ is a natural abelian category associated with a scheme X. This allows us to do homological algebra on schemes (e.g., *sheaf cohomology*). The following reconstruction theorem states that, indeed, **Quasi-Coh**$_X$ captures all the information about X.

Proposition 4.4. (Gabriel–Rosenberg Reconstruction Theorem [Ro]) *A scheme* X *can be reconstructed, up to isomorphism, from the abelian category* **Quasi-Coh**$_X$*.*

More generally, Rosenberg [Ro], building on the work of Gabriel, associates to an abelian category A a topological space $\operatorname{Spec} \mathsf{A}$ together with a sheaf of rings $\mathcal{O}_{\mathsf{A}}$ on it. In the case where $\mathsf{A} =$ **Quasi-Coh**$_X$, the pair $(\operatorname{Spec} \mathsf{A}, \mathcal{O}_{\mathsf{A}})$ is naturally isomorphic to $(X^{Zar}, \mathcal{O}_X)$.

The above theorem is a starting point in non-commutative algebraic geometry. It means that one can think of the abelian category **Quasi-Coh**$_X$ itself as a space.

5. Morita Equivalence of Rings ([We], [GeMa])

We saw that, by the Gabriel–Rosenberg Reconstruction Theorem, we can regard the abelian category **Quasi-Coh**$_X$ as being "the same" as the scheme X itself. The Gabriel–Rosenberg Reconstruction Theorem has a more classical precursor.

Theorem 5.1. (Gabriel [Ga, Fr]) *Let* R *be a unital ring* (*not necessarily commutative*)*. Then* R-**Mod** *has a small projective generator* (*e.g.*, R *itself*), *and is closed under arbitrary coproducts. Conversely, let* A *be an abelian category with a small projective generator* P *which is closed under arbitrary coproducts. Let* $R = (\operatorname{End} P)^{op}$*. Then* $\mathsf{A} \cong R$-**Mod***.*

Remark 5.2. This, however, is not exactly a reconstruction theorem: the projective generator P is never unique, so we obtain various rings $S = \operatorname{End} P$ such that $\mathsf{A} \cong S$-**Mod**. Nevertheless, all such rings are regarded as giving the *same* "noncommutative scheme". So, from the point of view of noncommutative geometry they are the same. By the Gabriel-Rosenberg Reconstruction Theorem, if such R and S are both commutative, then they are necessarily isomorphic.

Two rings R and S are called **Morita equivalent** if R-**Mod** $\cong S$-**Mod**.

Example 5.3. For any ring R, the ring $S = M_n(R)$ of $n \times n$ matrices over R is Morita equivalent to R. (**Proof.** In the above theorem take $P = R^{\oplus n}$, or use the next theorem.)

The following characterization of Morita equivalence is important. The proof is not difficult.

Theorem 5.4. *Let R and S be rings. Then the following are equivalent*:

(i) *The categories R-**Mod** and S-**Mod** are equivalent.*
(ii) *There is an $S - R$ bimodule M such that the functor $M \otimes_R -\colon$ R-**Mod** $\to S$-**Mod** is an equivalence of categories.*
(iii) *There is a finitely generated projective generator P for R-**Mod** such that $S \cong \operatorname{End} P$.*

6. Appendix: Injective and Projective Objects in Abelian Categories

We will need to deal with injective and projective objects in the next lecture, so we briefly recall their definition.

An object P in an abelian category is called **projective** if it has the following lifting property:

$$\boxed{\begin{tikzcd} & B \arrow[d, "\text{epi}"] \\ P \arrow[ur, dotted, "\exists"] \arrow[r] & C \end{tikzcd}}$$

Equivalently, P is projective if the functor $\operatorname{Hom}_{\mathsf{A}}(P, -)\colon \mathsf{A} \to \mathsf{Ab}$ takes exact sequences in A to exact sequences of abelian groups.

An object I in an abelian category is called **injective** if it has the following extension property:

$$\begin{tikzcd} & B \arrow[dl, dotted, "\exists"'] \\ I & A \arrow[l] \arrow[u, hook, "\text{mono}"'] \end{tikzcd}$$

Equivalently, I is injective if the functor $\mathrm{Hom}_{\mathsf{A}}(-, I)\colon \mathsf{A}^{op} \to \mathsf{Ab}$ takes exact sequences in A to exact sequences of abelian groups.

We say that A has **enough projectives** (respectively, **enough injectives**), if for every object A there exists an epimorphism $P \to A$ where P is projective (respectively, a monomorphism $A \to I$ where I is injective).

In a category with enough projectives (respectively, enough injectives) one can always find *projective resolutions* (respectively, *injective resolutions*) for objects.

The category R-**Mod** has enough injectives and enough projectives. The categories **PreSh**$(\mathsf{C}, \mathsf{Ab})$, **PreSh**$(\mathsf{C}, R$-**Mod**$)$, **Sh**$(\mathsf{C}, \mathsf{Ab})$, **Sh**$(\mathsf{C}, R$-**Mod**$)$, and **Quasi-Coh**$_X$ have enough injectives, but in general they do not have enough projectives.

Exercise. Show that the abelian category of finite abelian groups has no injective or projective object other than 0. Show that in the category of vector spaces every object is both injective and projective.

Lecture 2: Chain Complexes

Overview. Various cohomology theories in mathematics are constructed from chain complexes. However, taking the cohomology of a chain complex kills a lot of information contained in that chain complex. So, it is desirable to elevate the cohomological constructions to the chain complex level. In this lecture, we introduce the necessary machinery for doing so. The main tool here is the *mapping cone* construction, which should be thought of as a *homotopy cokernel* construction. We discuss in some detail the basic properties of the cone construction; we will see, in particular, how it allows us to construct long exact sequences on the chain complex level, generalizing the usual cohomology long exact sequences.

In practice, one is only interested in chain complexes up to *quasi-isomorphism.* This leads to the notion of the *derived category* of chain complexes. The cone construction is well adapted to the derived category. Derived categories provide the correct setting for manipulating chain complexes; for instance, they allow us to construct *derived functors* on the chain complex level.

1. Why Chain Complexes?

Chain complexes arise naturally in many areas of mathematics. There are two main sources for (co)chain complexes: *chains on spaces* and *resolutions*. We give an example for each.

Chains on spaces. Let X be a topological space. There are various ways to associate a chain complex to X. For example, *singular chains*, *cellular chains* (if X is a CW complex), *simplicial chains* (if X is triangulated), de Rham complex (if X is differentiable), and so on. These complexes encode a great deal of topological information about X in terms of algebra (e.g., homology and cohomology).

Observe the following two facts:

- Such constructions usually give rise to a chain complex of *free* modules.
- Various chain complexes associated to a given space are chain homotopy equivalent (hence, give rise to the same homology/cohomology).

Resolutions. Let us explain this with an example. Let R be a ring, and M and N R-modules. Recall how we compute $\mathrm{Tor}_i(M, N)$. First, we choose a free resolution

$$\underbrace{\cdots \to P^{-2} \to P^{-1} \to P^0}_{P^\bullet} \to M.$$

Then, we define

$$\mathrm{Tor}_i(M, N) = H^{-i}(P^\bullet \otimes N).$$

Observe the following two facts:

- The complex $P^\bullet$ is a complex of *free* modules.
- Any two resolutions $P^\bullet$ and $Q^\bullet$ of M are chain homotopy equivalent (hence Tor is well-defined).

Conclusion. In both examples above, we replaced our object with a chain complex of free modules that was unique up to chain homotopy. We could then extract information about our object by doing algebraic manipulations (e.g., taking homology) on this complex. Therefore, *the real object of interest is the* (*chain homotopy class of*) *a chain complex* (*of, say, free modules*). Of course, instead of working with the complex itself, one could choose to work with its (co)homology, but one loses some information this way.

Remark 1.1. Complexes of *projective* modules (e.g., projective resolutions) work equally well as complexes of free modules. Sometimes, we are in a dual situation where complexes of *injective* modules are more appropriate (e.g., in computing Ext groups).

2. Chain Complexes ([We], [GeMa], [Ha2], [HiSt], [KaSch], [Iv], and any book on algebraic topology)

We quickly recall a few definitions. We prefer to work with *cohomological* indexing, so we work with *cochain* complexes.

A **cochain complex** C in an abelian category A is a sequence of objects in A

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n-1} \longrightarrow \cdots, \qquad d^2 = 0.$$

Such a sequence is in general indexed by $\mathbb{Z}$, but in many applications one works with complexes that are bounded below, bounded above, or bounded on both sides.

A **chain map** $f\colon B \to C$ between two cochain complexes is a sequence of $f^n\colon B^n \to C^n$ of morphisms such that the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^{n-2}} & B^{n-1} & \xrightarrow{d^{n-1}} & B^n & \xrightarrow{d^n} & B^{n+1} & \xrightarrow{d^{n+1}} & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \xrightarrow{d^{n-2}} & C^{n-1} & \xrightarrow{d^{n-1}} & C^n & \xrightarrow{d^n} & C^{n+1} & \xrightarrow{d^{n+1}} & \cdots
\end{array}$$

A **null homotopy** for a chain map $f\colon B \to C$ is a sequence $s^n\colon B^n \to C^{n-1}$ such that

$$\boxed{f^n = s^{n+1} \circ d^n + d^{n-1} \circ s^n, \quad \forall\, n}$$

We say two chain maps $f, g\colon B \to C$ are **chain homotopic** if $f - g$ is null homotopic. We say $f\colon B \to C$ is a **chain homotopy equivalence** if there exists $g\colon C \to B$ such that $f \circ g$ and $g \circ f$ are chain homotopy equivalent to the corresponding identity maps.

To any chain complex C one can associate the following objects in A:

$$\boxed{\begin{aligned}
&Z^n(C) = \ker(C^n \xrightarrow{d} C^{n+1}) \\
&B^n(C) = \operatorname{im}(C^{n-1} \xrightarrow{d} C^n) \\
&H^n(C) = Z^n(C)/B^n(C)
\end{aligned}}$$

Exercise. Show that a chain map $f\colon B \to C$ induces morphisms on each of the above objects. In particular, we have induced morphisms $H^n(f)\colon H^n(B) \to H^n(C)$ for all n. Show that if f and g are chain homotopic, then they induce the same map on cohomology. In particular, a chain homotopy equivalence induces isomorphisms on cohomologies.

A chain map $f\colon B \to C$ is called a **quasi-isomorphism** if it induces isomorphisms on all cohomologies. (So every chain homotopy equivalence is a quasi-isomorphism.)

Exercise. When is a chain complex C quasi-isomorphic to the zero complex? (Hint: exact.) When is C chain homotopy equivalent to the zero complex? (Hint: split exact.)

A **short exact sequence** of chain complexes is a sequence

$$0 \to A \to B \to C \to 0$$

of chain maps which is exact at every $n \in \mathbb{Z}$. Such an exact sequence gives rise to a long exact sequence in cohomology

$$\cdots \xrightarrow{\partial} H^n(A) \to H^n(B) \to H^n(C) \xrightarrow{\partial} H^{n+1}(A) \to H^{n+1}(B) \to H^{n+1}(C) \xrightarrow{\partial} \cdots$$

A short exact sequence as above is called **pointwise split** (or **semi-split**) if every epimorphism $B^n \to C^n$ admits a section, or equivalently, every monomorphism $A^n \to B^n$ is a direct summand.

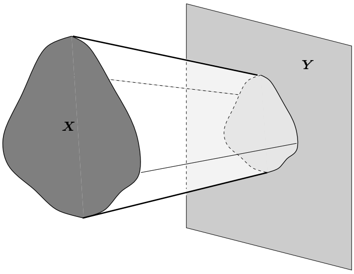
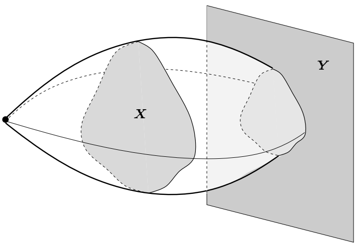
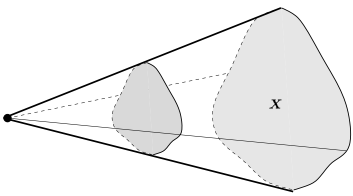
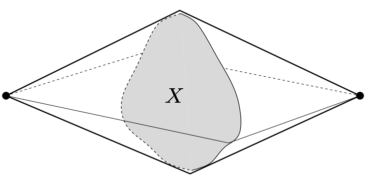
Exercise. Does pointwise split imply that the maps ∂ in the above long exact sequence are zero? Is the converse true? (Answer: both implications are false.)

3. Constructions on Chain Complexes ([We], [GeMa], [Ha2], [HiSt], [KaSch], [Iv])

In the following table we list a few constructions that are of great importance in algebraic topology. The left column presents the topological construction, and the right column is the cochain counterpart.

The right column is obtained from the left column as follows: imagine that the spaces are triangulated and translate the topological constructions in terms of the simplices; this gives the *chain complex* picture. The *cochain complex* picture is obtained by the appropriate change in indexing (from homological to cohomological).

The reader is strongly encouraged to do this as an exercise. *It is extremely important to pay attention to the orientation of the simplices, and keep track of the signs accordingly.*

<table>
<tr><th>Spaces</th><th>Cochains</th></tr>
<tr><td>$f\colon X \to Y$ continuous map</td><td>$f\colon B \to C$ cochain map</td></tr>
<tr><td>$\mathrm{Cyl}(f) = (X \times I) \cup_f Y$

</td><td>Mapping cylinder
$\mathrm{Cyl}(f)^n := B^n \oplus B^{n+1} \oplus C^n$
$B^n \oplus B^{n+1} \oplus C^n \xrightarrow{d} B^{n+1} \oplus B^{n+2} \oplus C^{n+1}$
$(b', b, c) \mapsto \big(d_B(b') - b, -d_B(b), f(b) + d_C(c)\big)$
$\begin{pmatrix} d_B & -\mathrm{id}_B & 0 \\ 0 & -d_B & 0 \\ 0 & f & d_C \end{pmatrix}$</td></tr>
<tr><td>$\mathrm{Cone}(f) = \mathrm{Cyl}(f)/X$

</td><td>Mapping cone
$\mathrm{Cone}(f) = \mathrm{Cyl}(f)/B$
$\mathrm{Cone}(f)^n := B^{n+1} \oplus C^n$
$B^{n+1} \oplus C^n \xrightarrow{d} B^{n+2} \oplus C^{n+1}$
$(b, c) \mapsto \big(-d_B(b), f(b) + d_C(c)\big)$</td></tr>
<tr><td>$\mathrm{Cone}(X) = \mathrm{Cone}(\mathrm{id}_X) = X \times I / X \times \{0\}$

</td><td>Cone
$\mathrm{Cone}(B) := \mathrm{Cone}(\mathrm{id}_B)$
$\mathrm{Cone}(B)^n = B^{n+1} \oplus B^n$
$B^{n+1} \oplus B^n \xrightarrow{d} B^{n+2} \oplus B^{n+1}$
$(b, b') \mapsto \big(-d(b), b + d(b')\big)$</td></tr>
<tr><td>$\Sigma(X) = \mathrm{Cone}(X)/X = X \times I / X \times \{0, 1\}$

</td><td>Shift (Suspension)
$B[1] := \mathrm{Cone}(B)/B$
$B[1]^n := B^{n+1}$
$B[1]^n \xrightarrow{d} B[1]^{n+1}$
$b \mapsto -d(b)$</td></tr>
</table>

Some history. These constructions were originally used by topologists (e.g., Puppe [Pu]) as a unified way of treating various cohomology theories for topological spaces (the so-called *generalized cohomology theories* [Ad]), and soon they became a standard tool in *stable homotopy theory.* Around the same time, Grothendieck and Verdier developed the same machinery for cochain complexes, to be used in Grothendieck's formulation of duality theory and the Riemann–Roch theorem for schemes [Ha2]. This led to the invention of *derived categories* and *triangulated categories* [Ve]. As advocated by Grothendieck, it has become more and more apparent over time that working in *derived categories* is a better alternative to working with cohomology. Nowadays people consider even richer structures such as *dg-categories* [BoKa, Ke3], A_∞*-categories* [Ke4], (*stable*) model categories ([Ho], Section 7), etc. The idea is that, the higher one goes in this hierarchy, the more *higher-order* cohomological information (e.g., Massey products, Steenrod operations, or other cohomology operations) is retained. In these lectures we will not go beyond derived categories.

An important fact. Taking the mapping cone of $f\colon X \to Y$ is somewhat like taking cokernels: we are essentially killing the image of f, but softly (i.e., we make it contractible). That is why $\mathrm{Cone}(f)$ is sometimes called the *homotopy cofiber* or *homotopy cokernel* of f; see [May1], §8.4, especially, the first lemma on p. 58.

Proposition 3.1. *In the following diagram of cochain complexes, if g and h are chain equivalences, then so is the induced map on the mapping cones*

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & \mathrm{Cone}(f) \arrow[d] \\ B' \arrow[r, "f'"] & C' \arrow[r] & \mathrm{Cone}(f') \end{tikzcd}$$

A similar statement is true for topological spaces.

Remark 3.2. The above proposition is still true if we use *quasi-isomorphisms* instead of chain homotopy equivalences. Similarly, the topological version remains valid if we use *weak equivalences* (i.e., maps that induce isomorphisms on all homotopy groups) instead of homotopy equivalences.

Exercise. Give an example (in both the topological and chain complex settings) to show that the above proposition is not true if we use the strict cofiber $Y/f(X)$ instead of $\mathrm{Cone}(f)$. (Therefore, since we are interested in chain homotopy equivalence classes, or quasi-isomorphism classes, of cochain complexes, $\mathrm{Cone}(f)$ is a better-behaved notion than $Y/f(X)$.)

Exercise. Formulate a universal property for the mapping cone construction.

4. Basic Properties of Cofiber Sequences ([We], [GeMa], [KaSch], [Ha2], [Ve])

The sequence

$$\boxed{B \xrightarrow{f} C \longrightarrow \mathrm{Cone}(f)}$$

(or any sequence quasi-isomorphic to such a sequence) is called a **cofiber sequence**. The same definition can be made with topological spaces. We list the basic properties of cofiber sequences.

1. Exact sequence, basic form. An important property of cofiber sequences is that they give rise to long exact sequences. Here is a baby version which is very easy to prove and is left as an exercise.

Proposition 4.1. *Let $B \to C \to \mathrm{Cone}(f)$ be a cofiber sequence. Then the sequence*

$$H^n(B) \to H^n(C) \to H^n(\mathrm{Cone}(f))$$

is exact for every n.

Remark 4.2. This is of course part of a long exact sequence

$$\cdots \xrightarrow{\partial} H^n(B) \to H^n(C) \to H^n(\mathrm{Cone}(f)) \xrightarrow{\partial} H^{n+1}(B)$$
$$\to H^{n+1}(C) \to H^{n+1}(\mathrm{Cone}(f)) \xrightarrow{\partial} \cdots$$

that we will discuss shortly. If we use the fact that a short exact sequence of cochain complexes gives rise to a long exact sequence of cohomology groups, this is easily proven by showing that the cofiber sequence $B \to C \to \mathrm{Cone}(f)$ is chain homotopy equivalent to the short exact sequence $B \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f)$. The equivalence $C \sim \mathrm{Cyl}(f)$ is given by

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Cyl}(f) \\ c & \mapsto & (0,0,c) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Cyl}(f) & \longrightarrow & C \\ B^n \oplus B^{n+1} \oplus C^n & \longrightarrow & C^n \\ (b', b, c) & \mapsto & f(b') + c. \end{array}$$

Remark 4.3. We have a similar statement for a cofiber sequence $X \to Y \to \mathrm{Cone}(f)$ of topological spaces. The corresponding long exact sequence in (co)homology should then be interpreted as the long exact sequence of *relative* (co)homology groups. More specifically, if f is an inclusion, we have

$$H_*(\mathrm{Cone}(f)) \cong H_*(Y,X) \quad \text{and} \quad H^*(\mathrm{Cone}(f)) \cong H^*(Y,X),$$

where the right-hand sides are relative (co)homology groups.

2. A short exact sequence is a cofiber sequence. We pointed out in Remark 4.2 that a cofiber sequence is chain equivalent (hence quasi-isomorphic) to a pointwise split short exact sequence. The converse is also true.

Proposition 4.4. *Consider a short exact sequence of cochain complexes*

$$0 \to B \xrightarrow{f} C \to C/B \to 0.$$

Then the natural chain map $\varphi\colon \mathrm{Cone}(f) \to C/B$ *defined by*

$$\begin{array}{ccc} B^{n+1} \oplus C^n & \to & C^n/B^n \\ (b,c) & \mapsto & \bar{c} \end{array}$$

is a quasi-isomorphism. $B^n \hookrightarrow C^n$ *is split, If the sequence is pointwise split, with* $s\colon C^n/B^n \to C^n$ *a choice of splitting, then* φ *is a chain equivalence. The inverse is given by* $\psi\colon C/B \to \mathrm{Cone}(f)$

$$\begin{array}{ccc} C^n/B^n & \to & B^{n+1} \oplus C^n \\ x & \mapsto & \big(sd(x) - ds(x), s(x)\big). \end{array}$$

Exercise. Let $f\colon B \hookrightarrow C$ be an inclusion of R-modules, viewed as cochain complexes concentrated in degree 0. Show that $\varphi\colon \mathrm{Cone}(f) \to C/B$ defined above is a chain equivalence if and only if f is split.

3. Iterate of a cofiber sequence. Consider a cofiber sequence

$$B \xrightarrow{f} C \xrightarrow{g} \mathrm{Cone}(f).$$

Observe that $g\colon C \to \mathrm{Cone}(f)$ is a pointwise split inclusion. This implies that the mapping cone of g is naturally chain equivalent to $\mathrm{Cone}(f)/C = B[1]$. More precisely, we have the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "f"] & C \arrow[r, hook, "g"] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[r, "\partial"] \arrow[d, equal] & B[1] \\ & C \arrow[r, hook, "g"] & \mathrm{Cone}(f) \arrow[r, hook, "h"] & \mathrm{Cone}(g). \arrow[u, "\varphi", "\text{ch. eq}"'] \end{tikzcd}$$

Note that we have natural chain equivalences

$$\mathrm{Cone}(\partial) \sim \mathrm{Cone}(h) \sim \mathrm{coker}(h) = C[1].$$

Indeed, a careful chasing through the above chain equivalences (for which we have given explicit formulas) shows that the sequence

$$\mathrm{Cone}(f) \xrightarrow{\partial} B[1] \xrightarrow{-f[1]} C[1]$$

is a cofiber sequence. We can now iterate this process forever and produce a long sequence

$$\boxed{\cdots \xrightarrow{\partial[-1]} B \xrightarrow{f} C \xrightarrow{g} \mathrm{Cone}(f) \xrightarrow{\partial} B[1] \xrightarrow{-f[1]} C[1] \xrightarrow{-g[1]} \mathrm{Cone}(f)[1] \xrightarrow{-\partial[1]} \cdots} \quad (\bigstar)$$

in which every three consecutive terms form a cofiber sequence.

Remark 4.5. A similar long exact sequence can be constructed for a map $f\colon X \to Y$ of topological spaces, and it is called a *Puppe sequence* (or *cofiber sequence* [May1] §8.4). A Puppe sequence, however, only extends to the right. The reason for this is that the suspension functor on the category of topological spaces is not invertible (as opposed to the shift functor for cochain complexes). The Puppe sequence can be used, among other things, to give a natural construction of the long exact (co)homology sequence of a pair.

4. Long exact sequence of a cofiber sequence. Applying Proposition 4.1 to the above long exact sequence of chain complexes, and observing that $H^i(B[n]) = H^{i+n}(B)$, we obtain a long exact sequence of cohomology groups

$$\begin{aligned} \cdots \xrightarrow{\partial} H^n(B) \to H^n(C) \to H^n(\mathrm{Cone}(f)) \xrightarrow{\partial} H^{n+1}(B) \to H^{n+1}(C) \\ \to H^{n+1}(\mathrm{Cone}(f)) \xrightarrow{\partial} \cdots . \end{aligned}$$

Moral. By exploiting the notion of cone of a map in a systematic way, we can elevate many basic cohomological constructions to the level of chain complexes. For this to work conveniently, one needs to *invert* chain equivalences so that one can treat such morphisms as isomorphisms. Indeed, for various reasons, inverting only chain equivalences is usually not enough. For example, recall that one of our motivations for working with chain complexes was that we wanted to be able to replace an object $M \in \mathsf{A}$ with a better behaved resolution (projective or injective) of it. Since the resolution map is only a quasi-isomorphism and not a chain equivalence in general, it is only after inverting quasi-isomorphisms that we can regard an $M \in \mathsf{A}$ and its resolution as the "same".

5. Derived Categories ([We], [GeMa], [Ha2], [KaSch], [Iv], [Ve], [Ke1], [Ke2])

Let A be an abelian category.

The **homotopy category** $\mathcal{K}(\mathsf{A})$ is the category obtained by inverting all chain equivalences in the category $\mathrm{Ch}(\mathsf{A})$ of chain complexes. More precisely, there is a natural functor $\mathrm{Ch}(\mathsf{A}) \to \mathcal{K}(\mathsf{A})$ which has the following universal property:

If a functor $F\colon \mathrm{Ch}(\mathsf{A}) \to \mathsf{C}$ sends chain equivalences to isomorphisms, then

$$\begin{tikzcd} \mathrm{Ch}(\mathsf{A}) \arrow[r, "F"] \arrow[d] & \mathsf{C} \\ \mathcal{K}(\mathsf{A}) \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

If in the above definition we replace "chain equivalence" by "quasi-isomorphism", we arrive at the definition of the **derived category** $\mathcal{D}(\mathsf{A})$. We list the basic properties of $\mathcal{K}(\mathsf{A})$ and $\mathcal{D}(\mathsf{A})$.

1. Explicit construction of $\mathcal{K}(\mathsf{A})$. We can, alternatively, define the homotopy category $\mathcal{K}(\mathsf{A})$ to be the category whose objects are the ones of $\mathrm{Ch}(\mathsf{A})$ and whose morphisms are defined by

$$\boxed{\mathrm{Hom}_{\mathcal{K}(\mathsf{A})}(B,C) := \mathrm{Hom}_{\mathrm{Ch}(\mathsf{A})}(B,C)/N}$$

where N stands for the group of null homotopic maps. (*Exercise.* Show that this category satisfies the required universal property.) This, in particular, implies that $\mathcal{K}(\mathsf{A})$ is an additive category.

2. Cohomology functors. The cohomology functors $H^n\colon \mathrm{Ch}(\mathsf{A}) \to \mathsf{A}$ factor through $\mathcal{K}(\mathsf{A})$ and $\mathcal{D}(\mathsf{A})$:

$$\begin{tikzcd} \mathrm{Ch}(\mathsf{A}) \arrow[r] \arrow[drr, "H^n"'] & \mathcal{K}(\mathsf{A}) \arrow[r] \arrow[dr, "H^n"] & \mathcal{D}(\mathsf{A}) \arrow[d, "H^n"] \\ & & \mathsf{A} \end{tikzcd} .$$

3. Explicit construction of $\mathcal{D}(\mathsf{A})$. The objects of $\mathcal{D}(\mathsf{A})$ can again be taken to be the ones of $\mathrm{Ch}(\mathsf{A})$. Description of morphisms in $\mathcal{D}(\mathsf{A})$ is, however, slightly more involved than the case of $\mathcal{K}(\mathsf{A})$. It requires the notion of *calculus of fractions* in a category, which we will not get into for the sake of brevity.

Just to give an idea how this works, first we observe that $\mathcal{D}(\mathsf{A})$ can, equivalently, be defined by inverting quasi-isomorphisms in $\mathcal{K}(\mathsf{A})$. The class of quasi-isomorphisms in $\mathcal{K}(\mathsf{A})$ satisfies the axioms of the calculus of fractions ([GeMa], Definition III.2.6), and this allows one to give an explicit description of morphisms in $\mathcal{D}(\mathsf{A})$ (see [GeMa], Section III.2, [Ha2], Section I.3, or [We], Section 10.3). A consequence of this is that, for every morphism $f\colon B \to C$ in $\mathcal{D}(\mathsf{A})$, one can find morphisms s, t, g and h in $\mathcal{K}(\mathsf{A})$ such that s and t are quasi-isomorphisms and the following diagrams commute in $\mathcal{D}(\mathsf{A})$:

$$\begin{tikzcd} A \arrow[d, "s"'] \arrow[dr, "g"] & \\ B \arrow[r, "f"'] & C \end{tikzcd} \qquad \begin{tikzcd} & A' \\ B \arrow[ur, "h"] \arrow[r, "f"'] & C \arrow[u, "t"'] \end{tikzcd}$$

Also, for any two morphisms $f, f'\colon B \to C$, now in $\mathcal{K}(\mathsf{A})$, which become equal in $\mathcal{D}(\mathsf{A})$, there are s and t as above such that $f \circ s = f' \circ s$ and $t \circ f = t \circ f'$.

4. Cofiber sequences. There is a notion of cofiber sequence in both $\mathcal{K}(\mathsf{A})$ and $\mathcal{D}(\mathsf{A})$. Every morphism $f\colon B \to C$ fits in a sequence ★ as on page 403 in which every three consecutive terms form a cofiber sequence.

6. Variations on the Theme of Derived Categories ([We], [GeMa], [Ha2], [KaSch], [Iv], [Ke1], [Ke2])

To an abelian category A we can associate various types of homotopy or derived categories by imposing certain boundedness conditions on the chain complexes in question.

The **bounded below** derived category $\mathcal{D}^+(\mathsf{A})$ is the category obtained from inverting the quasi-isomorphisms in the category of bounded below chain complexes $\mathrm{Ch}^+(\mathsf{A})$. Equivalently, $\mathcal{D}^+(\mathsf{A})$ is obtained from inverting the quasi-isomorphisms in the homotopy category $\mathcal{K}^+(\mathsf{A})$ of $\mathrm{Ch}^+(\mathsf{A})$.

In the same way one can define **bounded** derived categories $\mathcal{D}^-(\mathsf{A})$ and $\mathcal{D}^b(\mathsf{A})$, where b stands for bounded on two sides.

Remark 6.1. One can use the calculus of fractions in $\mathcal{K}^+(\mathsf{A})$ (respectively, $\mathcal{K}^-(\mathsf{A})$, $\mathcal{K}^b(\mathsf{A})$) to give a description of morphisms in $\mathcal{D}^+(\mathsf{A})$ (respectively, $\mathcal{D}^-(\mathsf{A})$, $\mathcal{D}^b(\mathsf{A})$). It follows that each of these bounded derived categories can be identified with a full subcategory of $\mathcal{D}(\mathsf{A})$. In particular, $\mathcal{D}^b(\mathsf{A}) = \mathcal{D}^+(\mathsf{A}) \cap \mathcal{D}^-(\mathsf{A})$. We will say more on this in the next lecture.

There is an alternative way of computing morphisms in *bounded* derived categories using injective or projective resolutions. It is summarized in the following theorems.

Theorem 6.2. ([GeMa], § III.5.21, [We], Theorem 10.4.8) *Let* $\mathcal{I}^+(\mathsf{A}) \subset \mathcal{K}^+(\mathsf{A})$ *be the full subcategory consisting of complexes whose terms are all injective. Then the composition*

$$\begin{tikzcd} \mathcal{I}^+(\mathsf{A}) \arrow[r, hook] \arrow[rr, bend right] & \mathcal{K}^+(\mathsf{A}) \arrow[r] & \mathcal{D}^+(\mathsf{A}) \end{tikzcd}$$

is fully faithful. It is an equivalence if A *has enough injectives. The same thing is true if we replace* $+$ *by* $-$ *and injective by projective*:

$$\begin{tikzcd} \mathcal{P}^-(\mathsf{A}) \arrow[r, hook] \arrow[rr, bend right] & \mathcal{K}^-(\mathsf{A}) \arrow[r] & \mathcal{D}^-(\mathsf{A}). \end{tikzcd}$$

The (first part of the) above theorem says that, if I and J are bounded below complexes of injective objects, then we have

$$\boxed{\mathrm{Hom}_{\mathcal{D}(\mathsf{A})}(J, I) = \mathrm{Hom}_{\mathcal{K}(\mathsf{A})}(J, I).}$$

(*This is actually true if* J *is an arbitrary bounded below complex*; see [We], Corollary 10.4.7.) In particular, if I and J are quasi-isomorphic, then they are chain equivalent. This is good news, because we have seen that computing Hom is much easier in $\mathcal{K}(\mathsf{A})$. Now if A has enough injectives, it can be shown that for every bounded C, there exists a bounded below complex of injectives I, and a quasi-isomorphism $C \xrightarrow{\sim} I$. This is called an *injective resolution* for C. So, by virtue of the above fact, injective resolutions can be used to compute morphisms in derived categories:

$$\boxed{\mathrm{Hom}_{\mathcal{D}(\mathsf{A})}(B, C) = \mathrm{Hom}_{\mathcal{K}(\mathsf{A})}(B, I).}$$

Exercise. Set $\mathrm{Hom}^i_{\mathcal{D}(\mathsf{A})}(B, C) := \mathrm{Hom}(B, C[i])$. Let B and C be objects in A, viewed as complexes concentrated in degree 0. Assume A has enough injectives. Show that

$$\boxed{\mathrm{Hom}^i_{\mathcal{D}(\mathsf{A})}(B, C) \cong \mathrm{Ext}^i(B, C).}$$

Use the second part of the theorem to give a way of computing Ext groups using a projective resolution for C, if they exist. Compute $\mathrm{Hom}^i_{\mathcal{K}(\mathsf{A})}(B, C)$ and compare it with $\mathrm{Hom}^i_{\mathcal{D}(\mathsf{A})}(B, C)$.

Exercise. Give an example of a non-zero morphism in $\mathcal{D}(\mathsf{A})$, with A your favorite abelian category, which induces zero maps on all cohomologies.

7. Derived Functors ([We], [GeMa], [Ha2], [KaSch], [Iv], [Ke1], [Ke2])

To keep the lecture short, we will skip the very important topic of derived functors and confine ourselves to an example: the *derived tensor* $\overset{L}{\otimes}$. The reader is encouraged to consult the given references for the general discussion of derived functors, especially the all important *derived hom* RHom.

Let $\mathsf{A} = R$-**Mod**. The idea is that we want to have a notion of tensor product for chain complexes which is well defined on $\mathcal{D}(A)$.

The usual tensor product of chain complexes (Definition [We], §2.7.1) *does not pass to derived categories.* This is because tensor product is not exact. More precisely, if $A \to A'$ is a quasi-isomorphism, then $A \otimes B \to A' \otimes B$ may no longer be. (For a counterexample, let A be a short exact sequence, $A' = 0$, and B a complex concentrated in degree 0.)

The usual tensor product of chain complexes DOES pass to homotopy categories. This is because a null homotopy s for a chain map $A \to A'$ gives rise to a null homotopy $s \otimes B$ for $A \otimes B \to A' \otimes B$ (exercise).

In particular, taking all complexes to be (bounded-above) complexes of projective modules, we get a well-defined tensor product $- \otimes - \colon \mathcal{P}^-(\mathsf{A}) \times \mathcal{P}^-(\mathsf{A}) \to \mathcal{P}^-(\mathsf{A})$. Since R-**Mod** has enough projectives, we obtain, via the equivalence $\mathcal{P}^-(\mathsf{A}) \cong \mathcal{D}^-(\mathsf{A})$ of Theorem 6, the desired tensor product on the bounded above derived category:

$$\boxed{- \overset{L}{\otimes} - \colon \mathcal{D}^-(\mathsf{A}) \times \mathcal{D}^-(\mathsf{A}) \to \mathcal{D}^-(\mathsf{A}).}$$

More explicitly, $A \overset{L}{\otimes} B$ is defined to be $P \otimes Q$, where $P \to A$ and $Q \to B$ are certain chosen projective resolutions. (In fact, it is enough to resolve only one of A or B.)

Exercise. Show that if $M, N \in \mathsf{A}$ are viewed as complexes concentrated in degree 0, then

$$\boxed{H^{-i}(M \overset{L}{\otimes} N) \cong \operatorname{Tor}_i(M, N).}$$

Lecture 3: Triangulated Categories

Overview. The main properties of the cone construction for chain complexes can be formalized into a set of axioms. This leads to the notion of a *triangulated* category, the main topic of this lecture. A triangulated category is an additive category in which there is an abstract notion of mapping cone. The cohomology functors on chain complexes can also be studied at an abstract level in a triangulated category.

The main example of a triangulated category is the derived category $\mathcal{D}(\mathsf{A})$ of chain complexes in an abelian category A. Using a *t-structure* on a triangulated category, we can produce an abelian category, called the *heart* of the t-structure. For example, there is a standard t-structure on $\mathcal{D}(\mathsf{A})$ whose heart is A. The t-structure is not unique, and by varying it we can produce new abelian categories. We show how using a *torsion theory* on an abelian category A we can produce a new t-structure on $\mathcal{D}(\mathsf{A})$. The heart of this t-structure is then a new abelian category B. For example, this method has been used in noncommutative geometry to "deform" the abelian category A of coherent sheaves on a torus X. The new abelian category B can then be thought of as the category of coherent sheaves on a "noncommutative deformation" of X, a noncommutative torus.

1. Triangulated Categories ([We], [GeMa], [Ha2], [Ne], [BeBeDe], [KaSch], [Iv], [Ve], [Ke1], [Ke2])

We formalize the main properties of the mapping cone construction and define triangulated categories.

Let $\mathcal{T}$ be an additive category equipped with an auto-equivalence $X \mapsto X[1]$ called **shift** (or **translation**).

By a **triangle** in $\mathcal{T}$ we mean a sequence $X \to Y \to Z \to X[1]$. We sometimes write this as

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] & & Z \arrow[ll, "+1"] \end{tikzcd}$$

A **triangulation** on $\mathcal{T}$ is a collection of triangles, called **exact** (or **distinguished**) triangles, satisfying the following axioms:

TR1. a) $X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow X[1]$ is an exact triangle.
b) Any morphism $f\colon X \to Y$ is part of an exact triangle.
c) Any triangle isomorphic to an exact triangle is exact.

TR2. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ exact $\Leftrightarrow$ $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ exact.

TR3. In the diagram

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \\ X' \arrow[r] \arrow[u, "u"] & Y' \arrow[r] \arrow[u, "v"] & Z' \arrow[r] \arrow[u, dotted, "\exists"] & X'[1] \arrow[u, "{u[1]}"'] \end{tikzcd}$$

if the horizontal rows are exact, and the left square commutes, then the dotted arrow can be filled (not necessarily uniquely) to make the diagram commute.

TR4 (*short imprecise version*). Given $f: X \to Y$ and $g: Y \to Z$, we have the following commutative diagram in which every pair of same color arrows is part of an exact triangle

$$\begin{tikzcd}
& & & Xf \arrow[dl, Rightarrow, "f"'] \arrow[dd, dashed, "g"] & \\
& & Y \arrow[dll, Rightarrow] \arrow[dr, "g"'] & & \\
& & & Z \arrow[d, dashed] \arrow[dr] & \\
U \arrow[rrr, dotted] & & & V \arrow[r, dotted] & W
\end{tikzcd}$$

Idea: "$(Z/X)/(Y/X) = Z/Y$". See Appendix II for a more precise statement.

A **triangulated category** is an additive category $\mathcal{T}$ equipped with an auto-equivalence $X \mapsto X[1]$ and a triangulation.

A **triangle functor** (or an **exact functor**)[a] is a functor between triangulated categories which commutes with the translation functors (up to natural transformation) and sends exact triangles to exact triangles.

Remark 1.1. A triangulated category can be thought of as a category in which there are well-behaved notions of *homotopy kernel* and *homotopy cokernel*[b] (hence also various other types of homotopy limits and colimits).

Some immediate consequences. Let $\mathcal{T}$ be a triangulated category, and let $X \to Y \to Z \to X[1]$ be an exact triangle in $\mathcal{T}$. Then, the following are true:

0. The opposite category $\mathcal{T}^{op}$ is naturally triangulated.

1. Any length three portion of the sequence

$$\cdots \longrightarrow Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]} \cdots$$

is an exact triangle.

[a]This is not a very good terminology, because there is also a notion of t-*exact* functor with respect to a t-structure.

[b]Indeed not quite well-behaved (read, functorial), essentially due to the fact that the morphism whose existence is required in **TR3** may not be unique. This turns out to be problematic. A way to remedy this is to work with DG categories; see [BoKa].

2. In the above exact sequence, any two consecutive morphisms compose to zero. *Proof.* Enough to check $g \circ f = 0$. Apply **TR1**.a and **TR3** to

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1] \\ X \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] & X \arrow[u, "f"] \arrow[r] & 0 \arrow[u, dotted, "\exists"] \arrow[r] & X[1] \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

3. Let T be any object. Then,

$$\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y) \to \mathrm{Hom}(T, Z)$$

is exact. Therefore, the long sequence of (**1**) gives rise to a long exact sequence of abelian groups. The same thing is true for

$$\mathrm{Hom}(X, T) \to \mathrm{Hom}(Y, T) \to \mathrm{Hom}(Z, T).$$

Remark 1.2. In fact, **TR3** and half of **TR2** follow from the rest of the axioms. For this, see [May2]. Also see [Ne] for another formulation of **TR4**; especially, Remark 1.3.15 and Proposition 1.4.6.

The main example. The proof of the following theorem can be found in [GeMa] or [KaSch]. (We have sketched some of the ideas in Lecture 2.)

Theorem 1.3. *Let* A *be an abelian category. Then* $\mathcal{K}(\mathsf{A})$ *and* $\mathcal{D}(\mathsf{A})$ *are triangulated categories.*

In both cases, the distinguished triangles are the ones obtained from cofiber sequences. Equivalently, the distinguished triangles are the ones obtained from the pointwise split short exact sequences. (To see the equivalence, note that one implication follows from Proposition 4.4 of Lecture 2. The other follows from the fact that the cofiber sequence $B \to C \to \mathrm{Cone}(f)$ is chain homotopy equivalent to the pointwise split short exact sequence $B \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f)$; see Remark 4.2 of Lecture 2.) In the case of $\mathcal{D}(\mathsf{A})$ we get the same triangulation if we take *all* short exact sequences. This also follows from Proposition 4.4 of Lecture 2.

The following proposition allows us to produce more triangulated categories from the above basic ones.

Proposition 1.4. *Let* A *be an abelian category, and let* $\mathcal{C}$ *be a full additive subcategory of* $\mathrm{Ch}(\mathsf{A})$. *Let* $\mathcal{K} \subset \mathcal{K}(\mathsf{A})$ *be the corresponding quotient category and* $\mathcal{D}$ *the localization of* $\mathcal{K}$ *with respect to quasi-isomorphisms. Assume* $\mathcal{C}$ *is closed under translation, quasi-isomorphisms*, *and forming mapping* *cones. Then $\mathcal{K}$ and $\mathcal{D}$ are triangulated categories and we have fully faithful triangle functors*

$$\mathcal{K} \to \mathcal{K}(\mathsf{A})$$
$$\mathcal{D} \to \mathcal{D}(\mathsf{A}).$$

Proof. The proof in the case of $\mathcal{K}$ is straightforward. The case of $\mathcal{D} \to \mathcal{D}(\mathsf{A})$ follows easily from the existence of calculus of fractions on $\mathcal{K}(\mathsf{A})$. (Our discussion of calculus of fractions in Lecture 2 is enough for this.) $\square$

Exercise. Show that $\mathcal{D}^-(\mathsf{A})$ can, equivalently, be defined by inverting the quasi-isomorphisms in the category $\mathcal{C}$ of chain complexes with bounded above cohomology. The similar statement is true in the case of $\mathcal{D}^+(\mathsf{A})$ and $\mathcal{D}^b(\mathsf{A})$. (Hint: construct a right inverse to the inclusion $\iota\colon \mathrm{Ch}^-(\mathsf{A}) \hookrightarrow \mathcal{C}$ by choosing an appropriate truncation of each complex in C; see page 413 to learn how to truncate. Show that this becomes an actual inverse to ι when we pass to derived categories.)

Corollary 1.5. *We have the following fully faithful triangle functors*:

$$\mathcal{K}^-(\mathsf{A}), \mathcal{K}^+(\mathsf{A}), \mathcal{K}^b(\mathsf{A}) \hookrightarrow \mathcal{K}(\mathsf{A}),$$
$$\mathcal{D}^-(\mathsf{A}), \mathcal{D}^+(\mathsf{A}), \mathcal{D}^b(\mathsf{A}) \hookrightarrow \mathcal{D}(\mathsf{A}).$$

Proof. This is an immediate corollary of the previous proposition. (We also need to use the previous exercise.) $\square$

Important examples to keep in mind:

1. $\mathsf{A} = R$-**Mod**.

2. $\mathsf{A} =$ sheaves of R-modules over a topological space.

3. $\mathsf{A} =$ presheaves of R-modules over a topological space.

4. $\mathsf{A} =$ sheaves of $\mathcal{O}_X$-modules on a scheme X.

5. $\mathsf{A} =$ quasi-coherent sheaves on a scheme X.

2. Cohomological Functors ([We], [GeMa], [BeBeDe], [Ha2], [KaSch], [Ve])

Let $\mathcal{T}$ be a triangulated category and A an abelian category. An additive functor $H\colon \mathcal{T} \to \mathsf{A}$ is called a **cohomological functor** if for every exact triangle $X \to Y \to Z \to X[1]$,

$$\boxed{H(X) \to H(Y) \to H(Z) \text{ is exact in } \mathsf{A}.}$$

If we set $H^n(X) := H(X[n])$, we obtain the following long exact sequence:

$$\boxed{\begin{aligned}\cdots \longrightarrow H^{n-1}(Z) \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X)\\ \to H^{n+1}(Y) \to H^{n+1}(Z) \to \cdots\end{aligned}}$$

Example 2.1.

1. Let $\mathcal{T}$ be any of $\mathcal{K}^*(\mathsf{A})$ or $\mathcal{D}^*(\mathsf{A})$, $* = \emptyset, -, +, b$. Then the functor $H\colon \mathcal{T} \to \mathsf{A}$, $X \mapsto H^0(X)$ is a cohomological functor.
2. For any T, $\mathrm{Hom}(T, -)\colon \mathcal{T} \to \mathsf{A}$ is a cohomological functor. Similarly, $\mathrm{Hom}(-, T)\colon \mathcal{T} \to \mathsf{A}^{op}$ is a cohomological functor.

Exercise. Show that (**1**) is a special case of (**2**).

3. Abelian Categories Inside Triangulated Categories; *t*-Structures ([GeMa], [KaSch], [BeBeDe])

The triangulated categories associated to an abelian category A contain A as a full subcategory. More precisely,

Proposition 3.1. *Let $\mathcal{T}$ be any of $\mathcal{K}^*(\mathsf{A})$ or $\mathcal{D}^*(\mathsf{A})$, $* = \emptyset, -, +, b$. Then the functor $\mathsf{A} \to \mathcal{T}$ defined by*

$$A \quad \mapsto \quad \cdots \to 0 \to A \to 0 \to \cdots$$

(A is sitting in degree zero) is fully faithful.

Observe that this is not the only abelian subcategory of $\mathcal{T}$. For example, we have $\mathbb{Z}$ many copies of A in $\mathcal{T}$ (simply take shifts of the above embedding). We may also have abelian categories in $\mathcal{T}$ that are not isomorphic to A.

As we saw above, the triangulated structure of $\mathcal{T}$ is not sufficient to recover the abelian subcategory A. What extra structure do we need on $\mathcal{T}$ in order to reconstruct A?

A ***t*-structure** on a triangulated category $\mathcal{T}$ is a pair $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ of saturated (i.e. closed under isomorphism) full subcategories such that:

t1. If $X \in \mathcal{T}^{\le 0}$, $Y \in \mathcal{T}^{\ge 1}$, then $\mathrm{Hom}(X, Y) = 0$.
t2. $\mathcal{T}^{\le 0} \subseteq \mathcal{T}^{\le 1}$, and $\mathcal{T}^{\ge 1} \subseteq \mathcal{T}^{\ge 0}$.
t3. For every $X \in \mathcal{T}$, there is an exact triangle

$$A \to X \to B \to A[1]$$

such that $A \in \mathcal{T}^{\le 0}$ and $B \in \mathcal{T}^{\ge 1}$.

Notation: $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$.

Main example. Let $\mathcal{T} = \mathcal{D}(\mathsf{A})$. Set

$$\boxed{\begin{aligned} \mathcal{T}^{\leq 0} &= \{X \in \mathcal{D}(\mathsf{A}) \mid H^i(X) = 0,\ \forall\, i > 0\}, \\ \mathcal{T}^{\geq 0} &= \{X \in \mathcal{D}(\mathsf{A}) \mid H^i(X) = 0,\ \forall\, i < 0\}. \end{aligned}}$$

The proof that this is a t-structure is not hard. The axioms (**t1**) and (**t2**) are straightforward. To verify (**t3**), we define *truncation functors*

$$\tau_{\leq 0} \colon \mathcal{T} \to \mathcal{T}^{\leq 0}$$

$$\tau_{\geq 1} \colon \mathcal{T} \to \mathcal{T}^{\geq 1}$$

by

$$\boxed{\begin{array}{rl} \tau_{\leq 0}(X) := & \cdots \to X^{-2} \to X^{-1} \to \ker d \to 0 \to 0 \to \cdots \\ \tau_{\geq 1}(X) := X/\tau_{\leq 0}(X) = & \cdots \to 0 \to 0 \to X_0/\ker d \to X^1 \to X^2 \to \cdots . \end{array}}$$

It is easy to see that

$$\tau_{\leq 0}X \to X \to \tau_{\geq 1}X \to \tau_{\leq 0}X[1]$$

is exact (Lecture 2, Proposition 4.4). So, we can take $A = \tau_{\leq 0}X$ and $B = \tau_{\geq 1}X$.

Remark 3.2. This t-structure induces t-structures on each of $\mathcal{D}^{-,+,b}(\mathsf{A})$.

Exercise. Show that this does not give a t-structure on $\mathcal{K}(\mathsf{A})$. (Hint: show that (**t1**) fails by taking $X = Y$ to be an exact complex that is not contractible, e.g., a non-split short exact sequence.)

The truncation functors discussed above can indeed be defined for any t-structure.

Proposition 3.3. ([BeBeDe], §1.3) *Let $\mathcal{T}$ be a triangulated category equipped with a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$.*

(i) *The inclusion $\mathcal{T}^{\leq n} \hookrightarrow \mathcal{T}$ admits a right adjoint $\tau_{\leq n} \colon \mathcal{T} \to \mathcal{T}^{\leq n}$, and the inclusion $\mathcal{T}^{\geq n} \hookrightarrow \mathcal{T}$ admits a left adjoint $\tau_{\geq n} \colon \mathcal{T} \to \mathcal{T}^{\geq n}$.*

(ii) *For every $X \in \mathcal{T}$ and every $n \in \mathbb{Z}$, there is a unique d that makes the triangle*

$$\tau_{\leq n}X \longrightarrow X \longrightarrow \tau_{\geq n+1}X \xrightarrow{d} \tau_{\leq n}X[1]$$

exact.

Remark 3.4. An *aisle* in a triangulated category is a full saturated subcategory $\mathcal{U}$ that is closed under extensions, $\mathcal{U}[1] \subseteq \mathcal{U}$, and such that $\mathcal{U} \to \mathcal{T}$

admits a right adjoint. For example, $\mathcal{U} = \mathcal{T}^{\leq 0}$ is an aisle. Conversely, any aisle $\mathcal{U}$ is equal to $\mathcal{T}^{\leq 0}$ for a unique t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$; see [KeVo].

Proposition 3.5. ([BeBeDe], §1.3) *Let $a \leq b$. Then $\tau_{\geq a}$ and $\tau_{\leq b}$ commute, in the sense that*

$$\begin{tikzcd}
\tau_{\leq b}X \arrow[r] \arrow[d] & X \arrow[r] & \tau_{\geq a}X \arrow[d] \\
\tau_{\geq a}\tau_{\leq b}X \arrow[rr, dotted, "\exists!", "\varphi"'] & & \tau_{\leq b}\tau_{\geq a}X
\end{tikzcd}$$

is commutative. Furthermore, φ is necessarily an isomorphism.

We set $\tau_{[a,b]} := \tau_{\geq a}\tau_{\leq b}X$. We call $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ the **heart** (or **core**) of the t-structure.

Theorem 3.6. ([BeBeDe], §1.3) *The heart $\mathsf{C} := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian category. The functor $H^0 := \tau_{[0,0]} \colon \mathcal{T} \to \mathsf{C}$ is a cohomological functor.*

Example 3.7. The heart of the standard t-structure on $\mathcal{D}(\mathsf{A})$ is A, and the functor $\tau_{[0,0]}$ is the usual H^0.

Remark 3.8. In the above theorem, $\mathcal{T}$ may not be equivalent to any of $\mathcal{D}^{\emptyset,-,+,b}(\mathsf{C})$.

The moral of the story is that, by varying t-structures inside a triangulated category (e.g., $\mathcal{D}(\mathsf{A})$), we can produce new abelian categories.

4. Producing New Abelian Categories ([HaReSm])

One can use t-structures to produce new abelian categories out of given ones. This technique appeared for the first time in [BeBeDe] where they alter the t-structures on various derived categories of sheaves on a space to produce new abelian categories of *perverse sheaves* (see *loc. cit.* §2 and also Theorem 1.4.10).

Another way to produce t-structures is via *torsion theories*. For an application of this in noncommutative geometry see [Po].

A **torsion theory** on an abelian category A is a pair $(\mathbb{T}, \mathbb{F})$ of full additive subcategories of A such that:

- For every $T \in \mathbb{T}$ and $F \in \mathbb{F}$, we have $\mathrm{Hom}(T, F) = 0$.
- For every $A \in \mathsf{A}$, there is a (necessarily unique) exact sequence

$$0 \to T \to A \to F \to 0, \qquad T \in \mathbb{T},\ F \in \mathbb{F}.$$

Example 4.1. Let $\mathsf{A} = \mathbf{Ab}$ be abelian categories. Take $\mathbb{T}$ to be the torsion abelian groups and $\mathbb{F}$ the torsion-free abelian groups.

Exercise. Prove that for a torsion theory $(\mathbb{T}, \mathbb{F})$ we have $\mathbb{T}^{\perp} = \mathbb{F}$ and ${}^{\perp}\mathbb{F} = \mathbb{T}$, that is,

$$\mathbb{F} = \{F \in \mathsf{A} \mid \forall T \in \mathbb{T},\ \mathrm{Hom}(T, F) = 0\},$$

$$\mathbb{T} = \{T \in \mathsf{A} \mid \forall F \in \mathbb{F},\ \mathrm{Hom}(T, F) = 0\}.$$

Out of a given torsion theory $(\mathbb{T}, \mathbb{F})$ on an abelian category A we can construct a new abelian category B in which the roles of $\mathbb{T}$ and $\mathbb{F}$ are interchanged.

We construct B as follows. Consider the following t-structure on $\mathcal{D} = \mathcal{D}(\mathsf{A})$:

$$\mathbb{D}^{\leq 0} = \{B \in \mathcal{D}^{\leq 0} \mid H^0(B) \in \mathbb{T}\},$$

$$\mathbb{D}^{\geq 0} = \{B \in \mathcal{D}^{\geq -1} \mid H^{-1}(B) \in \mathbb{F}\}.$$

This is easily seen to be a t-structure, so, by Theorem 3.6, $\mathsf{B} := \mathbb{D}^{\leq 0} \cap \mathbb{D}^{\geq 0}$ is an abelian category. More precisely, B is equivalent to the full subcategory of $\mathcal{D}(\mathsf{A})$ consisting of complexes $d\colon A^{-1} \to A^0$ such that $\ker d \in \mathbb{F}$ and $\operatorname{coker} d \in \mathbb{T}$.

Remark 4.2. Let $\mathbb{T}'$ and $\mathbb{F}'$ be essential images of $\mathbb{F}[1]$ and $\mathbb{T}$ in B. Then $(\mathbb{T}', \mathbb{F}')$ is a torsion theory for B. (However, if we repeat the same process as above for this torsion theory, we may not get back A.)

5. Appendix I: Topological Triangulated Categories ([Ho], Sec. 7, [Ma], [We], Sec. 10.9)

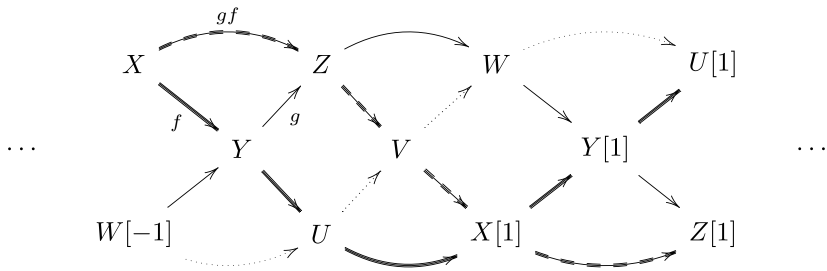
In our examples, we considered only triangulated categories that come from algebra (e.g, derived categories). There are certain triangulated categories that arise from topology (e.g., stable homotopy categories) that we have not discussed in these notes but which are worthy of attention. In line with our topological-vs-chain-complex comparison picture of Lecture 2, we say a few words about the homotopy category of spectra.

Spectra. Recall from Lecture 2 that the cone construction (which is the main input in the construction of derived categories) was motivated by a topological construction on spaces. One may wonder then whether one could repeat the construction of the derived categories in the topological setting. The problem that arises here is that *the suspension functor* $X \mapsto SX$ *is not an auto-equivalence* of the category of topological spaces (while its chain complex counterpart, the shift functor $C \mapsto C[1]$, is). To remedy this, one can formally "invert" the suspension functor. This gives rise to the category of Spanier-Whitehead spectra ([Ma], Chap. 1) from which we can construct a triangulated category by inverting weak equivalences, in the same way we obtained the derived category by inverting quasi-isomorphisms.

What made topologists unhappy about the category of Spanier-Whitehead spectra is that it is not closed under arbitrary colimits.[c] Nowadays there are better categories of spectra that do not have this deficiency. They go under various names such as *spectra*, Ω*-spectra*, *symmetric spectra*, etc. All these categories come equipped with a notion of weak equivalence, and inverting the weak equivalences in any of these categories gives rise to the *same* triangulated category, the *homotopy category of spectra*. Understanding the homotopy category of spectra is an area of active research that is called *stable homotopy theory*. To see how triangulated categories emerge in this context consult the given references.

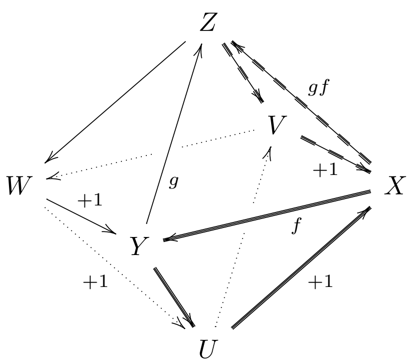
6. Appendix II: Different Illustrations of TR4

One way of illustrating **TR4** is via the following "braid" diagram:



The axiom **TR4** can be stated as saying that, given the three solid strands of braids, the dotted strand can be filled so that the diagram commutes. It is understood that each of the four long sequences is exact (i.e., obtained from a distinguished triangle). Note that we are requiring the existence of only three consecutive dotted arrows; the rest of the sequence is uniquely determined by these three.

Another depiction of the axiom **TR4** is via the following octahedron:



[c]Actually neither are the *bounded* derived categories!

The axiom requires that the dotted arrows could be filled. It is understood that the four mixed-color triangles are commutative, and the four unicolor triangles are exact.

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References

[Ad] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995.

[BeBeDe] A. A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[BoKa] A. I. Bondal and M. M. Kapranov, *Framed triangulated categories*, (Russian) Mat. Sb. **181** (1990), no. 5, 669–683; translation in Math. USSR-Sb. **70** (1991), no. 1, 93–107.

[Fr] P. Freyd, *Abelian categories*, Harper, 1966.

[Ga] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[GeMa] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, Springer-Verlag, Berlin, 1996.

[Gr] A. Grothendieck, *Sur quelques points d'algébre homolgiques*, Tôhoku Math. J. **9** (1957), 119–221.

[HaReSm] D. Happel, I. Reiten, S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575.

[Ha1] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York–Heidelberg, 1977.

[Ha2] R. Hartshorne, *Residues and duality*, Lecture Notes in Math. 20, Springer-Verlag, 1966.

[HiSt] P. Hilton, U. Stammbach, *A course in homological algebra*, Graduate Texts in Mathematics 4, Springer-Verlag, New York, 1997.

[Ho] M. Hovey, *Model categories*, Mathematical Surveys and Monographs 63, American Mathematical Society, Providence, RI, 1999.

[Iv] B. Iversen, *Cohomology of sheaves*, Universitext, Springer-Verlag, Berlin, 1986.

[Bernhard Keller's Webpage] http://www.math.jussieu.fr/%7Ekeller/publ/index.html

[Ke1] B. Keller, *Introduction to abelian and derived categories*, in *Representations of reductive groups*, edited by R. W. Carter and M. Geck, Cambridge University Press 1998, 41–62 (available on Keller's webpage).

[Ke2] B. Keller, *Derived categories and their uses*, in *Handbook of algebra*, edited by M. Hazewinkel, Elsevier 1996 (available on Keller's webpage).

[Ke3] B. Keller, *On differential graded categories*, preprint (available on Keller's webpage).

[Ke4] B. Keller, *Introduction to A_∞-algebras and modules*, Homology, Homotopy, and Applications **3** (2001), 1–35 (available on Keller's webpage).

[KeVo] B. Keller, D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Ser. A 40 (1988), no. 2, 239–253.

[KaSch] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Grundlehren der Mathematichen Wissenschaften 292, Springer-Verlag, 1990.

[Li] J. Lipman, *Notes on derived categories and derived functors*, available at http://www.math.purdue.edu/∼lipman.

[Ma] S. Mac Lane, *Categories for the working mathematician*, Second edition. Graduate Texts in Mathematics 5, Springer-Verlag, New York, 1998.

[Ma] H. R. Margolis, *Spectra and the Steenrod algebra* North-Holland Mathematical Library 29, North-Holland Publishing Co., Amsterdam, 1983.

[May1] J. P. May *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999.

[May2] J. P. May, *The axioms for triangulated categories*, preprint.

[Mi1] B. Mitchel, *The full embedding theorem*, Am. J. Math. **86** (1964), 619–637.

[Mi2] B. Mitchel, *Rings with several objects*, Adv. in Math. **8** (1972), 1–161.

[Ne] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies, 148. Princeton University Press, Princeton, NJ, 2001.

[Po] A. Polishchuk, *Noncommutative two-tori with real multiplication as noncommutative projective varieties*, J. Geom. Phys. **50**, no. 1–4, 162–187.

[Pu] D. Puppe, *On the formal structure of stable homotopy theory*, in *Colloq. Alg. Topology*, Aarhus University (1962), 65–71.

[Ro] A. Rosenberg, *The spectrum of abelian categories and reconstructions of schemes*, in *Rings, Hopf Algebras, and Brauer groups*, 257–274, Lectures Notes in Pure and Appl. Math. 197, Marcel Dekker, New York, 1998.

[Ve] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque 239, 1996.

[We] C. A. Weibel, *Introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, 1994.

An Invitation to Noncommutative Geometry

EXAMPLES OF NONCOMMUTATIVE MANIFOLDS: COMPLEX TORI AND SPHERICAL MANIFOLDS

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Contents

We survey some aspects of the theory of noncommutative manifolds focusing on the noncommutative analogs of two-dimensional tori and low-dimensional spheres. We are particularly interested in those aspects of the theory that link the differential geometry and the algebraic geometry of these spaces.

Keywords: Noncommutative tori; noncommutative spherical manifolds.

AMS Subject Classification: Primary 58B34, 46L87; Secondary 46L89, 16S38

0. Introduction

These notes are based on a series of lectures given at the International Workshop on Noncommutative Geometry held at Tehran in September 2005. The aim of the lectures was to introduce some examples of noncommutative manifolds as well as some of the tools and notions needed for their study. We focused on those examples in which recent developments have provided links between the smooth and the algebraic aspects of noncommutative geometry.

In the first part of the lectures, some aspects of the theory of noncommutative two-dimensional tori are treated. Noncommutative tori are standard prototypes of noncommutative spaces. Since the early stages of noncommutative geometry these spaces have been central examples arising naturally in various contexts (cf. [3, 35]). Their topology and differential geometry are by now well understood. In particular, vector bundles on these spaces and the corresponding theory of Morita equivalences can be characterized. Connections on these vector bundles arising as liftings of the natural derivations on the noncommutative torus give rise to a rich theory (cf. [3, 15]). Recently this theory has been recast in the context of complex algebraic geometry. The study of categories of holomorphic bundles has thrown some light on the underlying algebraic structures of these spaces (cf. [18, 32, 34, 38, 31]). Noncommutative tori can be viewed as noncommutative limits of elliptic curves. Moreover, it is believed that noncommutative tori may play a role in number theory analogous to the role played by elliptic curves (cf. [24, 25]). This opens a new perspective for the fruitful study of these spaces. We begin by discussing the differential topology of noncommutative tori. We then discuss how the theory of vector bundles on noncommutative tori and the corresponding K-theory lead to the existence of a class of noncommutative tori whose structure is related to arithmetic data. Finally we study holomorphic connections on these spaces and the homogeneous coordinate rings arising from them.

The second class of examples we treat are noncommutative spaces defined in terms of a K-theoretic equation whose solutions in the commutative case correspond to the classical spheres; we call a noncommutative space of this type a spherical manifold. The characterization of the geometry of spheres from a spectral point of view raised the question about the existence of such examples (cf. [6]). The first noncommutative examples of spherical manifolds were constructed as a one-parameter family of deformations of the algebra of smooth functions on the sphere S^4 (cf. [10]). These noncommutative spheres are a part of a broader class of noncommutative manifolds known as θ-deformations. Shortly afterwards the case of spherical manifolds in dimension 3 was carried out systematically (cf. [7, 8]). For this purpose new tools have to be introduced; these tools provide links between the algebraic and the differential aspects of noncommutative geometry.

I am happy to thank the organizers of the conference and the Institute for Studies in Theoretical Physics and Mathematics at Tehran for giving me the opportunity to take part in this very nice conference. I am also grateful to Arthur Greenspoon, Masoud Khalkhali and Matilde Marcolli; their comments helped me to greatly improve the present manuscript.

1. Noncommutative Tori

In what follows we will denote by $\mathbb{T}^2$ the two-dimensional torus $\mathbb{T}\times\mathbb{T}$, where $\mathbb{T} = S^1 = \mathbb{R}/\mathbb{Z}$. We can consider different structures on $\mathbb{T}^2$; these different structures will give rise to algebras of functions of various levels of regularity. For instance, considering $\mathbb{T}^2$ as a topological space leads to the study of its algebra of continuous functions and considering $\mathbb{T}^2$ as a smooth manifold leads to the study of its algebra of smooth functions. Once we endow the smooth manifold $\mathbb{T}^2$ with a complex structure it becomes a Riemann surface of genus one, i.e. an elliptic curve over $\mathbb{C}$. Once enriched with this algebro-geometric structure $\mathbb{T}^2$ plays a fundamental role in number theory.

In this section we will study the noncommutative analogs of $\mathbb{T}^2$. These noncommutative tori are defined in terms of noncommutative algebras which can be obtained as deformations of some algebras of functions on $\mathbb{T}^2$ and share many of their structural properties. In particular it makes sense to study the geometric behavior of these spaces in a framework which generalizes the classical setting. One advantage of passing to the noncommutative setting comes from the fact that in many cases the presence of noncommutativity enriches the classical picture. For instance, as will be seen below, noncommutative tori already exhibit at the topological and smooth levels an arithmetic behavior close to the arithmetic behavior of elliptic curves.

1.1. *Noncommutative tori as topological spaces*

As a topological space $\mathbb{T}^2$ it is characterized by $C(\mathbb{T}^2)$, the algebra of continuous functions from $\mathbb{T}^2$ to the complex numbers. In general the algebra of continuous complex-valued functions on any compact Hausdorff space is a unital commutative C^*-algebra. Conversely, by a theorem of Gelfand, any unital commutative C^*-algebra can be realized as the algebra of continuous functions on some compact Hausdorff space. This is one of the departure points of noncommutative geometry. We can drop the commutativity condition and consider a noncommutative C^*-algebra as the algebra of continuous functions on some noncommutative topological space defined by duality with this algebra. In this section we will describe the noncommutative analogs of $C(\mathbb{T}^2)$. The reader may refer to [17], [20] or [22] for the basic concepts and definitions of the theory of C^*-algebras.

Let us start by taking a closer look at $C(\mathbb{T}^2)$. The algebra $C(\mathbb{T}^2)$ can be realized as the universal C^*-algebra generated by two commuting unitaries U and V. One may for instance take U and V to be the functions $e^{2\pi \imath \varphi_1}$ and $e^{2\pi \imath \varphi_2}$ where $\varphi_j \in \mathbb{R}/\mathbb{Z},\ j = 1, 2$ give the coordinate functions on $\mathbb{T}^2$. Any element of $C(\mathbb{T}^2)$ admits a Fourier expansion in terms of powers of these unitaries. Moreover, the algebra $C(\mathbb{T}^2)$ maps to any other C^*-algebras generated by two commuting unitaries. Therefore $C(\mathbb{T}^2)$ is the *universal C^*-algebra generated by two commuting unitaries U and V.*

At a topological level noncommutative tori are defined by their algebras of continuous functions, which are noncommutative deformations of $C(\mathbb{T}^2)$ defined in terms of a similar universal property where we have dropped the commutativity condition on the generating unitaries:

Definition 1.1. Given $\theta \in \mathbb{R}$ we define $A_\theta = C(\mathbb{T}^2_\theta)$, the *algebra of continuous functions on the noncommutative torus* $\mathbb{T}^2_\theta$, as the universal C^*-algebra generated by two unitaries U and V subject to the relation:

$$UV = e^{2\pi \imath \theta} VU. \tag{1.1}$$

Note that if we take $\theta = 0$ we recover $C(\mathbb{T}^2)$.

Fix $\theta \in \mathbb{R}$. We can use a standard procedure to construct the C^*-algebra A_θ starting from the unitaries U and V together with the relation (1.1). First we construct a $*$-algebra $C_{\mathrm{alg}}(\mathbb{T}^2_\theta)$ by taking all $\mathbb{C}$-linear combinations of products of the generators U, U^*, V and V^* and imposing the relations:

$$\begin{aligned} UU^* &= U^*U = 1, \\ VV^* &= V^*V = 1, \\ UV &= e^{2\pi \imath \theta} VU. \end{aligned}$$

Equivalently $C_{\mathrm{alg}}(\mathbb{T}^2_\theta)$ is the quotient of the free algebra $\mathbb{C}\langle U, U^*, V, V^* \rangle$ by the above relations. The $*$-algebra structure is the obvious one defined by $* : U \mapsto U^*$ and $* : V \mapsto V^*$. Then we consider representations of $C_{\mathrm{alg}}(\mathbb{T}^2_\theta)$ as an algebra of bounded operators. These are given by $*$-morphisms

$$\rho : C_{\mathrm{alg}}(\mathbb{T}^2_\theta) \to \mathcal{B}(\mathcal{H}_\rho),$$

where $\mathcal{B}(\mathcal{H}_\rho)$ is the algebra of bounded operators on a Hilbert space $\mathcal{H}_\rho$. In this way we may endow $C_{\text{alg}}(\mathbb{T}^2_\theta)$ with the norm

$$\|a\| = \sup_\rho \|\rho(a)\|, \ \ a \in C_{\text{alg}}(\mathbb{T}^2_\theta)$$

where the supremum is taken over all the representations of $C_{\text{alg}}(\mathbb{T}^2_\theta)$ and the norm on the right is the operator norm. This supremum will be finite since unitary operators have norm one and the images of the generators U, U^*, V and V^* are all unitary. The C^*-completion of $C_{\text{alg}}(\mathbb{T}^2_\theta)$ under this norm is the algebra $A_\theta = C(\mathbb{T}^2_\theta)$.

For the above construction to work we have to show the existence of nontrivial representations of $C_{\text{alg}}(\mathbb{T}^2_\theta)$. For this purpose we consider the unitary operators u and v on $L^2(S^1)$ given by

$$(uf)(t) = e^{2\pi\imath\theta} f(t), \tag{1.2}$$

$$(vf)(t) = f(x - \theta). \tag{1.3}$$

Since $vu = e^{2\pi\imath\theta} uv$ the map $U \mapsto u$, $V \mapsto v$ gives a representation of $C_{\text{alg}}(\mathbb{T}^2_\theta)$ as a subalgebra of the algebra of bounded operators on $L^2(S^1)$.

From the definition we see that if $\theta = \theta' + n$ for some integer n then the corresponding algebras A_θ and $A_{\theta'}$ are isomorphic as C^*-algebras. So whenever it is necessary we can assume $0 \leq \theta < 1$. Moreover, by universality of the construction we obtain an isomorphism between A_θ and $A_{1-\theta}$ by interchanging the roles of the generators. Thus we may equally well assume that $0 \leq \theta \leq \frac{1}{2}$. It can be shown that for different values of θ in this interval one obtains nonisomorphic C^*-algebras.

Many of the tools used for the study of topological spaces can be extended to the study of C^*-algebras. One important phenomenon that arises in this context is that topological invariants like K-theory and cyclic cohomology are invariant under strong Morita equivalence of C^*-algebras. This notion coincides in the case of unital C^*-algebras with the notion of Morita equivalence for rings which identifies rings whose categories of right modules are equivalent (cf. [20]). In the following sections we will look more closely at Morita equivalences between noncommutative tori. In the context of C^*-algebras Morita equivalences between noncommutative tori are characterized by the following result:

Theorem 1.2. ([35]) *Let $SL_2(\mathbb{Z})$ act on $\mathbb{R}$ by fractional linear transformations*; *thus*

$$g\theta = \frac{a\theta + b}{c\theta + d},$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \qquad \theta \in \mathbb{R}.$$

Then the C^*-algebras A_θ and $A_{\theta'}$ are strongly Morita equivalent if and only if there exists a matrix $g \in SL_2(\mathbb{Z})$ such that $\theta' = g\theta$.

Taking into account that any two rational numbers are conjugate to each other by a fractional linear transformation in $SL_2(\mathbb{Z})$, we obtain the following corollary:

Corollary 1.3. *If $\theta \in \mathbb{Q}$, then the C^*-algebra A_θ is strongly Morita equivalent to $A_0 = C(\mathbb{T}^2)$.*

In what follows we will assume unless otherwise stated that θ is an irrational number.

The compact group $\mathbb{T}^2$ acts on the algebra A_θ. This action can be given in terms of the generators U and V by:

$$\alpha_\varphi(U) = e^{2\pi\imath\varphi_1}U,$$
$$\alpha_\varphi(V) = e^{2\pi\imath\varphi_2}V,$$

where $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T}^2$. This action is continuous in the sense that for any $a \in A_\theta$ the function

$$\begin{aligned} \mathbb{T}^2 &\to A_\theta \\ \varphi &\mapsto \alpha_\varphi(a) \end{aligned} \tag{1.4}$$

is continuous.

By continuity of the above action, the following integral makes sense as a limit of Riemann sums for any $a \in C(\mathbb{T}^2_\theta)$:

$$\int_{\mathbb{T}^2} \alpha_\varphi(a)\, d\varphi.$$

Denote by $\mathbf{1}_{A_\theta}$ the unit of the algebra A_θ. It can be shown that this integral takes values in $\mathbb{C}\mathbf{1}_{A_\theta}$ and thus it defines a linear functional which once normalized induces a trace

$$\chi : A_\theta \to \mathbb{C}. \tag{1.5}$$

Theorem 1.4. ([19]) *Let θ be an irrational number. Then χ is the unique normalized trace on A_θ invariant under the action of $\mathbb{T}^2$.*

This fact has the following important consequence (cf. [17]).

Theorem 1.5. ([19]) *Let θ be an irrational number. Then A_θ is simple, i.e. it has no nontrivial two-sided ideals.*

The C^*-algebras A_θ arise naturally in various contexts. Let us finish this section by briefly mentioning two of them.

It is possible to associate a crossed product C^*-algebra to an action of a discrete group on a topological space. Let θ be irrational and consider the action $\tilde{\theta}$ of $\mathbb{Z}$ on S^1 generated by the rotation by an angle θ. The corresponding crossed product algebra is then isomorphic to A_θ. Because of this fact the algebras A_θ are referred to as *irrational rotation algebras* (cf. [35]).

It is also possible to associate a C^*-algebra to a smooth foliation on a manifold. Consider the Kronecker foliation $d\varphi_1 = \theta d\varphi_2$ on $\mathbb{T}^2$. The C^*-algebra associated to this foliation is isomorphic to A_θ (cf. [3]).

1.2. *Smooth functions on noncommutative tori*

In this section we study the noncommutative analogs of the algebra $C^\infty(\mathbb{T}^2)$ consisting of smooth complex-valued functions on $\mathbb{T}^2$. As before we start by taking a closer look at the commutative case. In this case any element of $C(\mathbb{T}^2)$ admits a Fourier expansion and smooth functions are characterized among continuous functions as those functions whose coefficients in the corresponding Fourier expansion decay rapidly at infinity. In view of this fact we make the following definition:

Definition 1.6. Given $\theta \in \mathbb{R}$ we define $\mathcal{A}_\theta$, the *algebra of smooth functions on the noncommutative torus* $\mathbb{T}^2_\theta$, as the algebra of formal power series in two unitaries U and V with rapidly decreasing coefficients and multiplication given by the relation $UV = e^{2\pi\imath\theta}VU$:

$$\begin{aligned}\mathcal{A}_\theta &= C^\infty(\mathbb{T}^2_\theta)\\ &= \left\{ a = \sum_{n,m\in\mathbb{Z}} a_{n,m}U^nV^m \mid \{a_{n,m}\} \in \mathcal{S}(\mathbb{Z}^2) \right\}.\end{aligned}$$

The algebra $\mathcal{A}_\theta$ is a pre-C^*-algebra whose C^*-completion is isomorphic to A_θ. As a subalgebra $\mathcal{A}_\theta$ can be obtained as the set of *smooth elements of A_θ for the action of* $\mathbb{T}^2$; that is, elements $a \in A_\theta$ for which the map (1.4) is smooth.

In the algebra $\mathcal{A}_\theta$ the unique trace χ defined by (1.5) is given by

$$\chi\Big(\sum a_{n,m}U^nV^m\Big) = a_{0,0}.$$

The action of $\mathbb{T}^2$ on A_θ induces an action of its Lie algebra $L = \mathbb{R}^2$ on $\mathcal{A}_\theta$ given by the derivations

$$\delta_1(U) = 2\pi\imath U; \qquad \delta_1(V) = 0 \tag{1.6}$$

$$\delta_2(U) = 0; \qquad \delta_2(V) = 2\pi\imath V. \tag{1.7}$$

These derivations determine the Fréchet structure of $\mathcal{A}_\theta$. The relation between $\mathcal{A}_\theta$ and A_θ parallels the relation between $C^\infty(\mathbb{T}^2)$ and $C(\mathbb{T}^2)$. We refer the reader to the seminal paper [3] and the survey [36] for the main results about the algebras $A_\theta = C(\mathbb{T}^2_\theta)$ and $\mathcal{A}_\theta = C^\infty(\mathbb{T}^2_\theta)$. In the next section we exploit the corresponding relation between smooth vector bundles and continuous vector bundles over the noncommutative torus.

1.3. *Morita equivalences and real multiplication*

By the Serre-Swan theorem the theory of vector bundles over $\mathbb{T}^2$ is equivalent to the theory of finite type projective modules over the algebra $C(\mathbb{T}^2)$. To each complex vector bundle over $\mathbb{T}^2$ one associates the $C(\mathbb{T}^2)$-module of its global sections. Smooth bundles correspond to finite type projective modules over $C^\infty(\mathbb{T}^2)$ and every continous vector bundle over $\mathbb{T}^2$ is equivalent to a smooth vector bundle.

We consider projective finite type right A_θ-modules as continuous vector bundles over $\mathbb{T}^2_\theta$ and projective finite type right $\mathcal{A}_\theta$-modules as smooth vector bundles over $\mathbb{T}^2_\theta$. If $\tilde{E}$ is a projective finite type right A_θ-module, then there exists a projective finite type right $\mathcal{A}_\theta$-module E such that one has an isomorphism of right A_θ-modules:

$$\tilde{E} \simeq E \otimes_{\mathcal{A}_\theta} A_\theta.$$

Therefore, as in the commutative case, the categories of smooth and continuous vector bundles over $\mathbb{T}^2_\theta$ are equivalent (cf. [3]). In what follows we will restrict to $\mathcal{A}_\theta$-modules.

For any positive integer n the trace χ can be extended to a trace Tr_χ on the matrix algebra

$$\begin{aligned}\mathcal{M}_n(\mathcal{A}_\theta) &= \mathcal{A}_\theta \otimes \mathcal{M}_n(\mathbb{C}) \\ &\simeq \mathrm{End}(\mathcal{A}_\theta^n).\end{aligned}$$

We can use this trace to define a rank function for vector bundles over $\mathbb{T}^2_\theta$. A right $\mathcal{A}_\theta$-module E is projective of finite type if and only if there exists some n and an idempotent $e = e^2 = e^*$ in $M_n(\mathcal{A}_\theta)$ such that $E \simeq e\mathcal{A}_\theta^n$; thus we can define the rank of E by

$$\mathrm{rk}(E) = \mathrm{Tr}_\chi(e). \tag{1.8}$$

In what follows unless otherwise stated a right (resp. left) $\mathcal{A}_\theta$-module will always mean a right (resp. left) projective finite type $\mathcal{A}_\theta$-module.

Let $\theta \in \mathbb{R}$ be irrational. Following [3] we define, for any pair $c, d \in \mathbb{Z}$, $c > 0$, a right $\mathcal{A}_\theta$-module $E_{d,c}(\theta)$ given by the following action of $\mathcal{A}_\theta$ on the Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z}) = \mathcal{S}(\mathbb{R})^c$:

$$(fU)(x, \alpha) = f\left(x - \frac{c\theta + d}{c}, \alpha - 1\right), \tag{1.9}$$

$$(fV)(x, \alpha) = \exp\left(2\pi\imath\left(x - \frac{\alpha d}{c}\right)\right) f(x, \alpha). \tag{1.10}$$

Theorem 1.7. ([3]) *The rank of $E_{d,c}(\theta)$ is $|c\theta + d|$. If E is any right $\mathcal{A}_\theta$-module with* $\mathrm{rk}(E) = |c\theta + d|$ *then $E \simeq E_{d,c}(\theta)$.*

The K_0 group of $\mathcal{A}_\theta$, $K_0(\mathcal{A}_\theta)$, is by definition the enveloping group of the abelian semigroup given by isomorphism classes of right $\mathcal{A}_\theta$-modules together with direct sum. The rank function rk extends to an injective morphism

$$\mathrm{rk} : K_0(\mathcal{A}_\theta) \to \mathbb{R} \tag{1.11}$$

whose image is $\mathbb{Z} \oplus \theta\mathbb{Z}$. Therefore, one gets an ordered structure on $K_0(\mathcal{A}_\theta)$ given by the isomorphism

$$K_0(\mathcal{A}_\theta) \simeq \mathbb{Z} \oplus \theta\mathbb{Z} \subset \mathbb{R}. \tag{1.12}$$

By a cancellation theorem due to Rieffel the abelian semigroup of isomorphism classes of right $\mathcal{A}_\theta$-modules is a cancellation semigroup (cf. [37]). This fact together with the fact that rk is injective imply that right $\mathcal{A}_\theta$-modules are classified up to isomorphism by their rank and that any finite type projective right $\mathcal{A}_\theta$-module is either free or isomorphic to a right module of the form $E_{d,c}(\theta)$.

If c and d are relatively prime we say that $E_{d,c}(\theta)$ is a *basic* $\mathcal{A}_\theta$-module. This being the case the pair d, c can be completed to a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{1.13}$$

We write $E_g(\theta)$ for the module $E_{d,c}(\theta)$. By definition the degree of $E_g(\theta)$ is taken to be c. We also define the degree of a matrix $g \in SL_2(\mathbb{Z})$, given as above, by $\deg(g) = c$.

Let $SL_2(\mathbb{Z})$ act on $\mathbb{R}$ by fractional linear transformations. Let $g \in SL_2(\mathbb{Z})$ be as above and denote by U' and V' two generating unitaries of the algebra $\mathcal{A}_{g\theta}$. We can define a left action of the algebra $\mathcal{A}_{g\theta}$ on E_g

by:

$$(U'f)(x,\alpha) = f\left(x - \frac{1}{c}, \alpha - a\right), \tag{1.14}$$

$$(V'f)(x,\alpha) = \exp\left(2\pi\imath\left(\frac{x}{c\theta + d} - \frac{\alpha}{c}\right)\right) f(x,\alpha). \tag{1.15}$$

This action gives an identification:

$$\mathrm{End}_{\mathcal{A}_\theta}(E_g(\theta)) \simeq \mathcal{A}_{g\theta}. \tag{1.16}$$

The tensor product of basic modules is again a basic module. More precisely, given $g_1, g_2 \in SL_2(\mathbb{Z})$ there is a well-defined pairing of right $\mathcal{A}_\theta$-modules:

$$t_{g_1,g_2} : E_{g_1}(g_2\theta) \otimes_{\mathbb{C}} E_{g_2}(\theta) \to E_{g_1 g_2}(\theta). \tag{1.17}$$

This map gives rise to an isomorphism of $\mathcal{A}_{g_1 g_2 \theta} - \mathcal{A}_\theta$ bimodules:

$$E_{g_1}(g_2\theta) \otimes_{\mathcal{A}_{g_2\theta}} E_{g_2}(\theta) \to E_{g_1 g_2}(\theta). \tag{1.18}$$

In particular, if $g\theta = \theta$ one has an isomorphism

$$\underbrace{E_g(\theta) \otimes_{\mathcal{A}_\theta} \cdots \otimes_{\mathcal{A}_\theta} E_g(\theta)}_{n} \simeq E_{g^n}(\theta). \tag{1.19}$$

Taking into account the remarks at the beginning of this section we can now go back to the setting of Sec. 1.2. We say that two noncommutative tori $\mathbb{T}^2_{\theta'}$ and $\mathbb{T}^2_\theta$ are Morita equivalent if their corresponding algebras of smooth functions are Morita equivalent. Thus $\mathbb{T}^2_{\theta'}$ and $\mathbb{T}^2_\theta$ are Morita equivalent if and only if there exist an $\mathcal{A}_{\theta'}$-$\mathcal{A}_\theta$-bimodule which is projective and of finite type both as a left $\mathcal{A}_{\theta'}$-module and as a right $\mathcal{A}_\theta$-module. We will consider the category whose objects are noncommutative tori and whose morphisms are given by isomorphism classes of finite type projective bimodules over the corresponding algebras of smooth functions. Composition is provided by tensor product over the corresponding algebra. An isomorphism in this category is called a *Morita equivalence*. From the discussion above we see that given a real number θ and a matrix $g \in SL_2(\mathbb{Z})$ the noncommutative torus $\mathbb{T}^2_{g\theta}$ and $\mathbb{T}^2_\theta$ are Morita equivalent. The inverse of the morphism represented by the $\mathcal{A}_{g\theta}$- $\mathcal{A}_\theta$-bimodule $E_g(\theta)$ is the morphism represented by the $\mathcal{A}_\theta$-$\mathcal{A}_{g\theta}$-bimodule$E_{g^{-1}}(g\theta)$. By a result of Rieffel these are the only possible Morita equivalences in the category of noncommutative tori (cf. [35]). More precisely, two noncommutative tori $\mathbb{T}^2_{\theta'}$ and $\mathbb{T}_\theta$ are Morita equivalent if and only if there exists $g \in SL_2(\mathbb{Z})$ such that $\theta' = g\theta$.

If $g\theta = \theta$, then $E_g(\theta)$ has the structure of $\mathcal{A}_\theta$-bimodule. An irrational number $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is a fixed point of a fractional linear transformation $g \in SL_2(\mathbb{Z})$ if and only if it generates a quadratic extension of $\mathbb{Q}$.

Definition 1.8. The noncommutative torus $\mathbb{T}^2_\theta$ with algebra of smooth functions $\mathcal{A}_\theta$ is a *real multiplication noncommutative torus* if the parameter θ generates a quadratic extension of $\mathbb{Q}$.

Thus $\mathbb{T}^2_\theta$ is a real multiplication noncommutative torus if and only if it has nontrivial Morita autoequivalences.

In [24] Manin proposed the use of noncommutative tori as a geometric framework under which to attack the explicit class field theory problem for real quadratic extensions of $\mathbb{Q}$. The explicit class field theory problem for number fields asks for explicit generators of the maximal abelian extension of a given number field and the corresponding Galois action of the abelianization of the absolute Galois group on these generators. The only number fields for which a complete solution of this problem is known are the imaginary quadratic extensions of $\mathbb{Q}$ and $\mathbb{Q}$ itself. Elliptic curves whose endomorphism ring is nontrivial correspond to lattices generated by imaginary quadratic irrationalities and play a central role in the solution of the explicit class field theory problem for the corresponding imaginary quadratic extensions. It is believed that noncommutative tori with real multiplication may play an analogous role in the study of real quadratic extensions of $\mathbb{Q}$. In order to achieve arithmetical applications, it is important to realize algebraic structures underlying noncommutative tori. This is our main motivation for the study of the homogeneous coordinate rings described below.

1.4. *Complex structures on tori and holomorphic connections*

A complex structure on the noncommutative torus $\mathbb{T}^2_\theta$ is determined through the derivations δ_1 and δ_2 on $\mathcal{A}_\theta$ by choosing a complex structure on the Lie algebra $L = \mathbb{R}^2$ of $\mathbb{T}^2$. For this we make a decomposition of the complexification of L into two complex conjugate subspaces. This can be done by choosing a complex parameter τ with nonzero imaginary part and taking $\{1, \tau\}$ as a basis for the holomorphic part of this decomposition. The resulting derivation $\delta_\tau = \tau\delta_1 + \delta_2$ is by definition a complex structure on $\mathbb{T}^2_\theta$. Explicitly we have

$$\delta_\tau : \sum_{n,m\in\mathbb{Z}} a_{n,m}U^nV^m \mapsto (2\pi\imath) \sum_{n,m\in\mathbb{Z}} (n\tau + m)a_{n,m}U^nV^m. \tag{1.20}$$

This derivation should be viewed as an analog of the operator $\bar\partial$ on a complex elliptic curve. We will denote by $\mathbb{T}^2_{\theta,\tau}$ the noncommutative torus $\mathbb{T}^2_\theta$ equipped with this complex structure. In what follow we will assume that $\mathrm{Im}(\tau) < 0$. We will also freely refer to τ as the complex structure on $\mathbb{T}^2_{\theta,\tau}$.

Complex structures on noncommutative tori were introduced by Connes in relation with the Yang–Mills equation and positivity in Hochschild cohomology for noncommutative tori (cf. [4]). The study of the structure of the space of connections associated to the above derivations was carried out in [15]. An approach through noncommutative analogs of theta functions was developed in [38, 18], where these are viewed as holomorphic sections of vector bundles on noncommutative tori. The resulting categories were studied thoroughly in [34] and [33].

A holomorphic structure on a right $\mathcal{A}_\theta$-module E is given by an operator $\bar{\nabla} : E \to E$ which is compatible with the complex structure δ_τ in the sense that it satisfies the following Leibniz rule:

$$\bar{\nabla}(ea) = \bar{\nabla}(e)a + e\delta_\tau(a), \quad e \in E, a \in \mathcal{A}_\theta. \tag{1.21}$$

Given a holomorphic structure $\bar{\nabla}$ on a right $\mathcal{A}_\theta$-module E the corresponding set of holomorphic sections is the space

$$H^0(\mathbb{T}^2_{\theta,\tau}, E_{\bar{\nabla}}) := \mathrm{Ker}(\bar{\nabla}). \tag{1.22}$$

On every basic module $E_{d,c}$ one can define a family of holomorphic structures $\{\bar{\nabla}_z\}$ depending on a complex parameter $z \in \mathbb{C}$:

$$\bar{\nabla}_z(f) = \frac{\partial f}{\partial x} + 2\pi\imath \left(\frac{d\tau}{c\theta + d} x + z \right) f. \tag{1.23}$$

By definition a *standard holomorphic vector bundle on* $\mathbb{T}^2_{\theta,\tau}$ is given by a basic module $E_{d,c} = E_g$ together with one of the holomorphic structures $\bar{\nabla}_z$.

The spaces of holomorphic sections of a standard holomorphic vector bundle on $\mathbb{T}^2_{\theta,\tau}$ are finite dimensional (cf. [34], Sec. 2). If $c\theta + d > 0$ then $\dim H^0(E_g, \bar{\nabla}_0) = c$. In what follows we will consider the spaces of holomorphic sections corresponding to $\bar{\nabla}_0$:

$$\mathcal{H}_g := H^0(\mathbb{T}^2_{\theta,\tau}, E_{g,\bar{\nabla}_0}). \tag{1.24}$$

A basis of $\mathcal{H}_g$ is given by the Schwartz functions:

$$\varphi_\alpha(x,\beta) = \exp\left(-\frac{c\tau}{c\theta + d} \frac{x^2}{2} \right) \delta_\alpha^\beta \qquad \alpha = 1, ..., c. \tag{1.25}$$

The tensor product of holomorphic sections is again holomorphic. Using the above basis the product can be written in terms of the corresponding structure constants.

Theorem 1.9. ([34], *Sec.* 2) *Suppose g_1 and g_2 have positive degree. Then $g_1 g_2$ has positive degree and t_{g_1,g_2} induces a well-defined linear map*

$$t_{g_1,g_2} : \mathcal{H}_{g_1}(g_2\theta) \otimes_{\mathbb{C}} \mathcal{H}_{g_2}(\theta) \to \mathcal{H}_{g_1 g_2}(\theta). \tag{1.26}$$

Let g_1, g_2 and $g_1 g_2$ be given by

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad g_1 g_2 = \begin{pmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{pmatrix}$$

and let $\{\varphi_\alpha\}$, $\{\varphi'_\beta\}$ *and* $\{\psi_\gamma\}$ *be respectively the bases of* $\mathcal{H}_{g_1}(g_2\theta)$, $\mathcal{H}_{g_2}(\theta)$ *and* $\mathcal{H}_{g_1g_2}(\theta)$ *as given in* (1.25). *Then*

$$t_{g_1,g_2} : \varphi_\alpha \otimes \varphi'_\beta \mapsto C^\gamma_{\alpha,\beta}\psi_\gamma, \tag{1.27}$$

where for $\alpha = 1, ..., c_1$, $\beta = 1, ..., c_2$ *and* $\gamma = 1, ..., c_{12}$ *we have*

$$C^\gamma_{\alpha,\beta} = \sum_{m\in I_{g_1,g_2}(\alpha,\beta,\gamma)} \exp\left[-\pi\imath\frac{\tau m^2}{2c_1c_2c_{12}}\right] \tag{1.28}$$

with

$$\begin{aligned} I_{g_1,g_2}(\alpha,\beta,\gamma) = \{n \in \mathbb{Z}\,|\, n &\equiv -c_1\gamma + c_{12}\alpha \mod c_{12}c_1, \\ n &\equiv c_2d_{12}\gamma - c_{12}d_2\beta \mod c_{12}c_2\}\end{aligned}$$

(we use the convention of summing over repeated indices).

1.5. *Homogeneous coordinate rings*

Given a projective scheme Y over a field k together with an ample line bundle $\mathcal{L}$ on Y one can construct the homogeneous coordinate ring

$$B = \bigoplus_{n\geq 0} H^0(Y, \mathcal{L}^{\otimes n}).$$

This ring plays a prominent role in the study of the geometry of Y.

In [32] Polishchuk proposed an analogous definition of the homogeneous coordinate ring of a real multiplication noncommutative torus $\mathbb{T}^2_\theta$ in terms of holomorphic sections of tensor powers of a standard holomorphic vector bundle on $\mathbb{T}^2_{\theta,\tau}$. As mentioned above the real multiplication condition is fundamental in order to be able to perform the tensor power operation on holomorphic bundles over $\mathbb{T}^2_{\theta,\tau}$.

Assume that $\theta \in \mathbb{R}$ is a quadratic irrationality. So there exists some $g \in SL_2(\mathbb{Z})$ with $g\theta = \theta$ and $\mathbb{T}^2_\theta$ has real multiplication. Fix a complex structure τ on $\mathbb{T}^2_\theta$. In the case $E = E_g(\theta)$ we can extend a holomorphic structure on E_g to a holomorphic structure on the tensor powers $E_g^{\otimes n}$. Following [32] we define a *homogeneous coordinate ring for* $\mathbb{T}^2_{\theta,\tau}$ by

$$\begin{aligned} B_g(\theta,\tau) &= \bigoplus_{n\geq 0} H^0(\mathbb{T}^2_{\theta,\tau}, E^{\otimes n}_{\bar\nabla_0}) \\ &= \bigoplus_{n\geq 0} \mathcal{H}_{g^n}. \end{aligned} \tag{1.29}$$

The category of holomorphic vector bundles on $\mathbb{T}^2_{\theta,\tau}$ is equivalent to the heart $\mathcal{C}^\theta$ of a t-structure on $\mathcal{D}^b(E_\tau)$, the derived category of the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. In [32] Polishchuk exploits this equivalence in order to study the properties of the algebra $B_g(\theta,\tau)$ by studying the corresponding image under this equivalence (cf. [23]).

The following result characterizes some structural properties of $B_g(\theta,\tau)$ in terms of the matrix elements of g:

Theorem 1.10. ([32] Theorem 3.5) *Assume $g \in SL_2(\mathbb{Z})$ has positive real eigenvalues*:

(1) *If $c \geq a+d$ then $B_g(\theta,\tau)$ is generated over $\mathbb{C}$ by $\mathcal{H}_g$.*
(2) *If $c \geq a+d+1$ then $B_g(\theta,\tau)$ is a quadratic algebra.*
(3) *If $c \geq a+d+2$ then $B_g(\theta,\tau)$ is a Koszul algebra.*

Let us recall these definitions. If $A = \bigoplus_{n\geq 0} A_n$ is a connected graded algebra over a field k generated by its degree one piece A_1, then A is isomorphic to a quotient $T(A_1)/\mathcal{I}$ where $T(A_1) = \bigoplus_{n\geq 0} A_1^{\otimes n}$ is the tensor algebra of the vector space A_1 and $\mathcal{I}$ is a two-sided ideal in $T(A_1)$. The algebra A is a *quadratic algebra* if the ideal $\mathcal{I}$ can be generated by homogeneous elements of degree two. Since A is connected, we can consider $A_0 = k$ as a left module over A. A quadratic algebra A is a *Koszul algebra* if the graded k-algebra $\bigoplus_{n\geq 0} \mathrm{Ext}^n_A(k,k)$ is generated by $\mathrm{Ext}^1_A(k,k) \simeq A_1^*$.

In a different but related perspective homogeneous coordinate rings on noncommutative tori have been recently studied by Vlasenko in [43] where the theory of rings of quantum theta functions is developed.

1.6. *Arithmetic structures*

As in the last section let θ be a real quadratic irrationality, assume $g \in SL_2(\mathbb{Z})$ is a fractional linear transformation fixing θ and let τ be a complex structure on $\mathbb{T}^2_\theta$. The restriction to $\mathcal{H}_g \otimes \mathcal{H}_g$ of the multiplication map in the homogeneous coordinate ring $B_g(\theta,\tau)$ is given by the map $t_{g,g}$ defined in (1.27). The corresponding structure constants $C^\gamma_{\alpha,\beta}$ are of the form

$$\vartheta_r(l\tau), \tag{1.30}$$

where $\vartheta_r(l\tau)$ is the theta constant with rational characteristic r defined by the series

$$\vartheta_r(l\tau) = \sum_{n\in\mathbb{Z}} \exp[\pi\imath(n+r)^2 l\tau] \tag{1.31}$$

and $r \in \mathbb{Q}$, $l \in \mathbb{N}$ depend on γ,α,β and g (the reader may refer to [29] and [30] for the main results about theta constants). The arithmetic consequences of this fact were studied in [31]. In this section we survey some of these results. Throughout the section we assume g satisfies condition (3) of Theorem 1.10 and so $B_g(\theta,\tau)$ is a quadratic Koszul algebra.

First we relate the field of definition the quadratic algebra $B_g(\theta, \tau)$ with the field of definition of the elliptic curve with period lattice generated by the complex structure τ on $\mathbb{T}^2_\theta$:

Theorem 1.11. ([31]) *Let E_τ be the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. Let k be its minimal field of definition. Then the algebra $B_g(\theta, \tau)$ admits a rational presentation over a finite algebraic extension K of k.*

The minimal field of definition k of the elliptic curve E_τ is given by adjoining to $\mathbb{Q}$ the value of the absolute invariant $j(E_\tau)$ of the curve E_τ. By (1.25) there is an isomorphism $\mathcal{H}_g \simeq \sum_{i=1}^c \mathbb{C}x_i$; then by Theorem 1.11 there is a finite algebraic extension K of k and an embedding $K \hookrightarrow \mathbb{C}$ such that the algebra $B_g(\theta, \tau)$ admits a presentation

$$B_g(\tau, \theta) = \mathbb{C}\langle x_1, ..., x_c\rangle/\mathcal{R} \tag{1.32}$$

with $\mathcal{R}$ a two-sided ideal in the free algebra $\mathbb{C}\langle x_1, ..., x_c\rangle$ generated by elements in $(\sum_{i=1}^c Kx_i)^{\otimes 2}$. In particular it makes sense to restrict scalars and consider the K-algebra.

$$B_g(\tau, \theta)_K = K\langle x_1, ..., x_c\rangle/\mathcal{R}. \tag{1.33}$$

Starting with Theorem 1.11 we can study the properties of $B_g(\tau, \theta)$ for special values of τ giving interesting fields of definition. For instance, one can start with an elliptic curve defined over a number field; in this case $j(E_\tau)$ is algebraic over $\mathbb{Q}$ and the field K is again a number field.

The fact that the homogeneous coordinate ring $B_g(\tau, \theta)$ can be described in terms of theta constants also has some nice consequences when considering $B_g(\tau, \theta)$ as a family of algebras parametrized by τ. Theta constant with rational characteristics are half-weight modular forms. From this fact a presentation of the algebra $B_g(\tau, \theta)$ can be given in such a way that each of the coefficients in the defining relations is a modular function of certain weight and level. Being more explicit:

Theorem 1.12. ([31]) *A presentation*

$$B_g(\tau, \theta) = \mathbb{C}\langle x_1, ..., x_c\rangle/\mathcal{R} \tag{1.34}$$

can be given in such a way that the two-sided ideal $\mathcal{R}$ is generated by elements of the form:

$$f = v^1 x_{i_1} x_{j_1} + \cdots + v^c x_{i_c} x_{j_c}$$

where, as a function of τ, each v^k is a modular form of cusp type.

This result can be used in order to define quadratic algebras associated to $\mathbb{T}^2_\theta$ independently of the choice of a particular complex structure τ. Modular forms of cusp type define cohomology classes in universal covers of modular curves and therefore can be paired with canonical cycles on these covers. By averaging the coefficients of the defining relations over these cycles one obtains algebras which correspond to averages of the homogeneous coordinate rings $B_g(\tau,\theta)$ along curves in the space $\mathbb{H} = \{\tau \in \mathbb{C}|\mathrm{Im}(\tau) < 0\}$ parametrizing complex structures on $\mathbb{T}^2_\theta$. Of particular interest are those algebras coming from averages along geodesics in $\mathbb{H}$ connecting points in the set of cusps $\mathbb{Q} \cup \{\infty\}$. These geodesics correspond to homology classes known as modular symbols (cf. [26, 39]). Considering geodesics in $\mathbb{H}$ with limits in $\mathbb{R} \setminus \mathbb{Q}$ the theory of modular symbols can be extended in order to define homology classes corresponding to asymptotic limits of modular symbols (cf. [27]). In particular Yu. Manin and M. Marcolli have shown that the limiting modular symbol associated to the quadratic irrationality θ exists and can be computed as a linear combinations of classical modular symbols corresponding to cusps in $\mathbb{Q} \cup \{\infty\}$. Applying the averaging process described above along this linear combination of modular symbols one may obtain a quadratic algebra $B_g(\theta)$ canonically associated to g and θ.

2. Spherical Manifolds

The fact that the geometry of a Riemannian manifold is encoded in terms of spectral data plays a central role in noncommutative geometry (cf. [6, 5, 12]). This spectral point of view was exemplified by A. Connes in the survey article [6] through the study and characterization of spheres. The non-triviality of the pairing between the K-theory of the spheres and the K-homology class of the Dirac operator is given in terms of a simple polynomial equation. This non-triviality is fundamental in spectral geometry. The formula of this pairing owes its simple form to the fact that lower terms of the Chern character of canonical elements in the K-theory of spheres vanish. The vanishing of these lower terms encode to a large extent the algebraic properties of the algebras of functions over the sphere. One remarkable fact is that there are noncommutative spaces that share the geometric and algebraic characteristics exhibited by spheres.

2.1. *Chern characters*

We start by recalling the definition of the Chern character in K-theory. Let $\mathcal{A}$ be a unital $*$-algebra; we denote by $\tilde{\mathcal{A}}$ the quotient of $\mathcal{A}$ by the subspace $\mathbb{C}\mathbf{1}_A$. Recall that $K_0(\mathcal{A})$, the even K-theory of $\mathcal{A}$, can be defined in terms of equivalence classes of self-adjoint idempotents in the matrix algebras $M_q(\mathcal{A})$. Recall also that $K_1(\mathcal{A})$, the odd K-theory group of $\mathcal{A}$, can be defined in terms of equivalence classes of unitaries in the matrix algebras $M_q(\mathcal{A})$ (cf. [22]). As before we use the convention of summing over repeated indices.

Definition 2.1. The (even) Chern character of a self-adjoint idempotent $e = e^2 = e^* \in M_q(\mathcal{A})$,

$$\mathrm{ch}(e) = \mathrm{ch}_0(e) + \mathrm{ch}_1(e) + \mathrm{ch}_2(e) + \cdots \tag{2.1}$$

is given by the homogeneous components $\mathrm{ch}_k(e) \in \mathcal{A} \otimes (\tilde{\mathcal{A}}^{\otimes 2k})$ defined by

$$\begin{aligned}\mathrm{ch}_k(e) &= \mathrm{Tr}\left(\left(e - \frac{1}{2}\right) \underbrace{e \otimes \cdots \otimes e}_{2k}\right)\\ &= \left(e^{i_0}_{i_1} - \frac{1}{2}\delta^{i_0}_{i_1}\right) \otimes e^{i_1}_{i_2} \otimes e^{i_2}_{i_3} \otimes \cdots \otimes e^{i_k}_{i_0}.\end{aligned} \tag{2.2}$$

Definition 2.2. The (odd) Chern character of a unitary matrix $U \in M_q(\mathcal{A})$, $U^*U = UU^* = \mathbf{1}_A$,

$$\mathrm{ch}(U) = \mathrm{ch}_{\frac{1}{2}}(U) + \mathrm{ch}_{\frac{3}{2}}(U) + \mathrm{ch}_{\frac{5}{2}}(U) + \cdots \tag{2.3}$$

is given by the homogeneous components $\mathrm{ch}_{k+\frac{1}{2}}(U) \in \mathcal{A} \otimes (\tilde{\mathcal{A}}^{\otimes(2k+1)})$ defined by

$$\begin{aligned}\mathrm{ch}_{k+\frac{1}{2}}(U) &= \mathrm{Tr}(\underbrace{(U-1) \otimes (U^*-1) \otimes \cdots \otimes (U-1) \otimes (U^*-1)}_{2k+2})\\ &= U^{i_0}_{i_1} \otimes U^{*i_1}_{i_2} \otimes \cdots \otimes U^{*i_{2k+1}}_{i_0} - U^{*i_0}_{i_1} \otimes U^{i_1}_{i_2} \otimes \cdots \otimes U^{i_{2k+1}}_{i_0}\end{aligned} \tag{2.4}$$

(we use the convention of summing over repeated indices).

The above Chern characters define maps between the K-theory groups of the algebra $\mathcal{A}$ and its cyclic homology groups (cf. [4, 6]). In the commutative case (corresponding to the algebra of functions on a smooth manifold) one recovers the classical Chern character from the Atiyah–Hirzebruch K-theory to the de Rham homology of the manifold. It is also possible to define a Chern character in K-homology. The pairing between the K-theory of the algebra $\mathcal{A}$ and its K-homology can be computed via the corresponding Chern characters in terms of the pairing between the cyclic homology of the algebra and its cyclic cohomology. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ the operator D defines an element in the K-homology of the algebra $\mathcal{A}$. This class plays the role of the fundamental class of the geometry encoded by the triple $(\mathcal{A}, \mathcal{H}, D)$ (cf. [12]).

2.2. *The two sphere* S^2 ([6])

Consider the following projector over the sphere S^2:

$$e = \frac{1}{2}\begin{pmatrix} 1+z & x - \imath y \\ x + \imath y & 1 - z\end{pmatrix} \in M_2(C^\infty(S^2)), \tag{2.5}$$

where x, y and z are the coordinate functions of $S^2 \subset \mathbb{R}^3$ (so $x^2+y^2+z^2 = 1$). One checks that $e = e^2 = e^*$ and so e is actually a projection. Also $\mathrm{ch}_0(e) = 0$ and the element $\mathrm{ch}_1(e) = \frac{\imath}{2}[x \otimes (y \otimes z - z \otimes y) - y \otimes (x \otimes z - z \otimes x) + z \otimes (x \otimes y - x \otimes y)]$ corresponds (up to the constant $\frac{\imath}{2}$) to the volume form of the round metric on S^2.

One can go the other way around and note that the algebra $C^\infty(S^2)$ is generated by the matrix elements of the projection e. Moreover, it can be shown that the conditions $e = e^2 = e^*$, $\mathrm{ch}_0(e) = 0$ and $\mathrm{ch}_1(e) \neq 0$ determine completely the algebra $C^\infty(S^2)$ (cf. [42]). Once we impose this condition on a 2×2 matrix with entries in a $*$-algebra it follows that such matrix can be written in the form

$$e = \frac{1}{2}\begin{pmatrix} 1+z & x - \imath y \\ x + \imath y & 1 - z \end{pmatrix}, \tag{2.6}$$

where x, y and z are mutually commuting self-adjoint elements of the algebra and satisfy the relation $x^2 + y^2 + z^2 = 1$. The joint spectrum of the $*$-algebra generated by x, y and z in any faithful representation is S^2.

2.3. *The noncommutative spheres* S^4_θ ([10])

In dimension four interesting phenomena appear. This was first noticed by studying projections satisfying the same K-theoretic equations as the corresponding analogs of the above projection for S^4. These equations impose relations on the algebras generated by their matrix elements. A. Connes and G. Landi studied the geometry of the corresponding noncommutative spaces as part of their work on noncommutative instantons. Below we recall their construction of noncommutative analogs of S^4.

Let θ be real number and let $C_{\mathrm{alg}}(S^4_\theta)$ be the unital $*$-algebra generated by 3 elements a, b and x together with the relations

$$\begin{aligned} x &= x^* \\ a^*a &= aa^* \\ b^*b &= bb^* \\ ab &= e^{2\pi\imath\theta}ba \\ a^*b &= e^{-2\pi\imath\theta}ba^* \\ a^*a + bb^* + x^2 &= 1. \end{aligned}$$

We call $C_{\mathrm{alg}}(S^4_\theta)$ the algebra of polynomial functions on the noncommutative 4-sphere S^4_θ.

Theorem 2.3. ([10]) *Consider the matrix*

$$e = \begin{pmatrix} 1+x & 0 & a & b \\ 0 & 1+x & -\lambda b^* & a^* \\ a^* & -\bar{\lambda} b & 1-x & 0 \\ b^* & a & 0 & 1-x \end{pmatrix} \in M_4(C_{\mathrm{alg}}(S^4_\theta)), \qquad \lambda = e^{2\pi\imath\theta}. \tag{2.7}$$

Then

- $e = e^2 = e^*$.
- $\mathrm{ch}_0(e) = \mathrm{ch}_1(e) = 0$.
- $\mathrm{ch}_2(e) \neq 0$.

The algebra $C_{\mathrm{alg}}(S^4_\theta)$ can be completed to a pre-C^*-algebra $C^\infty(S^4_\theta)$, this algebra acts on the Hilbert space $\mathcal{H}$ of L^2 spinors over on S^4 and together with the corresponding Dirac operator D forms an even spectral triple $(C^\infty(S^4_\theta), \mathcal{H}, D)$ that satisfies all the axioms of a noncommutative spin geometry.

More generally one can consider a compact manifold M whose automorphism group has rank 2. In this case the torus $\mathbb{T}^2$ acts on the algebra $C^\infty(M)$ and we may twist the product by considering also the action of $\mathbb{T}^2$ on the noncommutative torus $\mathbb{T}^2_\theta$. More precisely, one defines the pre-C^*-algebra $C^\infty(M_\theta)$ as the sub-algebra of the product $C^\infty(M)\hat{\otimes}C^\infty(\mathbb{T}^2_\theta)$ consisting of elements which are invariant under the diagonal action of the group $\mathbb{T}^2$. We consider then the algebra $C^\infty(M_\theta)$ as the algebra of smooth function on the noncommutative manifold M_θ. We will refer to the class of noncommutative manifolds obtained in this way and their higher dimensional analogs as *θ-deformations* of classical manifolds (cf. [10, 7]).

If the manifold M is a spin manifold and $(C^\infty(M), \mathcal{H}, D)$ is the corresponding commutative spectral triple then $C^\infty(M_\theta)$ acts on the Hilbert space $\mathcal{H}$ and it can be shown that $(C^\infty(M_\theta), \mathcal{H}, D)$ is a spectral triple. These kinds of geometries are called *isospectral deformations.*

2.4. *Noncommutative 3-spheres*

In this section we survey the results of [8, 9] where a complete classification of the algebras corresponding to the analog in dimension 3 of the above situation was given. The study of the moduli space that parametrizes these noncommutative three-dimensional spheres was also carried out in detail in these articles. The tools developed by A. Connes and M. Dubois-Violette for the study of the geometric properties of these noncommutative spaces provide a bridge between the purely algebraic framework of quadratic algebras and the analytic framework of C^*-algebras. The notion of central quadratic form plays a fundamental role in order to control the behavior of the corresponding norms.

By definition *the algebra of polynomial functions on a noncommutative spherical manifold of dimension* 3 is a complex unital $*$-algebra A generated by the elements of a unitary matrix $U \in \mathcal{M}_2(A)$ satisfying $\mathrm{ch}_{\frac{1}{2}}(U) = 0$ and $\mathrm{ch}_{\frac{3}{2}}(U) \neq 0$. One first looks at the restrictions imposed on the structure of the algebra by the fact that U is a unitary and $\mathrm{ch}_{\frac{1}{2}}(U) = 0$. It will turn out that the non-vanishing of $\mathrm{ch}_{\frac{1}{2}}(U)$ follows from these conditions. Thus, being more explicit, A is a unital $*$-algebra generated by the elements of a matrix $U \in \mathcal{M}_2(A)$ satisfying

$$UU^* = U^*U = 1, \tag{2.8}$$

$$U^i_j \otimes U^{*j}_i - U^{*i}_j \otimes U^j_i = 0. \tag{2.9}$$

As before we use the convention of summing over repeated indices.

It is convenient to consider also the corresponding homogeneous problem coming from relaxing a little the unitarity condition and considering "unitaries up to scale". This corresponds to studying the algebra B of polynomial functions on a noncommutative plane of dimension 4 spanned by a spherical manifold of dimension 3. Explicitly B is a unital $*$-algebra generated by the elements of a matrix $U \in \mathcal{M}_2(B)$ satisfying

$$UU^* = U^*U \in \mathbf{1}_2 \otimes B, \tag{2.10}$$

$$U^i_j \otimes U^{*j}_i - U^{*i}_j \otimes U^j_i = 0, \tag{2.11}$$

where $\mathbf{1}_2$ is the identity matrix in $\mathcal{M}_2(\mathbb{C})$.

Once we have chosen a basis $\{\tau_\mu |\ \mu = 0, \ldots, 3\}$ for $\mathcal{M}_2(\mathbb{C})$ we can write any matrix $U \in \mathcal{M}_2(A)$ (resp. $\mathcal{M}_2(B)$) as

$$U = \tau_\mu z^\mu, \quad z^\mu \in A \text{ (resp. } B) \tag{2.12}$$

and the above relations become relations between the elements $z^\mu \in B$. We may for instance take the basis

$$\tau_0 = \mathbf{1}_2, \quad \tau_1 = \imath\sigma_1, \quad \tau_2 = \imath\sigma_2, \quad \tau_3 = \imath\sigma_3, \tag{2.13}$$

where the σ_k are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.14}$$

The basis $\{\tau_0, \tau_1, \tau_2, \tau_3\}$ is orthonormal for the inner product $\langle a, b\rangle = \frac{1}{2}\mathrm{Trace}(a^*b)$ in $\mathcal{M}_2(\mathbb{C})$. If we now write $U = \tau_\mu z^\mu$ the equality (2.11) is equivalent to:

$$z^{\mu *} = \Lambda^\mu_\nu z^\nu, \tag{2.15}$$

where $\Lambda \in \mathcal{M}_2(\mathbb{C})$ is a unitary symmetric matrix. By imposing the conditions of unitarity and unitarity up to scale we are led to the following definitions (cf. [8, 9]):

Definition 2.4. Let $\Lambda \in \mathcal{M}_4(\mathbb{C})$ be a unitary symmetric matrix. We define $C_{\mathrm{alg}}(\mathbb{R}^4_\Lambda)$, *the algebra of polynomial functions on the noncommutative four-plane* $\mathbb{R}^4_\Lambda$ as the $*$-algebra generated by four elements $\{z^0, z^1, z^2, z^3\}$ subject to the relations

$$z^{\mu *} = \Lambda^\mu_\nu z^\nu, \tag{2.16}$$

$$z^k z^{0*} - z^0 z^{k*} - \sum \epsilon_{klm} z^l z^{m*} = 0, \tag{2.17}$$

$$z^{0*} z^k - z^{k*} z^0 - \sum \epsilon_{klm} z^{l*} z^m = 0, \tag{2.18}$$

where (k, l, m) runs over the cyclic permutations of $(1, 2, 3)$ and ϵ_{klm} is the totally antisymmetric tensor.

We define $C_{\mathrm{alg}}(S^3_\Lambda)$, *the algebra of polynomial functions on the noncommutative three-sphere* S^3_Λ as the quotient of $C_{\mathrm{alg}}(\mathbb{R}^4_\Lambda)$ by the two-sided ideal generated by

$$\sum z^\mu z^{\mu *} - 1. \tag{2.19}$$

The following theorem shows that S^3_Λ is a spherical manifold of dimension 3 and any spherical manifold can be embedded in some S^3_Λ.

Theorem 2.5. ([8, 9]) *For any unitary symmetric matrix $\Lambda \in \mathcal{M}_4(\mathbb{C})$ the unitary*

$$U = \tau_\mu z^\mu \quad \in \mathcal{M}_2(C_{\mathrm{alg}}(S^3_\Lambda)) \tag{2.20}$$

satisfies $\mathrm{ch}_{\frac{1}{2}}(U) = 0$

Conversely, for any complex unital $*$-algebra A generated by the elements of a unitary matrix $\tilde{U} \in \mathcal{M}_2(A)$ satisfying $\mathrm{ch}_{\frac{1}{2}}(\tilde{U}) = 0$ there exist a unitary symmetric matrix $\Lambda \in \mathcal{M}_4(\mathbb{C})$ and a surjective morphism

$$C_{\mathrm{alg}}(S^3_\Lambda) \to A \tag{2.21}$$

carrying U to $\tilde{U}$.

Given $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathbb{T}^3$ we will denote respectively by $\mathbb{R}^4_\varphi$ and S^3_φ the noncommutative 4-plane and 3-sphere corresponding to the matrix

$$\Lambda(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2\pi\imath\varphi_1} & 0 & 0 \\ 0 & 0 & e^{-2\pi\imath\varphi_2} & 0 \\ 0 & 0 & 0 & e^{-2\pi\imath\varphi_3} \end{pmatrix}. \tag{2.22}$$

It can be shown that for any unitary symmetric matrix $\Lambda \in \mathcal{M}_4(\mathbb{C})$ there exists a point in the torus $\varphi \in \mathbb{T}^3$ such that one has isomorphisms $\mathbb{R}^4_\Lambda \simeq \mathbb{R}^4_\varphi$ and $S^3_\Lambda \simeq S^3_\varphi$ given by isomorphisms of the corresponding algebras. Therefore spherical manifolds in dimension three are parametrized by the points in a 3-torus $\mathbb{T}^3 = S^1 \times S^1 \times S^1$.

In order to get Hermitian generators for the $*$-algebra $C_{\text{alg}}(\mathbb{R}^4_\varphi)$ we take $x^0 = z^0$ and $x^k = e^{-\pi \imath \varphi_k} z^k$, $k = 1, 2, 3$. Thus

$$x^\mu = x^{\mu *}, \qquad \mu = 0, \ldots, 3. \tag{2.23}$$

Therefore $C_{\text{alg}}(\mathbb{R}^4_\varphi)$ can also be realized as the unital $*$-algebra generated by four Hermitian elements x^0, x^1, x^2 and x^3 together with the relations

$$\cos(\varphi_k)(x^0 x^k - x^k x^0) = \imath \sin(\varphi_l - \varphi_m)(x^l x^m + x^m x^l), \tag{2.24}$$
$$\cos(\varphi_l - \varphi_m)(x^l x^m - x^m x^l) = -\imath \sin(\varphi_k)(x^0 x^k + x^k x^0), \tag{2.25}$$

where (k, l, m) runs over the cyclic permutations of $(1, 2, 3)$.

Note that the value $\varphi = (0, 0, 0)$ corresponds to the polynomial algebras of the usual 4-plane and 3-sphere.

The first step in the study of these algebras is the identification of a vector field on $\mathbb{T}^3$ corresponding to the foliation coming from the equivalence relation on $\mathbb{T}^3$ given by

$$\varphi \sim \varphi' \text{ if and only if } C_{\text{alg}}(\mathbb{R}^4_\varphi) \simeq C_{\text{alg}}(\mathbb{R}^4_{\varphi'}). \tag{2.26}$$

The algebras corresponding to critical values of the foliation will give rise either to spherical manifolds that are noncommutative 3-spheres whose suspension is of the form S^4_θ or to noncommutative 3-spheres arising from quantum groups. The study of the generic case of the algebras $C_{\text{alg}}(S^3_\varphi)$ corresponding to regular values of the foliation uses tools coming from noncommutative algebraic geometry and nontrivial refinements of these tools in the C^*-algebra context.

By construction the algebras $C_{\text{alg}}(\mathbb{R}^4_\varphi)$ are quadratic algebras with 4 generators of degree 1. More explicitly, the algebra $C_{\text{alg}}(\mathbb{R}^4_\varphi)$ is isomorphic to the quotient

$$\mathbb{C}\langle x^0, x^1, x^2, x^3 \rangle / \mathcal{R}, \tag{2.27}$$

where $\mathbb{C}\langle x^0, x^1, x^2, x^3 \rangle$ is the free $*$-algebra in four Hermitian generators and $\mathcal{R}$ is the two-sided ideal generated by the relations (2.24) and (2.25).

If we let $V := \sum \mathbb{C} x_i \simeq \mathbb{C}^4$ then the algebra $\mathbb{C}\langle x^0, x^1, x^2, x^3 \rangle$ is isomorphic to the tensor algebra of the space V:

$$\mathcal{T}(V) = \bigoplus_{n \geq 0} V^{\otimes n} \tag{2.28}$$

and we can identify the relations (2.24) and (2.25) with elements in $V \otimes V$.

To each element $f \in V \otimes V$ we associate a bilinear form $\check{f} : V^* \times V^* \to \mathbb{C}$. We call $\check{f}$ the *multilinearization* of f. Since $\check{f}$ is bihomogeneous its zero locus defines a hypersurface in $\mathbb{P}(V) \times \mathbb{P}(V) \simeq \mathbb{P}^3(\mathbb{C}) \times \mathbb{P}^3(\mathbb{C})$. The locus of common zeroes of the multilinearizations of the elements of $\mathcal{R}$ defines a variety $\{\check{f}_i = 0\} = \Gamma \subset \mathbb{P}^3 \times \mathbb{P}^3$. Let Y_1 and Y_2 be the corresponding projections and $\sigma : Y_1 \to Y_2$ be the correspondence with graph Γ. Assume we can make an identification $Y_1 = Y_2 = Y$. If σ is an isomorphism we consider it as an automorphism of Y. Let $i : Y \hookrightarrow \mathbb{P}^3$ be the inclusion and take $\mathcal{L} = i^* \mathcal{O}_{\mathbb{P}^3}(1)$ where $\mathcal{O}_{\mathbb{P}^3}(1)$ is the canonical bundle on $\mathbb{P}^3$. The *geometric data associated to the quadratic algebra* $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ is by definition the triple

$$(Y, \sigma, \mathcal{L}). \tag{2.29}$$

The variety Y is called *the characteristic variety* $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$.

Starting from such a triple one can construct the graded algebra:

$$B(Y, \sigma, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(Y, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}), \tag{2.30}$$

where $\mathcal{L}^\sigma := \sigma^* \mathcal{L}$ and the multiplication of two sections $s_1 \in B(Y, \sigma, \mathcal{L})_n, s_2 \in B(Y, \sigma, \mathcal{L})_m$ is given by $s_1 s_2 := s_1 \otimes s_2^{\sigma^n}$.

The study of the geometric data of $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ for different values of φ is a fundamental step in the classification of these algebras. For generic values of the parameter φ the characteristic variety Y is given by an elliptic curve E_φ together with four isolated points and the morphism σ acts by translation on the elliptic curve E_φ and trivially on the four points (cf. [40, 8, 9]).

By construction there is a morphism

$$C_{\mathrm{alg}}(\mathbb{R}^4_\varphi) \to B(Y, \sigma, \mathcal{L}). \tag{2.31}$$

In order to refine this morphism A. Connes and M. Dubois-Violette introduced in [8] the notion of a *central quadratic form* on a connected component C of $Y \times Y$. Roughly speaking these are symmetric tensors on the space of generators of the algebra whose action on the points of C gives rise to a Hermitian structure on the line bundle $\mathcal{L}$ compatible with the action of σ on its sections.

By construction the algebras $C_{\mathrm{alg}}(\mathbb{R}^4_\varphi)$ are quadratic $*$-algebras, thus; there is an involution preserving the space V of generators and giving this space a real structure. For this kind of algebras it makes sense to ask whether a compact connected component C of the graph giving the geometric data is invariant under the involution coming from the real structure on V and, this being the case, whether on C the involution commutes with the action of the automorphism σ. When these compatibility conditions hold one may obtain a $*$-homomorphism to a twisted crossed product C^*-algebra $C(F) \times_{\sigma, \mathcal{L}} \mathbb{Z}$ where F is the image of C under the first projection.

In the case of the noncommutative spheres S^3_φ corresponding to generic values of φ this construction leads to a unital $*$-morphism

$$C_{\mathrm{alg}}(S^3_\varphi) \to C^\infty(E_\varphi) \times_{\sigma,\mathcal{L}} \mathbb{Z}. \tag{2.32}$$

The existence of this morphism is the basis for the study of the geometry of the spheres S^3_φ. In particular using this map we can pair canonical Hochschild classes in $C^\infty(E_\varphi) \times_{\sigma,\mathcal{L}} \mathbb{Z}$ with the corresponding pullback of $\mathrm{ch}_{\frac{3}{2}}(U)$; since the result obtained is non zero it follows that

$$\mathrm{ch}_{\frac{3}{2}}(U) \neq 0. \tag{2.33}$$

The computation of this pairing is expressed naturally in terms of modular functions, thus providing a bridge between these noncommutative spaces and arithmetic geometry (cf. [9] Theorem 12.1).

3. Epilogue

There is a wide variety of quadratic algebras arising naturally from geometric considerations in the framework of algebraic noncommutative geometry (cf. [1, 40, 41, 32]). Applying the tools developed by A. Connes and M. Dubois-Violette in [9] to the study of these algebras may provide valuable information about the structures which play a role in the interplay between algebraic noncommutative geometry and differential noncommutative geometry. In particular, applying these tools to the homogeneous coordinate rings $B_g(\tau,\theta)$ and $B_g(\theta)$ associated to real multiplication noncommutative tori could have potential applications. Nontrivial extensions of the theory may have to be developed in order to handle these quadratic algebras since typically they exhibit ill behaved growth properties (cf. [32]).

Being able to construct embeddings from algebraically defined rings into C^*-algebras gives important information about their representation theory. In the case of rings coming from arithmetic data these embeddings may provide crucial information towards applications. This can be seen by considering the recent developments relating class field theory to quantum statistical mechanics (cf. [2, 11, 13, 14, 21, 16]). The algebra of observables of a quantum statistical mechanical system is a C^*-algebra, and the time evolution of the system singles out particular states which are obtained as extremal points in the space of equilibrium states of the system. In quantum statistical mechanical systems describing the explicit class field theory of a particular global field K the extremal equilibrium states are evaluated at observables corresponding to an arithmetically defined rational subalgebra of observables. The values obtained in this way are generators of the abelian extensions of the field K. In our context the tools developed in [9] could provide a way to construct in a canonical way a quantum statistical mechanical system with a particular given ring as an arithmetic subalgebra. In particular, in view of Manin's real multiplication program outlined above, it seems that investigating the homogeneous coordinate rings of real multiplication noncommutative tori in this context could throw some light about the explicit class field theory problem for real quadratic fields.

References

[1] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves.* The Grothendieck Festschrift, Vol. I, Progr. Math. 86, Birkhäuser Boston (1990) 33–85.

[2] J. B. Bost and A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory.* Selecta Math. (N.S.) **1** (1995) 411–457.

[3] A. Connes, *C^*-algèbres et géométrie différentielle.* C. R. Ac. Sci. Paris, t. **290** (1980) 599–604.

[4] A. Connes, *Noncommutative differential geometry.* Inst. Hautes Études Sci. Publ. Math. No. **62** (1985) 257–360.

[5] A. Connes, *Noncommutative geometry.* Academic Press, 1994.

[6] A. Connes, *A short survey of noncommutative geometry.* J. Math. Phys. **41** (2000) 3832–3866.

[7] A. Connes and M. Dubois-Violette, *Noncommutative finite dimentional manifolds I. Spherical manifolds and related examples.* Comm. Math. Phys. **230** (2002) 539–579.

[8] A. Connes and M. Dubois-Violette, *Moduli space and structure of noncommutative 3-spheres.* Lett. Math. Phys. **66** (2003) 91–121.

[9] A. Connes and M. Dubois-Violette, *Noncommutative finite dimensional manifolds II. Moduli space and structure of non commutative 3-spheres*, math.QA/0511337.

[10] A. Connesand and G. Landi *Noncommutative manifolds, the instanton algebra and isospectral deformations.* Comm. Math. Phys. **221** (2001) 141–159.

[11] A. Connes and M. Marcolli, *From physics to number theory via noncommutative geometry. Part I. Quantum statistical mechanics of $\mathbb{Q}$-lattices.* In"Frontiers in Number Theory, Physics, and Geometry I." Springer Verlag (2006) 269–347.

[12] A. Connes and M. Marcolli, *A walk in the noncommutative garden.* Proceedings of the "International Workshop on Noncommutative Geometry", IPM, Tehran, Iran 2005.

[13] A. Connes, M. Marcolli and N. Ramachandran, *KMS states and complex multiplication.* Selecta Math. (N.S.) **11** (2005) 325–347.

[14] A. Connes, M. Marcolli and N. Ramachandran, *KMS states and complex multiplication. Part II*, to appear in the Abel Symposium Volume.

[15] A. Connes and M. Rieffel, *Yang–Mills for noncommutative two-tori. Operator algebras and mathematical physics.* Contemp. Math., **62**, Amer. Math. Soc. Providence, RI (1987) 237–266.

[16] C. Consani and M. Marcolli, *Quantum statistical mechanics over function fields*, math.QA/0607363.
[17] K. R. Davidson, *C^*-algebras by example.* Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, (1996).
[18] M. Dieng and A. Schwarz, *Differential and complex geometry of two-dimensional noncommutative tori.* Lett. Math. Phys. **61** (2002) 263–270.
[19] E. G. Effros and F. Hahn, *Locally compact transformation groups and C^*-algebras.* Bull. Amer. Math. Soc. **73** (1967) 222–226.
[20] J. M. Gracia-Bondía, J. Várilly and H. Figueroa, *Elements of noncommutative geometry.* Birkhäuser Boston, Inc., Boston, MA, (2001).
[21] B. Jacob, *Bost-Connes type systems for function fields*, math.OA/0602554.
[22] M. Khalkhali, *Lectures on noncommutative geometry.* Proceedings of the "International Workshop on Noncommutative Geometry", IPM, Tehran, Iran 2005.
[23] S. Mahanta, *Lecture notes on noncommutative algebraic geometry and noncommutative tori.* Proceedings of the "International Workshop on Noncommutative Geometry", IPM, Tehran, Iran 2005.
[24] Y. Manin, *Real multiplication and noncommutative geometry.* In "The legacy of Niels Henrik Abel", Springer Verlag, Berlin (2004).
[25] Y. Manin, *The notion of dimension in geometry and algebra.* Bull. Amer. Math. Soc. (N.S.) **43** (2006) 139–161.
[26] Y. Manin, *Parabolic points and zeta functions of modular curves.* Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972) 19–66.
[27] Y. Manin and M. Marcolli, *Continued fractions, modular symbols, and noncommutative geometry.* Selecta Math. (N.S.) **8** (2002) 475–521.
[28] M. Marcolli, *Arithmetic noncommutative geometry.* University Lecture Series, 36. Amer. Math. Soc., Providence (2005).
[29] D. Mumford, *Tata Lectures on Theta I.* Progr. Math. 28, Birkhäuser, Boston (1983).
[30] D. Mumford, *Tata Lectures on Theta III.* With the collaboration of M. Nori and P. Norman. Progr. Math. 97. Birkhäuser, Boston (1991).
[31] J. Plazas, *Arithmetic structures on noncommutative tori with real multiplication*, math.QA/0610127.
[32] A. Polishchuk, *Noncommutative two-tori with real multiplication as noncommutative projective varieties.* J. Geom. Phys. **50** (2004) 162–187.
[33] A. Polishchuk, *Classification of holomorphic vector bundles on noncommutative two-tori.* Doc. Math. **9** (2004) 163–181.
[34] A. Polishchuk and A. Schwarz, *Categories of holomorphic bundles on noncommutative two-tori.* Comm. Math. Phys. **236** (2003) 135–159.
[35] M. A. Rieffel, *C^*-algebras associated with irrational rotations.* Pacific J. Math. **93** (1981) 415–429.
[36] M. A. Rieffel, *Noncommutative tori - a case study of noncommutative differentiable manifolds.* Geometric and topological invariants of elliptic operators, Contemp. Math., 105, Amer. Math. Soc., Providence, RI, (1990) 191–211.
[37] M. A. Rieffel, *The cancellation theorem for projective modules over irrational rotation C^*-algebras.* Proc. London Math. Soc. (3) **47** (1983) 285–302.

[38] A. Schwarz, *Theta functions on noncommutative tori.* Lett. Math. Phys. **58** (2001) 81–90.

[39] V. V. Shokurov, *Modular symbols of arbitrary weight.* Funct. Anal. Appl. **10** (1976) 95–96.

[40] S. P. Smith and J. T. Stafford, *Regularity of the four dimensional Sklyanin algebra.* Compositio Math. **83** (1992) 259–289.

[41] J. T. Stafford and M. van den Bergh, *Noncommutative curves and noncommutative surfaces.* Bull. Amer. Math. Soc. (N.S.) **38** (2001) 171–216.

[42] J. Várilly, *Hopf algebras in noncommutative geometry.* In Geometric and topological methods for quantum field theory (Villa de Leyva, 2001), World Scientific Publ., River Edge, NJ (2003) 1–85.

[43] M. Vlasenko, *The graded ring of quantum theta functions for noncommutative torus with real multiplication.* Int. Math. Res. Not. Art. ID 15825 (2006).

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An Invitation to Noncommutative Geometry

D-BRANES IN NONCOMMUTATIVE FIELD THEORY

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Contents

A mathematical introduction to the classical solutions of noncommutative field theory is presented, with emphasis on how they may be understood as states of D-branes in Type II superstring theory. Both scalar field theory and gauge theory on Moyal spaces are extensively studied. Instantons in Yang–Mills theory on the two-dimensional noncommutative torus and the fuzzy sphere are also constructed. In some instances the connection to D-brane physics is provided by a mapping of noncommutative solitons into K-homology.

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1. Introduction and Background from String Theory

These lecture notes provide an introduction to some selected topics in noncommutative field theory that are motivated by the interaction between string theory and noncommutative geometry. The material is intended to be geared at an audience consisting of graduate students and beginning postdoctoral researchers in mathematics, and in general any mathematician interested in how string theory crops up in certain mathematical settings. The material also serves as an introduction to physicists into some of the more formal aspects of noncommutative field theory, revolving primarily around the geometric structure of their classical solutions and the mathematical interpretation of noncommutative solitons as D-branes.

The main theme that we will focus on revolves around the scenario that string theory with D-branes in the presence of "background fields" leads to noncommutative geometries on the worldvolumes of the branes. Some familiarity with this notion along with the basics of noncommutative geometry, the standard examples of noncommutative spaces, and the rudiments of K-theory are assumed, as they were discussed in other minicourses at the workshop. Nevertheless, to set the stage and terminology, we will take a short mathematical tour in this section through this story and define the relevant concepts in the correspondence to make our presentation as self-contained as possible. Throughout we will deal only with *classical* aspects of string theory and noncommutative field theory, and all of our definitions and explanations are to be understood in this context alone. Quantization introduces many complicated technical aspects that are out of the scope of these introductory notes.

We begin by explaining some basic concepts from classical string theory. Let X be an oriented Euclidean spin manifold with Riemannian metric G. We call X the *spacetime* or the *target space*. Let Σ be a Riemann surface. A *string* is a harmonic map $x : \Sigma \to X$, i.e. an immersion of the surface Σ in X of minimal area in the induced metric $x^*(G)$. The surface Σ is called the *worldsheet* of the string. The harmonic property can be described in the usual way by a variational principle based on a σ-model on Σ with target space X. The string is said to be of *Type II* if Σ is oriented, and of *Type I* if Σ is non-orientable. In the following we will only deal with Type II strings. A string is *closed* if its worldsheet is closed, $\partial\Sigma = \emptyset$, while it is *open* if its worldsheet has boundary, $\partial\Sigma \neq \emptyset$. The spin condition on X is assumed so that we can define spinors and ultimately *superstrings*, but we shall not go into any aspects of supersymmetry here.

In the simplest instance, a *D-brane* may be defined as a closed oriented submanifold $W \subset X$ which can be used as a boundary condition for open strings. This means that in the presence of D-branes the admissible open strings are the *relative* harmonic maps $x : (\Sigma, \partial\Sigma) \to (X, W)$. The submanifold W is called the *worldvolume* of the D-brane. Not all submanifolds are allowed as viable D-brane worldvolumes. For instance, a consistent choice of boundary conditions must preserve the fundamental conformal invariance of the string theory. Determining the allowed D-branes in a given spacetime X is an extremely difficult problem which requires having the quantum field theory of the worldsheet σ-model under control. For example, the cancellation of worldsheet anomalies for Type II strings requires W to be a spin^c manifold [29].

Let us next introduce the concept of a *supergravity background field*. In addition to the metric G on X, we assume the presence of an additional geometrical entity called the *Neveu–Schwarz B-field*. It is a two-form $B \in \bigwedge^2(T^*X)$, which we will locally regard as a skew-symmetric linear map $B_p : T_pX \to T_p^*X$ for $p \in X$. The B-field has curvature $H = \mathrm{d}B \in \bigwedge^3(T^*X)$ and characteristic class $[H] \in \mathrm{H}^3(X, \mathbb{Z})$. Of course as $H = \mathrm{d}B$ is an exact three-form it defines a trivial class in de Rham cohomology, but there can be torsion and other effects which yield a non-trivial characteristic class. The H-field is constrained to obey the *supergravity equation*

$$R(G) = \frac{1}{4}\, H \circ G^{-1} \neg H \,, \tag{1.1}$$

where $R(G)$ is the Ricci curvature two-form of the metric G and $\neg$ denotes contraction. Similarly to the B-field, the metric here and throughout is regarded locally as a symmetric non-degenerate homomorphism $G_p : T_pX \to T_p^*X$ and likewise $H_p : T_pX \to \bigwedge^2(T_p^*X)$ for $p \in X$. This equation ties the characteristic class of the B-field to the curvature of the spacetime X. The *semi-classical limit* is the one in which X "approaches" flat space, i.e. $R(G) \to 0$, and B becomes topologically trivial.

The important point is that both open and closed strings feel the presence of H, but in very different ways. Closed strings only "see" the cohomology class $[H]$. According to the supergravity equation (1.1), $[H] \to 0$ in the semi-classical limit. Thus a consistent semi-classical treatment of closed strings will be insensitive to the presence of a B-field. Aspects of the noncommutative geometry of closed strings can be found in [19, 30, 31, 59, 60, 54]. They will not be covered in these notes. In contrast, open strings are sensitive to a concrete choice of B with $H = \mathrm{d}B$. The induced two-form $x^*(B) \in \bigwedge^2(T^*\Sigma)$ does not vanish in the limit $R(G) \to 0$ and we can now explore the possibility of analysing the string geometry semi-classically in the background B-field, whereby one should be able to say some concrete things.

Consider a D-brane with embedding $\zeta : W \to X$. By a slight abuse of notation, we will denote the pullback $\zeta^*(B)$ of the B-field also by the symbol B, as this shouldn't cause any confusion in the following. If this pullback is non-degenerate, then we can define the *Seiberg–Witten bivector* [81]

$$\theta = (B + G\,B^{-1}\,G)^{-1}\,, \tag{1.2}$$

which is regarded locally as a non-degenerate skew-symmetric linear map $\theta_p : T_p^*X \to T_pX$ for $p \in X$. In the formal "limit" $B \to \infty$, the Seiberg–Witten bivector is $\theta = B^{-1}$. Thus when in addition $H = \mathrm{d}B = 0$, as happens in the semiclassical flat space limit, the B-field defines a symplectic structure on W and θ is the Poisson bivector corresponding to the symplectic two-form B. If B is degenerate (and hence *not* symplectic), then under favourable circumstances θ will still be a Poisson bivector. One only requires that the Jacobi identity for the corresponding Poisson brackets be fulfilled. This is equivalent to the closure condition $\mathrm{d}B = 0$ only when B is non-degenerate. The quantum theory of the open strings attached to the D-brane tells us that we should *quantize* this Poisson geometry [7, 20, 78, 81]. This leads to a string theoretic picture of the Kontsevich deformation quantization of Poisson manifolds [52]. An explicit realization of this picture is provided by the Cattaneo-Felder topological σ-model [18].

When $\mathrm{d}B \neq 0$ new phenomena occur. One encounters generalizations of ordinary Poisson structures, variations of quantum group algebras, and the like [3]. The generic situation leads to non-associative deformations, which in some instances can still be handled by the realization that they define A_∞-homotopy associative structures [23]. But there is no general notion of quantization for such geometries. We will therefore continue to work with the limits described above wherein one obtains true symplectic geometries on the worldvolumes of D-branes. This sequence of limits is often referred to as the *Seiberg–Witten limit* [81].

A D-brane, and more generally collections of several D-branes, also has much more structure to it than what we have described thus far. The most primitive definition we can take of a D-brane is as a *Baum–Douglas K-cycle* (W, E, ζ) in X [11], where W is a spin^c manifold, $\zeta : W \to X$ is a continuous map and $E \to W$ is a complex vector bundle called the *Chan–Paton bundle*. The description of D-branes in terms of K-cycles and K-homology [8, 46, 74, 87] will be central to our analysis later on. If we equip E with a smooth connection, then we can define a Yang–Mills gauge theory on W. We may also define more general field theories on W by considering smooth sections of this bundle (and other canonically defined bundles over W). The connections and sections in this context are referred to as *worldvolume fields*. The semiclassical motion of the D-brane is thereby described dynamically by a *worldvolume field theory* which is induced by the quantum theory of open strings on the brane. If the geometry of the D-brane is quantized in the manner explained above, then one finds a *noncommutative field theory* on the brane worldvolume W in the Seiberg–Witten limit [81]. In the particular case of Yang–Mills theory, the deformation gives rise to a *noncommutative gauge theory*.

In what follows, for us the interesting aspects of these noncommutative worldvolume field theories will lie in the property that they possess novel classical solitonic solutions which have no counterparts in ordinary field theory [35]. In many instances these solutions can themselves be interpreted as D-branes [1, 24, 47, 48, 90, 39, 58, 72, 82], quite unlike the usual worldvolume field theories for $B = 0$. Field theoretical constructions of BPS soliton solutions in noncommutative supersymmetric Yang–Mills theory, and their applications to D-brane dynamics, can be found in [39, 32, 44, 88] (see [33, 34] for BPS soliton solutions in other noncommutative field theories). The crucial point is that the Seiberg–Witten limit still retains a lot of stringy information, in contrast to the usual field theoretic or point particle limits of string theory. We can thus use the solitons of noncommutative field theory to teach us about aspects of D-branes. For example, they can provide insights into what sort of worldvolume geometries live in a given spacetime X. Moreover, their eventual quantization (which will not be covered here) could teach us a lot about the nonperturbative structure of quantum string theory. The purpose of these lecture notes is to explain this correspondence in some specific spacetimes X, and to illustrate how the techniques of noncommutative geometry can be used to construct the appropriate noncommutative field configurations. The classical noncommutative solitons will then admit an interpretation in terms of "branes within branes" [25] and are built solely from the properties of noncommutative field theory. A key point in our analysis will be the unveiling of the connections with the K-theory classification of D-brane charges [64, 91, 49, 67].

The outline of the remainder of these notes is as follows. In Section 2 we describe D-branes in flat Euclidean space and the corresponding noncommutative field theories. This section contains most of the elementary definitions used throughout these notes. In Section 3 we construct scalar field solitons on Moyal spaces in these settings, and give their interpretations as D-branes through an intimate connection to K-homology. In Section 4 we proceed to noncommutative gauge theory and explicitly construct instanton solutions in the two-dimensional case. In Section 5 we look at D-branes whose worldvolume is a noncommutative torus and compare the instantons in this case with those of the Moyal space. Finally, in Section 6 we consider D-branes in curved spaces described by group manifolds and the ensuing fuzzy spaces which describe the quantized worldvolume geometries, looking in particular at the classic example of fuzzy spheres in $X = \mathrm{SU}(2) \cong \mathrm{S}^3$.

2. Euclidean D-branes

We will spend most of our initial investigation working in the simplest cases of flat target spaces with $R(G) = H = 0$. Then we are automatically in the semi-classical regime of the string theory and we can proceed straightforwardly with the quantization of the brane worldvolume geometries as explained in Section 1. In this section we begin with some elementary definitions and then proceed to describe the construction of noncommutative field theories on the quantized worldvolumes. Extensive reviews of these sorts of noncommutative field theories with exhaustive lists of references can be found in [26, 51, 86].

2.1. *Moyal spaces*

We consider spacetimes which are either a d-dimensional Euclidean space $X = \mathbb{R}^d$ or a d-dimensional torus $X = \mathrm{T}^d$ for some integer $d \geq 2$. Let $2n \leq d$, and consider a D-brane localized along a $2n$-dimensional hyperplane $\mathbb{V}_{2n} \subset X$ with tangent space $T\mathbb{V}_{2n}$. Let $\theta : T^*\mathbb{V}_{2n} \to T\mathbb{V}_{2n}$ be a skew-symmetric non-degenerate linear form. It may be represented by a skew-symmetric $2n \times 2n$ constant real matrix $\theta = (\theta^{ij})_{1 \leq i,j \leq 2n}$ of maximal rank $2n$. There exists a linear transformation on $\mathbb{V}_{2n} \to \mathbb{V}_{2n}$ which brings θ into its canonical Jordan normal form

$$\theta = \begin{pmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \theta_n \\ -\theta_n & 0 \end{pmatrix} . \tag{2.1}$$

For definiteness we will take $\theta_k > 0$ for all $k = 1, \dots, n$. It is straightforward to extend all of our considerations below to matrices θ of rank $< 2n$ (and in particular to odd-dimensional hyperplanes), but for simplicity we will work only with non-degenerate θ.

We now define the appropriate quantization of the D-brane worldvolume $\mathbb{V}_{2n}$ in these instances. Throughout we will denote $\mathrm{i} := \sqrt{-1}$.

Definition 2.1. Let $F(\mathbb{V}_{2n}) = \mathbb{C}\langle \mathbb{1}, x^1, \dots, x^{2n} \rangle$ be the free unital algebra on $2n$ generators $x^1, \dots, x^{2n}$. Let $I_\theta(\mathbb{V}_{2n})$ be the two-sided ideal of $F(\mathbb{V}_{2n})$ generated by the $n\,(2n-1)$ elements $x^i\,x^j - x^j\,x^i - \mathrm{i}\,\theta^{ij}\,\mathbb{1}$, $1 \le i < j \le 2n$. The *Moyal $2n$-space* $\mathbb{V}^\theta_{2n}$ is the polynomial algebra $\mathbb{V}^\theta_{2n} = F(\mathbb{V}_{2n}) \,/\, I_\theta(\mathbb{V}_{2n})$.

The Moyal $2n$-space will be loosely regarded as the algebra of "functions generated by the coordinates" $x^1, \dots, x^{2n}$ satisfying the commutation relations of a degree n Heisenberg algebra

$$\left[x^i\,,\,x^j\right] := x^i\,x^j - x^j\,x^i = \mathrm{i}\,\theta^{ij}\,\mathbb{1}\,. \tag{2.2}$$

In the normal form (2.1), the only non-vanishing commutation relations are

$$\left[x^{2k-1}\,,\,x^{2k}\right] = \mathrm{i}\,\theta_k\,\mathbb{1}\,, \quad k = 1, \dots, n\,. \tag{2.3}$$

We will also formally regard $\mathbb{V}^\theta_{2n}$ as a completion of the polynomial algebra of Definition 2.1. There are various technical complications with this since the polynomial algebra is really an algebra of differential operators and so it has no completion. We will not concern ourselves with this issue. Our definition is such that the "commutative limit" $\mathbb{V}^{\theta=0}_{2n} = \mathcal{S}(\mathbb{V}_{2n})$ is the algebra of Schwartz functions on $\mathbb{V}_{2n} \to \mathbb{C}$, i.e. the algebra of infinitely differentiable functions whose derivatives all vanish at infinity faster than any Laurent polynomial. This definition can be made more precise through the formalism of deformation quantization which will be considered in Section 2.3.

2.2. *Fock modules*

To do concrete calculations later on, and in particular to define gauge theories, we need to look at representations of the algebra $\mathbb{V}^\theta_{2n}$. For the moment, we set $n = 1$ and $\theta := \theta_1 > 0$. In higher dimensions we can then "glue" the n independent skew-blocks in (2.1) together, as we explain later on.

Starting from the generators x^1, x^2 we introduce the formal complex linear combinations

$$a = \frac{1}{\sqrt{2\,\theta}}\big(x^1 + \mathrm{i}\,x^2\big)\,, \quad a^\dagger = \frac{1}{\sqrt{2\,\theta}}\big(x^1 - \mathrm{i}\,x^2\big)\,. \tag{2.4}$$

Then the Heisenberg commutation relation (2.3) is equivalent to

$$\big[a\,,\,a^\dagger\big] = \mathbb{1}\,. \tag{2.5}$$

Consider the separable Hilbert space $\mathcal{F} := \ell^2(\mathbb{N}_0)$ with orthonormal basis e_n, $n \in \mathbb{N}_0$. The dual space $\mathcal{F}^*$ has corresponding basis e_n^* with the canonical dual pairing

$$\big\langle e_n^*\,,\,e_m\big\rangle = \delta_{nm}\,. \tag{2.6}$$

Operators from $\mathcal{F} \to \mathcal{F}$ are elements of the endomorphism algebra $\mathrm{End}(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{F}^*$.

Definition 2.2. The Hilbert space $\mathcal{F}$ is a finitely generated left $\mathbb{V}_2^\theta$-module, called the *Fock module*, with left action $\mathbb{V}_2^\theta \times \mathcal{F} \to \mathcal{F}$ given by

$$a \cdot e_n = \sqrt{n}\;e_{n-1}\,, \quad a^\dagger \cdot e_n = \sqrt{n+1}\;e_{n+1}\,.$$

To ease notation, we will not distinguish between the algebra $\mathbb{V}_2^\theta$ and its representation as operators in $\mathrm{End}(\mathcal{F})$ acting on $\mathcal{F}$.

Remark 2.1. By the Stone–von Neumann theorem, $\mathcal{F}$ is the unique irreducible representation of the Heisenberg commutation relations. There is a natural isomorphism $\mathcal{F} \cong \mathrm{L}^2(\mathbb{R}, \mathrm{d}x)$ obtained by representing $x^2 =: M_x$ as multiplication of a function by $x \in \mathbb{R}$ and x^1 as the differential operator $\mathrm{i}\,\theta\,\frac{\mathrm{d}}{\mathrm{d}x}$. This is known as the *Schrödinger representation.*

With this action one has

$$\begin{aligned} a \cdot e_0 &= 0\,, \\ \big(a^\dagger\,a\big) \cdot e_n &= n\;e_n\,, \\ e_n &= \frac{1}{\sqrt{n!}}\big(a^\dagger\big)^n \cdot e_0\,. \end{aligned} \tag{2.7}$$

Thus the orthonormal basis e_n forms the complete set of eigenvectors for the action of the element $a^\dagger\,a \in \mathbb{V}_2^\theta$ and is called the *number basis.* The last identity in (2.7) implies that this is the natural basis for a given representation of $a^\dagger$ on some fixed vector $e_0 \in \mathcal{F}$. For definiteness we will mostly present our computations in the number basis, but it is possible to reformulate everything in a basis independent way.

Proposition 2.1. *The Fock module $\mathcal{F}$ is projective.*

Proof. The endomorphism

$$\Pi_n := e_n \otimes e_n^*$$

is the orthogonal projection of $\mathcal{F}$ onto the one-dimensional subspace spanned by the vector $e_n \in \mathcal{F}$. The operators Π_n, $n \in \mathbb{N}_0$ generate a complete system of mutually orthogonal projectors with

$$\Pi_n \, \Pi_m = \delta_{nm} \, \Pi_n \, , \quad \sum_{n \in \mathbb{N}_0} \Pi_n = \mathbb{1} \, .$$

For each $n \in \mathbb{N}_0$, this determines an orthogonal decomposition for the left action of the algebra $\mathbb{V}_2^\theta$ on $\mathcal{F}$ given by

$$\mathbb{V}_2^\theta = \Pi_n \cdot \mathbb{V}_2^\theta \ \oplus \ (\mathbb{1} - \Pi_n) \cdot \mathbb{V}_2^\theta \, .$$

There is a natural isomorphism $\mathcal{F} \cong \Pi_n \cdot \mathbb{V}_2^\theta$ given by $\Pi_n \cdot f \mapsto f \cdot e_n \in \mathcal{F}$ for $f \in \mathbb{V}_2^\theta$. Thus $\mathcal{F}$ is projective. □

Remark 2.2. By formally iterating the orthogonal decomposition above we have

$$\mathbb{V}_2^\theta = \bigoplus_{n \in \mathbb{N}_0} \Pi_n \cdot \mathbb{V}_2^\theta$$

with $\Pi_n \cdot \mathbb{V}_2^\theta \cong \mathcal{F}$ for each $n \in \mathbb{N}_0$. Heuristically, this means that the Fock module is the analog of a single "point" on the noncommutative space $\mathbb{V}_2^\theta$. This is analogous to what occurs in the commutative situation, whereby any function f can be decomposed formally as $f(x) = \int \mathrm{d}y \ \delta(x-y) \, f(y)$ with the evaluation maps $\delta_x(f) = f(x)$ being the analogs of the projectors Π_n above.

Finally, the generalization to higher dimensions $n > 1$ is obtained by defining

$$a_k = \frac{1}{\sqrt{2\,\theta_k}} \big(x^{2k-1} + \mathrm{i}\, x^{2k} \big) \, , \quad a_k^\dagger = \frac{1}{\sqrt{2\,\theta_k}} \big(x^{2k-1} - \mathrm{i}\, x^{2k} \big) \tag{2.8}$$

for each $k = 1, \dots, n$. Then the non-vanishing commutation relations are given by

$$\big[a_k \, , \, a_l^\dagger \big] = \delta_{kl} \, \mathbb{1} \, . \tag{2.9}$$

In this case a right module over the algebra $\mathbb{V}_{2n}^\theta$ is obtained by taking n independent copies of the basic Fock module above with $\mathcal{F}^{\oplus n} \cong \ell^2(\mathbb{N}_0^n) \cong \mathrm{L}^2(\mathbb{R}^n, \mathrm{d}x)$. By the Hilbert hotel argument there is a natural isomorphism $\mathcal{F}^{\oplus n} \cong \mathcal{F}$.

Remark 2.3. The commutation relations (2.9) show that all algebras $\mathbb{V}^\theta_{2n}$ for $\theta_k \neq 0$ are isomorphic and one can simply scale away the noncommutativity parameters θ_k to 1, as the redefinition (2.8) essentially does. At times it will be convenient to keep θ in as an explicit parameter for comparison of the new phenomena in noncommutative field theories to ordinary field theories.

2.3. *Deformation quantization*

The algebra $\mathbb{V}^\theta_{2n}$ can be regarded as a deformation quantization of the algebra of Schwartz functions $\mathcal{S}(\mathbb{V}_{2n})$ on the *ordinary* hyperplane $\mathbb{V}_{2n}$ in the standard way [12] with respect to the constant symplectic two-form $\omega = \theta^{-1}$. This is common practise in the string theory literature. Although it will not be used extensively in what follows, we will briefly describe here the basic features of the approach. For most of these notes we will stick to the more abstract setting with $\mathbb{V}^\theta_{2n}$ realized as operators on the Fock module $\mathcal{F}$ (or any other module), as everything can be straightforwardly constructed in this setting. The deformation quantization approach will only be used occasionally when it can provide a useful way of envisaging the "profiles" of noncommutative fields.

Consider a polynomial function $f : \mathbb{V}_{2n} \to \mathbb{C}$ with Fourier transform $\tilde{f} : T\mathbb{V}_{2n} \to \mathbb{C}$ defined through

$$f(x) = (2\pi)^{-2n} \int_{T\mathbb{V}_{2n}} \mathrm{d}k \ \tilde{f}(k) \ \exp\big(\mathrm{i}\,\langle k, x\rangle\big)\,, \tag{2.10}$$

where $\langle k, x\rangle := \sum_{i=1}^{2n} k_i\, x^i$ with $k = (k_1, \ldots, k_{2n}) \in T\mathbb{V}_{2n}$ and $x = (x^1, \ldots, x^{2n}) \in \mathbb{V}_{2n}$. Here we will distinguish between local coordinates x^i on $\mathbb{V}_{2n}$ and the generators $\hat{x}^i$ of the noncommutative algebra $\mathbb{V}^\theta_{2n}$ by drawing a hat over the latter symbols. Thus $[\hat{x}^i, \hat{x}^j] = \mathrm{i}\,\theta^{ij}\,\mathbb{1}$. We will usually consider $\mathbb{V}^\theta_{2n}$ in its concrete realization as linear operators in $\mathrm{End}(\mathcal{F})$ acting on the Fock module, but this is not necessary and the following construction also works at a more abstract level [86].

The *Weyl symbol* $\Omega(f) \in \mathbb{V}^\theta_{2n}$ is defined by

$$\Omega(f) = (2\pi)^{-2n} \int_{T\mathbb{V}_{2n}} \mathrm{d}k \ \tilde{f}(k) \ \exp\big(\mathrm{i}\,\langle k, \hat{x}\rangle\big)\,. \tag{2.11}$$

Heuristically, the Weyl symbol of $f(x)$ is obtained by substituting in the generators of $\mathbb{V}^\theta_{2n}$ to associate to the function a *noncommutative field* $\Omega(f) = f(\hat{x})$. This definition extends to Schwartz functions $f \in \mathcal{S}(\mathbb{V}_{2n})$, for which $\Omega(f)$ is a compact operator when acting on $\mathcal{F}$. The compact operators form a dense domain $\mathcal{K} = \mathcal{K}(\mathcal{F})$ in the endomorphism algebra $\mathrm{End}(\mathcal{F})$.

The exponential function of algebra elements appearing in Eq. (2.11) is defined formally through its power series expansion. It implies a particular ordering of the noncommutative variables called *Weyl ordering*.

The map $f \mapsto \Omega(f)$ is invertible. Its inverse allows one to go the other way and associate functions to noncommutative fields. The *Wigner function* $\Omega^{-1}(O) \in \mathcal{S}(\mathbb{V}_{2n})$ of an element $O \in \mathcal{K}$ is given by

$$\Omega^{-1}(O)(x) = \big|\mathrm{Pf}(2\pi\,\theta)\big|^{-1} \int_{T\mathbb{V}_{2n}} \mathrm{d}k \, \exp\big(-\mathrm{i}\,\langle k,x\rangle\big) \ \mathrm{Tr}\,\big[O \, \exp\big(\,\mathrm{i}\,\langle k,\hat{x}\rangle\big)\big]\,, \tag{2.12}$$

where the symbol Pf denotes the Pfaffian of an antisymmetric matrix and Tr is the trace defined on $\mathcal{K}$. It follows that the Weyl symbol determines a vector space isomorphism between appropriate subspaces of functions on $\mathbb{V}_{2n}$ and dense domains of elements in $\mathbb{V}_{2n}^{\theta}$ in an appropriate Fréchet algebra topology. This point of view is also useful for adding further structure to the algebra $\mathbb{V}_{2n}^{\theta}$. For instance, the *trace* Tr on $\mathbb{V}_{2n}^{\theta}$ is defined for $O \in \mathcal{K}$ by

$$\mathrm{Tr}\,(O) = \int_{\mathbb{V}_{2n}} \mathrm{d}x \ \Omega^{-1}(O)(x)\,. \tag{2.13}$$

However, the linear mapping $f \mapsto \Omega(f)$ is *not* an algebra isomorphism. This fact can be used to *deform* the pointwise multiplication of functions on $\mathbb{V}_{2n}$ and define the *Moyal star-product* by

$$f \star g = \Omega^{-1}\big(\Omega(f)\,\Omega(g)\big) \tag{2.14}$$

for $f,g \in \mathcal{S}(\mathbb{V}_{2n})$. For the domains of functions we are interested in, a convenient explicit expression for the star-product is

$$(f \star g)(x) = (2\pi)^{-2n} \int_{T\mathbb{V}_{2n}} \mathrm{d}k \int_{\mathbb{V}_{2n}} \mathrm{d}y \, f\left(x + \frac{1}{2}\,\theta\,k\right) g\big(x+y\big) \, \exp\big(\,\mathrm{i}\,\langle k,y\rangle\big)\,, \tag{2.15}$$

where $(\theta\,k)^i := \sum_{j=1}^{2n} \theta^{ij}\,k_j$. With this representation the star-product of two Schwartz functions is again a Schwartz function. There are other commonly used explicit expressions for the star-product in the string theory literature, such as a Fourier integral representation or a formal asymptotic expansion using a bidifferential operator [86], but these formulas do not necessarily return Schwartz functions. Let us conclude the present discussion with a class of examples that will be relevant to our later constructions.

Example 2.1. For $x = (x^1, x^2) \in \mathbb{V}_2$ with $|x|^2 := (x^1)^2 + (x^2)^2$ one can straightforwardly compute the basic Gaussian Wigner function [35, 56, 57]

$$\Omega^{-1}(e_0 \otimes e_0^*)(x) = 2\,\mathrm{e}^{\,-|x|^2/\theta}\,.$$

More generally, for any $n>m$ one finds the Wigner functions

$$\Omega^{-1}(e_n\otimes e_m^*)(r,\vartheta)=2\,(-1)^m\,\sqrt{\frac{m!}{n!}}\,\left(\frac{2\,r^2}{\theta}\right)^{\frac{n-m}{2}}\\ \times\ \mathrm{e}^{\,\mathrm{i}\,(n-m)\,\vartheta}\ \mathrm{e}^{\,-r^2/\theta}\ L_m^{n-m}\left(\frac{2\,r^2}{\theta}\right),$$

where $(r,\vartheta)\in[0,\infty)\times[0,2\pi)$ are plane polar coordinates on $\mathbb{V}_2$ and

$$L_k^j(t)=\sum_{l=0}^k(-1)^l\binom{k+j}{k-l}\,\frac{t^l}{l!}$$

for $j,k\in\mathbb{N}_0$ are the associated Laguerre polynomials. It is an instructive exercise to check, using the explicit integral representation (2.15), that the star-products of these functions obey the appropriate orthonormality relations required for both (2.6) and (2.14) to hold.

2.4. *Derivations*

The infinitesimal action of the translation group of $\mathbb{V}_{2n}$ induces automorphisms $\partial_i:\mathbb{V}_{2n}^\theta\to\mathbb{V}_{2n}^\theta$, $i=1,\dots,2n$. On generators they are given by

$$\partial_i\big(x^j\big)=\delta_i{}^j\,. \tag{2.16}$$

With this definition one can verify the Leibniz rule, so that the automorphisms ∂_i, $i=1,\dots,2n$ define a set of *derivations* of the algebra $\mathbb{V}_{2n}^\theta$. One also finds the expected commutation relations of the translation group

$$\big[\partial_i\,,\,\partial_j\big]=0\,. \tag{2.17}$$

It is possible to modify these relations to give a representation $[\partial_i,\partial_j]=-\,\mathrm{i}\,\Phi_{ij}\,\mathbb{1}$ of the centrally extended translation group of $\mathbb{V}_{2n}$ without affecting any of our later considerations [6], but we will stick to the setting of Eq. (2.17) for simplicity.

By using the Heisenberg commutation relations (2.2), one finds for the representation of $\mathbb{V}_{2n}^\theta$ on the Fock module $\mathcal{F}$ that the derivations can be represented as *inner* automorphisms

$$\partial_i(f)=\mathrm{i}\,\sum_{j=1}^{2n}\big(\theta^{-1}\big)_{ij}\,\big[x^j\,,\,f\big] \tag{2.18}$$

for $f\in\mathbb{V}_{2n}^\theta$. Moreover, they induce the Weyl symbols of ordinary coordinate derivatives $\partial F/\partial x^i$ of Schwartz functions F through

$$\Omega\left(\frac{\partial F}{\partial x^i}\right)=\partial_i\big(\Omega(F)\big)\,. \tag{2.19}$$

Finally, one can show that Eq. (2.13) defines an *invariant* trace for the action of the translation group since

$$\mathrm{Tr}\left(\partial_i(O)\right) = 0 \tag{2.20}$$

for any $O \in \mathcal{K}$, which is equivalent to the usual formula for integration by parts of Schwartz functions on $\mathbb{V}_{2n}$.

We now have all the necessary ingredients to study a broad class of field theories on the noncommutative space $\mathbb{V}_{2n}^\theta$. Elements of the noncommutative algebra provide noncommutative fields, the invariant trace gives us an integral, and the derivations introduced above yield derivatives. We begin this investigation of noncommutative field theories on the Euclidean D-brane worldvolume $\mathbb{V}_{2n}$ in the next section.

3. Solitons on $\mathbb{V}_{2n}^\theta$

In this section we will study some elementary noncommutative *scalar* field theories on the worldvolume $\mathbb{V}_{2n}$. We will construct two broad classes of noncommutative solitons and describe the rich geometric structure of the corresponding moduli spaces. We shall then demonstrate that these solutions naturally define elements of analytic K-homology [92, 63, 46], which leads into their worldvolume interpretation as D-branes in Type II string theory. Some reviews of noncommutative solitons in the contexts described here can be found in [43, 45, 77].

3.1. *Projector solitons*

Let $\mathrm{u}(\mathbb{V}_{2n}^\theta)$ be the Lie algebra of *Hermitean* elements in the noncommutative space, i.e. the set of $\phi \in \mathbb{V}_{2n}^\theta$ for which the corresponding endomorphisms of the Fock module are Hermitean operators with respect to the underlying Hilbert space structure of $\mathcal{F}$, or equivalently the corresponding Wigner functions $\Omega^{-1}(\phi) : \mathbb{V}_{2n} \to \mathbb{R}$ are real-valued. Let $V : \mathbb{R} \to \mathbb{R}$ be a polynomial function. We may then define an *action functional* $S : \mathrm{u}(\mathbb{V}_{2n}^\theta) \to \mathbb{R}$ by

$$\begin{aligned} S(\phi) :=&\ \mathrm{Tr}\left(\frac{1}{2}\sum_{i=1}^{2n} \partial_i(\phi)\,\partial_i(\phi) + V(\phi)\right) \\ =&\ \mathrm{Tr}\left(-\frac{1}{2}\sum_{i,j,k=1}^{2n} \left(\theta^{-1}\right)_{ij}\left(\theta^{-1}\right)_{ik}\left[x^j\,,\,\phi\right]\left[x^k\,,\,\phi\right] + V(\phi)\right), \end{aligned} \tag{3.1}$$

when this expression makes sense. An action of this form describes the dynamics of D-branes in *Type IIA* string theory.

We now apply the variational principle. Identify the tangent space $T\mathbb{V}_{2n}$ with the hyperplane $\mathbb{V}_{2n}$ itself. For any $\alpha \in \mathrm{u}(\mathbb{V}^\theta_{2n})$ and $t \in \mathbb{R}$, we may then compute the variation of the action functional (3.1) by using invariance of the trace to get

$$\frac{\delta}{\delta\phi}S(\phi) := \frac{\mathrm{d}}{\mathrm{d}t}S(\phi + t\,\alpha)\Big|_{t=0} = -\sum_{i=1}^{2n}(\partial_i \circ \partial_i)(\phi) + V'(\phi)\,. \tag{3.2}$$

Setting this equal to 0 thereby gives the *equation of motion*

$$V'(\phi) = \sum_{i=1}^{2n}(\partial_i \circ \partial_i)(\phi)\,. \tag{3.3}$$

We are interested in special classes of solutions to these equations.

Definition 3.1. A *soliton* on $\mathbb{V}^\theta_{2n}$ is a solution $\phi \in \mathrm{u}(\mathbb{V}^\theta_{2n})$ of the equation of motion (3.3) for which the action functional $S(\phi)$ is well-defined and finite.

Naively, it seems to be very simple to construct soliton solutions to the equations (3.3). If λ is a critical point of the polynomial V, i.e. $V'(\lambda) = 0$, then an obvious solution is the *constant* solution $\phi_0 = \lambda\,\mathbb{1}$. However, this solution is not trace-class and so the action $S(\phi_0)$ is is not well-defined. To obtain finite action solutions, we will first look for *static solitons* having $\partial_i(\phi) = 0$. From the inner derivation property (2.18) we see that such fields live in the center of the algebra $\mathbb{V}^\theta_{2n}$. We will soon lift this requirement and show how to extend the construction to the general case. The advantage of this restriction is that the equation of motion (3.3) for static fields takes the very simple form

$$V'(\phi) = 0\,. \tag{3.4}$$

Since V is a polynomial, it is easy to find finite action solutions of this equation [35].

Theorem 3.1. *Let $\lambda_1, \dots, \lambda_p$ be the critical points of the polynomial function $V(\lambda)$. Then to each collection $\mathsf{P}_1, \dots, \mathsf{P}_p$ of mutually orthogonal finite rank projectors on the Fock module $\mathfrak{F}$ there bijectively corresponds a static soliton*

$$\phi_{\{\mathsf{P}_i\}} = \sum_{i=1}^{p}\lambda_i\ \mathsf{P}_i$$

in $\mathrm{u}(\mathbb{V}^\theta_{2n})$ *of action*

$$S\big(\phi_{\{\mathsf{P}_i\}}\big) = \big|\mathrm{Pf}(2\pi\,\theta)\big|^{-1}\sum_{i=1}^{p}V(\lambda_i)\ \ \mathrm{Tr}\,(\mathsf{P}_i)\,.$$

Remark 3.1. In these notes all projectors are assumed to be Hermitean. The soliton solution corresponding to the collection of projectors with $k := \mathrm{Tr}\,(\mathsf{P}_i) > 0$ for some i and $\mathrm{Tr}\,(\mathsf{P}_j) = 0$ for all $j \neq i$ is interpreted as k non-interacting solitons sitting on top of each other at the origin of $\mathbb{V}_{2n}$.

Example 3.1. The simplest projector on $\mathcal{F}$ is $\mathsf{P}_{(1)} = e_0 \otimes e_0^*$. The corresponding Wigner function is the Gaussian field computed in Example 2.1. The soliton is localized within a width $\theta^{-1/2}$ around the origin of the hyperplane. Note that this width formally goes to infinity in the commutative limit $\theta \to 0$ and the action becomes infinite. The field "delocalizes" and spreads out to the constant solution of infinite action. Since $\mathsf{P}_{(1)}$ has rank 1, it describes a single soliton at the origin. More generally, the projector

$$\mathsf{P}_{(k)} = \sum_{n=0}^{k-1} e_n \otimes e_n^*$$

has rank k and describes k solitons at the origin $x = 0$. The corresponding Wigner functions are given by combinations of Gaussian fields and Laguerre polynomials as described in Example 2.1.

3.2. *Soliton moduli spaces*

In D-brane physics one would like to understand what are the parameters that label inequivalent configurations modulo symmetries. These configurations live in a moduli space which determines the effective worldvolume geometries and on which we can study the effective dynamics of the branes. This is also a crucial ingredient for the eventual quantization of the systems, which would require an integration over the moduli space. Let us introduce for each $k \in \mathbb{N}$ the complex *Grassmannian*

$$\mathrm{Gr}(k, \mathcal{F}) = \mathrm{U}(\mathcal{F}) \,/\, \mathrm{U}(k) \times \mathrm{U}(\mathcal{F})\,, \tag{3.5}$$

where $\mathrm{U}(\mathcal{F})$ is the group of unitary endomorphisms of the Hilbert space $\mathcal{F}$ and $\mathrm{U}(k)$ is the group of $k \times k$ unitary matrices.

Proposition 3.1. *The moduli space $\mathcal{M}_k^0(\mathbb{V}_{2n}^\theta)$ of static k-solitons is an infinite-dimensional Kähler manifold isomorphic to the Grassmannian*

$$\mathcal{M}_k^0\big(\mathbb{V}_{2n}^\theta\big) = \mathrm{Gr}\big(k, \mathcal{F}\big)\,.$$

Proof. For static fields, the action (3.1) is invariant under the unitary transformations

$$\phi \longmapsto \mathrm{Ad}_U(\phi)$$

where $U \in \mathrm{U}(\mathcal{F})$. Any two projectors on $\mathcal{F}$ of the same rank are homotopic under this action. A projector of rank k has image which is a k-dimensional linear subspace $\mathcal{F}_k \subset \mathcal{F}$ and thus specifies a point in the Grassmannian (3.5), where the first unitary group $\mathrm{U}(\mathcal{F})$ acts on the whole of $\mathcal{F}$, the group $\mathrm{U}(k)$ acts on the finite-dimensional subspace $\mathcal{F}_k$, and the last $\mathrm{U}(\mathcal{F})$ factor acts on the orthogonal complement $\mathcal{F} \ominus \mathcal{F}_k \cong \mathcal{F}$. Let E_k be the tautological hyperplane bundle over $\mathrm{Gr}(k, \mathcal{F})$. The inner product on E_k induces a natural metric on the determinant line bundle $\det(E_k)$. The curvature two-form of this line bundle is the natural Kähler form on the Grassmannian. □

All solitons ϕ in the infinite-dimensional moduli space $\mathcal{M}_k^0(\mathbb{V}_{2n}^\theta)$ have the same action $S(\phi) = |\mathrm{Pf}(2\pi\,\theta)|^{-1}\,k\,V(\lambda_i)$ (for some i). We can obtain a finite-dimensional soliton moduli space by "translating" the solitons obtained above away from the origin of $\mathbb{V}_{2n}$ [36]. Introduce a complex structure on $\mathbb{V}_{2n}$ with local complex coordinates $z^j = x^{2j} + \mathrm{i}\,x^{2j-1}$, $\bar z^{\bar j} = x^{2j} - \mathrm{i}\,x^{2j-1}$ for $j = 1, \dots, n$. For each $z = (z^1, \dots, z^n)$ we define the *coherent state* $\xi(z) \in \mathcal{F}$ by

$$\xi(z) = \exp\left(\sum_{j=1}^n z^j\,a_j^\dagger\right) \cdot e_0\,, \tag{3.6}$$

where again the exponential operator is understood through its formal power series expansion. These vectors diagonalize the operators a_j with

$$a_j \cdot \xi(z) = z^j\ \xi(z)\,, \quad j = 1, \dots, n\,. \tag{3.7}$$

For the k-soliton solution, we place the solitons at some chosen points $z_{(0)}, z_{(1)}, \dots, z_{(k-1)}$ in $\mathbb{V}_{2n}$ with $z_{(i)} = (z_{(i)}^1, \dots, z_{(i)}^n)$. Let $\mathsf{P}_{\{z_{(i)}\}}$ be the orthogonal projection onto the linear span of the corresponding vectors $\xi(z_{(0)}), \xi(z_{(1)}), \dots, \xi(z_{(k-1)})$. Then $\mathrm{Tr}\,(\mathsf{P}_{\{z_{(i)}\}}) = k$. The corresponding soliton solution is called a *separated soliton.*

Remark 3.2. One can compute the Wigner functions corresponding to these projectors. For $n = 1$ and $z_{(i)}$ all distinct, one finds [36]

$$\Omega^{-1}\big(\mathsf{P}_{\{z_{(i)}\}}\big)(w, \bar w) = 2\sum_{i,j=0}^{k-1} \exp\left(-\left(\frac{\bar w}{\sqrt\theta} - \bar z_{(i)}\right)\left(\frac{w}{\sqrt\theta} - z_{(j)}\right)\right)$$

for $(w, \bar w) \in \mathbb{V}_2$. This function has a natural interpretation in terms of separated solitons.

The operators $\mathsf{P}_{\{z_{(i)}\}}$ may be characterized as those projectors P obeying the equations

$$(\mathbb{1} - \mathsf{P})\,a_j\,\mathsf{P} = 0\,, \quad j = 1, \dots, n\,, \tag{3.8}$$

or equivalently that the image $\mathrm{im}(\mathsf{P}) \subset \mathcal{F}$ is an invariant subspace for the collection of operators $a_1, \dots, a_n$. As we will now show, they define a finite-dimensional subspace $\mathcal{M}_k(\mathbb{V}^\theta_{2n}) \subset \mathrm{Gr}(k, \mathcal{F})$. Introduce the *Hilbert scheme* $\mathrm{Hilb}^k(\mathbb{V}_{2n})$ of k points in $\mathbb{V}_{2n} \cong \mathbb{C}^n$ as the space of ideals $\mathcal{I}$ of codimension k in the polynomial ring $\mathbb{C}[y^1, \dots, y^n]$. It is easy to see at a heuristic level how the Hilbert scheme is related to the projectors constructed above. Since $f \in \mathcal{I}$ implies that $f\,g \in \mathcal{I}$ for all polynomial functions g, the polynomials in an ideal $\mathcal{I}$ may be thought of roughly as projections on $\mathbb{C}[y^1, \dots, y^n] \to \mathcal{I}$. Conversely, if $\mathsf{P} = \mathsf{P}_{\{z_{(i)}\}}$ with all $z_{(i)}$ distinct, then the corresponding ideal $\mathcal{I}$ consists of those polynomials $f \in \mathbb{C}[y^1, \dots, y^n]$ which vanish at the loci $z_{(i)}$, i.e. $f(z^1_{(i)}, \dots, z^n_{(i)}) = 0$ for each $i = 0, 1, \dots, k-1$. This correspondence can be made more precise.

Theorem 3.2. *The moduli space $\mathcal{M}_k(\mathbb{V}^\theta_{2n})$ of separated k-solitons is the Hilbert scheme*

$$\mathcal{M}_k\big(\mathbb{V}^\theta_{2n}\big) = \mathrm{Hilb}^k\big(\mathbb{V}_{2n}\big) \, .$$

Proof. We set up a one-to-one correspondence between projectors $\mathsf{P} \in \mathcal{M}_k(\mathbb{V}^\theta_{2n})$ obeying eq. (3.8) and ideals $\mathcal{I} \in \mathrm{Hilb}^k(\mathbb{V}_{2n})$. Define for each polynomial $f \in \mathbb{C}[y^1, \dots, y^n]$ the vector

$$e_f = f\big(a^\dagger_1, \dots, a^\dagger_n\big) \cdot e_0$$

in $\mathcal{F}$. If $\mathcal{I} \subset \mathbb{C}[y^1, \dots, y^n]$ is an ideal of codimension k, then we let $\mathbb{1} - \mathsf{P}$ be the orthogonal projection of $\mathcal{F}$ onto the linear span $\bigoplus_{f \in \mathcal{I}} \mathbb{C} \cdot e_f$. Conversely, if $\mathsf{P} \in \mathcal{M}_k(\mathbb{V}^\theta_{2n})$ we set

$$\mathcal{I} = \big\{ f \in \mathbb{C}[y^1, \dots, y^n] \bigm| \mathsf{P} \cdot e_f = 0 \big\} \, ,$$

which is an ideal since $\mathsf{P}\, a^\dagger_j = \mathsf{P}\, a^\dagger_j\, \mathsf{P}$ for each $j = 1, \dots, n$. □

Example 3.2. Theorem 3.2 allows us to work out some explicit soliton moduli spaces for low values of the integers k and n.

1. $\mathcal{M}_k(\mathbb{V}^\theta_2) = \mathrm{Hilb}^k(\mathbb{V}_2)$ is the k-th symmetric product orbifold $\mathrm{Sym}^k(\mathbb{V}_2) = (\mathbb{V}_2)^k / S_k$, where the symmetric group S_k acts on the soliton positions in $(\mathbb{V}_2)^k$ by permuting the entries of a k-tuple of elements of the plane $\mathbb{V}_2$. The Kähler metric inherited from the Grassmannian is smooth at the orbifold points corresponding to coinciding soliton positions in $(\mathbb{V}_2)^k$ [36, 41]. Thus noncommutativity smooths out the orbifold singularities of the symmetric product and as Kähler manifolds one has an isomorphism

$$\mathcal{M}_k\big(\mathbb{V}^\theta_2\big) = \big(\mathbb{V}_2\big)^k \, .$$

2. The two-soliton moduli space is [36]

$$\mathcal{M}_2\big(\mathbb{V}_{2n}^\theta\big) = \mathbb{V}_{2n} \times \mathcal{O}_{\mathbb{P}^{n-1}}\big(-1\big)\,,$$

   where $\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to \mathbb{P}^{n-1}$ is the Hopf bundle over the complex projective space $\mathbb{P}^{n-1}$. The first factor describes the center of mass position of the soliton configuration, while the second factor is the resolution of the singularity of the moduli space for the relative position which blows up the origin into $\mathbb{P}^{n-2}$. Again the moduli space is non-singular.

3. $\mathcal{M}_k(\mathbb{V}_4^\theta) = \mathrm{Hilb}^k(\mathbb{V}_4)$ is a smooth manifold which is a crepant resolution of the singular quotient variety $\mathrm{Sym}^k(\mathbb{V}_4)$. It also arises as the moduli space of k U(1) instantons on $\mathbb{V}_4^\theta$ [16, 36]. However, while the instanton moduli space is endowed with a hyper-Kähler metric, the soliton moduli space $\mathcal{M}_k(\mathbb{V}_4^\theta)$ is only a Kähler manifold.

4. For $n > 2$ and $k > 3$ the moduli space $\mathcal{M}_k(\mathbb{V}_{2n}^\theta)$ generically contains branches of varying dimension and so is not even a manifold [66].

3.3. *Partial isometry solitons*

We will now construct solitons corresponding to general complex elements $\phi \in \mathbb{V}_{2n}^\theta$. We call these *complex solitons.* We use the same polynomial function V as before, but now the action functional $S : \mathbb{V}_{2n}^\theta \to \mathbb{R}$ is defined by

$$S\big(\phi\,,\,\phi^\dagger\big) = \ \mathrm{Tr}\,\left(\sum_{i=1}^{2n} \partial_i\big(\phi\big)\,\partial_i\big(\phi^\dagger\big) + V\big(\phi^\dagger\phi - \mathbb{1}\big) + V\big(\phi\,\phi^\dagger - \mathbb{1}\big)\right). \quad (3.9)$$

Such an action describes the dynamics of D-branes in *Type IIB* string theory. As before it is straightforward to obtain soliton solutions of the equations of motion corresponding to (3.9).

Proposition 3.2. *To each partially isometric Fredholm operator on the Fock module $\mathcal{F}$ there bijectively corresponds a static complex soliton.*

Proof. Varying ϕ and $\phi^\dagger$ independently in the action functional (3.9) shows that the equations of motion for static fields are satisfied if $\phi, \phi^\dagger$ obey the defining equation

$$\phi\,\phi^\dagger\,\phi = \phi$$

for a partial isometry of $\mathcal{F}$. Equivalently, ϕ is an isometry in the orthogonal complement to a kernel and cokernel, or

$$\phi^\dagger\,\phi = \mathbb{1} - \mathsf{P}_{\ker(\phi)}\,, \quad \phi\,\phi^\dagger = \mathbb{1} - \mathsf{P}_{\mathrm{coker}(\phi)}\,,$$

where $\mathsf{P}_{\ker(\phi)}$ and $\mathsf{P}_{\mathrm{coker}(\phi)}$ are the orthogonal projections onto the kernel and cokernel of ϕ. Substituting these expressions into the action functional (3.9) and demanding that it be finite requires that both $\ker(\phi)$ and $\mathrm{coker}(\phi)$ be finite-dimensional subspaces of $\mathcal{F}$, i.e. that ϕ be also a Fredholm operator. □

Remark 3.3. Using Remark 2.2, the finite-dimensional subspaces $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are identified with the vanishing locus of the complex soliton in the corresponding Wigner function formulation on $\mathbb{V}_{2n}$.

Definition 3.2. The *topological charge* $Q(\phi)$ of a complex soliton $\phi \in \mathbb{V}_{2n}^{\theta}$ is its analytic index $Q(\phi) := \mathrm{index}(\phi) = \dim\ker(\phi) - \dim\,\mathrm{coker}(\phi)$.

To explicitly construct such solitons, let $\mathrm{C}\ell(\mathbb{V}_{2n})$ be the complex Clifford algebra of the vector space $\mathbb{V}_{2n}$ equipped with the canonical quadratic form. Let $\Delta_\pm$ be the irreducible half-spinor modules over $\mathrm{C}\ell(\mathbb{V}_{2n})$ of ranks $r := 2^{n-1}$. The half-spinor generators are denoted $\sigma_i : \Delta_+ \to \Delta_-$, $i = 1, \dots, 2n$ and they satisfy the algebra

$$\sigma_i{}^\dagger\,\sigma_j + \sigma_j{}^\dagger\,\sigma_i ~=~ 2\,\delta_{ij}\,\mathbb{1}_r ~=~ \sigma_i\,\sigma_j{}^\dagger + \sigma_j\,\sigma_i{}^\dagger\,. \tag{3.10}$$

We introduce the operator

$$\sigma_x := \sum_{i=1}^{2n} \sigma_i \otimes x^i \tag{3.11}$$

regarded as an element $\sigma_x \in \mathrm{Hom}(\Delta_+ \otimes \mathcal{F}, \Delta_- \otimes \mathcal{F})$.

Lemma 3.1. *The operator σ_x has one-dimensional kernel and no cokernel.*

Proof. From Eqs. (2.9) and (3.10) it follows that the operator (3.11) and its adjoint obey the identities

$$\sigma_x\,\sigma_x{}^\dagger = \sum_{i=1}^{n} \mathbb{1}_r \otimes 2\,\theta_i\,\left(a_i^\dagger\,a_i + \frac{1}{2}\,\mathbb{1}\right) - \sum_{i,j=1}^{2n} \mathrm{i}\,\theta^{ij}\,\Sigma_{ij} \otimes \mathbb{1}\,,$$

$$\sigma_x{}^\dagger\,\sigma_x = \sum_{i=1}^{n} \mathbb{1}_r \otimes 2\,\theta_i\,\left(a_i^\dagger\,a_i + \frac{1}{2}\,\mathbb{1}\right) - \sum_{i,j=1}^{2n} \mathrm{i}\,\theta^{ij}\,\Sigma_{ij}{}^\dagger \otimes \mathbb{1}\,,$$

where

$$\Sigma_{ij} = \frac{1}{4}\left(\sigma_i\,\sigma_j{}^\dagger - \sigma_j\,\sigma_i{}^\dagger\right)\,, \quad \Sigma_{ij}{}^\dagger = \frac{1}{4}\left(\sigma_i{}^\dagger\,\sigma_j - \sigma_j{}^\dagger\,\sigma_i\right)\,.$$

By elementary group theory, the last term in the second product is diagonalized by the lowest weight spinor ψ_0 of $\mathrm{SO}(2n)$ to $-\sum_{i=1}^{n}\,\theta_i\,\mathbb{1}_r \otimes \mathbb{1}$. Along with Eq. (2.7), this implies that the operator σ_x has a one-dimensional kernel in $\Delta_+ \otimes \mathcal{F}$ which is spanned by the vector $\psi_0 \otimes e_0$. On the other hand, the right-hand side of the first product can never vanish and so the kernel of ${\sigma_x}^\dagger$ is trivial. □

Theorem 3.3. *The surjection* $\mathsf{T} \in \mathrm{Hom}(\Delta_+ \otimes \mathcal{F}, \Delta_- \otimes \mathcal{F})$ *defined by*

$$\mathsf{T} = \left(\sigma_x \, {\sigma_x}^\dagger\right)^{-1/2} \sigma_x$$

is a complex soliton of topological charge $Q(\mathsf{T}) = 1$.

Proof. By Lemma 3.1 the positive operator $\sigma_x \, {\sigma_x}^\dagger$ is invertible and so the operator T is well-defined. Furthermore, it is a partial isometry, $\mathsf{T}\,\mathsf{T}^\dagger\,\mathsf{T} = \mathsf{T}$, and has one-dimensional kernel and trivial cokernel with

$$\mathsf{T}\,\mathsf{T}^\dagger = \mathbb{1}_r \otimes \mathbb{1}\,, \quad \mathsf{T}^\dagger\,\mathsf{T} = \mathbb{1}_r \otimes \mathbb{1} - \mathsf{P}_{\ker \sigma_x}$$

implying that $\mathrm{im}(\mathsf{T}) = \Delta_- \otimes \mathcal{F}$. □

Remark 3.4. The classical solution of Theorem 3.3 is interpreted as a single-soliton solution. More generally, a k-soliton solution is given by the power $(\mathsf{T})^k$ which has no cokernel and k-dimensional kernel by the Boutet de Monvel index theorem [15], i.e. $Q(\mathsf{T})^k = k$. One can also construct separated complex solitons by "translating" these operators away from the origin to points $x_{(0)}, x_{(1)}, \ldots, x_{(k-1)} \in \mathbb{V}_{2n}$ and defining [62]

$$\mathsf{T}_{\{x_{(i)}\}} = \prod_{i=0}^{k-1} \left(\sigma_{x-x_{(i)}\,\mathbb{1}} \, {\sigma_{x-x_{(i)}\,\mathbb{1}}}^\dagger\right)^{-1/2} \sigma_{x-x_{(i)}\,\mathbb{1}} \,.$$

The analysis of the moduli space of these separated complex solitons is similar to that carried out in Section 3.2.

3.4. *Topological charges*

We will now derive a geometric formula for the topological charge of a complex soliton by relating the analytic index of T to a topological index [46, 63]. Let $\mathrm{S}^{2n-1} = \{x \in \mathbb{V}_{2n} \mid |x| = 1\}$ be the unit sphere of odd dimension $2n-1$ in the hyperplane $\mathbb{V}_{2n}$. The restriction map $\mathbb{V}_{2n} \setminus \{0\} \to \mathrm{S}^{2n-1}$ is defined by $x \mapsto \frac{x}{|x|}$. The soliton T constructed in Theorem 3.3 can be thought of as a noncommutative version of the map $\mu : \mathrm{S}^{2n-1} \to \mathrm{GL}(r, \mathbb{C})$ defined by

$$\mu_x = \sum_{i=1}^{2n} \frac{x^i}{|x|} \, \sigma_i \,, \tag{3.12}$$

which is Clifford multiplication on the $C\ell(\mathbb{V}_{2n})$-module Δ_+ by the vector $x \in \mathbb{V}_{2n}$. To make this correspondence more precise, we have to explain what we mean by restricting an operator to a noncommutative sphere.

We begin by choosing another polarization of the Fock module $\mathcal{F}$ which can be regarded as the holomorphic version of the Schrödinger representation defined in Remark 2.1. The *Bargmann quantization* is the natural isomorphism

$$\mathcal{F} \cong \mathcal{F}_{\rm B} := {\rm Hol}\left(\mathbb{V}_{2n}\,,\,\exp\left(-2\sum_{k=1}^{n}\theta_k\,|z^k|^2\right)\,{\rm d}z\right) \tag{3.13}$$

of $\mathbb{V}^\theta_{2n}$-modules obtained by representing $a_k = M_{z^k}$ as multiplication by the complex coordinate z^k and $a_k^\dagger$ as the partial differential operator $-\frac{\partial}{\partial z^k}$. In Bargmann quantization, an orthogonal basis is provided by the collection of monomials

$$z^{\mathbf{k}} := \prod_{i=1}^{n}\left(z^i\right)^{k_i}\,, \quad \mathbf{k} = (k_1,\dots,k_n) \in \mathbb{N}_0^n\,. \tag{3.14}$$

The advantage of the diffeomorphism (3.13) is that there is a precise way to restrict vectors in $\mathcal{F}_{\rm B}$ to the sphere ${\rm S}^{2n-1}$. Using polar coordinates we decompose the canonical measure on $\mathbb{V}_{2n}$ as ${\rm d}z = r^{2n-1}~{\rm d}r~{\rm d}\Omega$, where $r \in \mathbb{R}_+ := [0,\infty)$ and ${\rm d}\Omega$ is the standard round measure on the unit sphere ${\rm S}^{2n-1}$. We use homogeneous complex coordinates $(z,\bar z)$ on ${\rm S}^{2n-1}$ with $\sum_{k=1}^n |z^k|^2 = 1$. The *Hardy space* ${\rm H}^2({\rm S}^{2n-1},{\rm d}\Omega)$ is the closed Hilbert subspace of ${\rm L}^2({\rm S}^{2n-1},{\rm d}\Omega)$ consisting of ${\rm L}^2$-functions on ${\rm S}^{2n-1}$ which admit a holomorphic extension to all of $\mathbb{V}_{2n}$. An orthogonal basis for the Hardy space ${\rm H}^2({\rm S}^{2n-1},{\rm d}\Omega)$ is again provided by the monomials [46]

$$\varphi_{\mathbf{k}} := z^{\mathbf{k}}\,. \tag{3.15}$$

To specify the restriction of the $\mathbb{V}^\theta_{2n}$-module structure, we need to make sense of the action of the classical coordinates $z^k, \bar z^{\bar k}$ on ${\rm H}^2({\rm S}^{2n-1},{\rm d}\Omega)$. Let $f : {\rm S}^{2n-1} \to \mathbb{C}$ be an ${\rm L}^2$-function. Let $P_+ : {\rm L}^2({\rm S}^{2n-1},{\rm d}\Omega) \to {\rm H}^2({\rm S}^{2n-1},{\rm d}\Omega)$ be the *Szegó projection* defined by

$$(P_+f)(z) = \int_{{\rm S}^{2n-1}} {\rm d}\Omega(w,\bar w)~\frac{f(w,\bar w)}{\left(1-\sum\limits_{k=1}^{n} z^k\,\bar w^{\bar k}\right)^n}\,, \tag{3.16}$$

and let $M_f : {\rm H}^2({\rm S}^{2n-1},{\rm d}\Omega) \to {\rm L}^2({\rm S}^{2n-1},{\rm d}\Omega)$ be the operator of multiplication by f. The *Toeplitz operator* $\mathsf{T}_f : {\rm H}^2({\rm S}^{2n-1},{\rm d}\Omega) \to {\rm H}^2({\rm S}^{2n-1},{\rm d}\Omega)$ is then defined by

$$\mathsf{T}_f = P_+ \circ M_f\,. \tag{3.17}$$

It is the *compression* of the multiplication operator M_f to $\mathrm{H}^2(\mathrm{S}^{2n-1},\mathrm{d}\Omega)$. The action of the Toeplitz operators corresponding to the coordinate functions on the basis (3.15) is straightforward to work out. Let $\mathbf{e}_i$ denote the ith standard unit vector in $\mathbb{R}^n$. Then for each $i=1,\dots,n$ one has

$$\begin{aligned} \mathsf{T}_{z^i}\varphi_{\mathbf{k}} &= \varphi_{\mathbf{k}+\mathbf{e}_i}\ , \\ \mathsf{T}_{\bar z^{\bar\imath}}\varphi_{\mathbf{k}} &= \begin{cases} 0 \ , & k_i=0 \\ 2\pi\,\dfrac{k_i}{|\mathbf{k}|+n-1}\,\varphi_{\mathbf{k}-\mathbf{e}_i}\ , & k_i>0 \end{cases} \end{aligned} \tag{3.18}$$

where $|\mathbf{k}|:=\sum_{i=1}^n k_i$ for $\mathbf{k}\in\mathbb{N}_0^n$.

Example 3.3. It is instructive at this stage to look explicitly at the two-dimensional case $n=1$. Then $r=1$, $\sigma_1=1$ and $\sigma_2=\mathrm{i}$ so that the standard complex soliton

$$\mathsf{T}=\left(a^\dagger\,a\right)^{-1/2}a=:\mathsf{S}^\dagger$$

coincides with the *shift operator* $\mathsf{S}:\mathcal{F}\to\mathcal{F}$ defined on the number basis by

$$\mathsf{S}e_m=e_{m+1}\ .$$

The shift operator is the basic partial isometry of $\mathcal{F}$ with

$$\mathsf{S}^\dagger\,\mathsf{S}=\mathbb{1}\ ,\quad \mathsf{S}\,\mathsf{S}^\dagger=\mathbb{1}-e_0\otimes e_0^*\ ,$$

and hence it has no kernel and a one-dimensional cokernel spanned by the lowest vector e_0 in the number basis of $\mathcal{F}$. The Hardy space in this instance $\mathrm{H}^2(\mathrm{S}^1,\mathrm{d}\Omega)=\mathcal{H}_+(\mathrm{S}^1)$ is the closed Hilbert subspace of $\mathrm{L}^2(\mathrm{S}^1,\mathrm{d}\Omega)$ spanned by the non-negative Fourier modes on the circle $\varphi_k=\mathrm{e}^{\,\mathrm{i}\,k\,\Omega}$, $k\geq 0$, $\Omega\in[0,2\pi)$. This is the positive eigenspace of the Dirac operator $-\,\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}\Omega}$ on S^1. By identifying the monomial φ_k with the number basis element $e_k\in\mathcal{F}$, one can identify the Toeplitz operators $\mathsf{T}_{\varphi_k}=(\mathsf{S})^k$ for $k>0$ (which have non-trivial kernels for $k<0$).

The correspondence between Toeplitz operators and complex solitons on $\mathbb{V}_{2n}^\theta$ given in Example 3.3 can be generalized to higher dimensions $n>1$ by replacing the Hardy space with $\mathrm{H}^2(\mathrm{S}^{2n-1},\mathrm{d}\Omega)\otimes\mathbb{C}^r$ (with $r:=2^{n-1}$ as before) and extending the Toeplitz operators T_f to matrix-valued L^2-functions $f:\mathrm{S}^{2n-1}\to\mathbb{M}_r(\mathbb{C})$. Then the complex soliton of Theorem 3.3 corresponds to a Toeplitz operator

$$\mathsf{T}_\mu=P_+\circ M_\mu \tag{3.19}$$

in $\mathrm{End}(\mathrm{H}^2(\mathrm{S}^{2n-1},\mathrm{d}\Omega)\otimes\Delta)$, where Δ is the unique irreducible spinor module of rank r over the Clifford algebra $\mathrm{C}\ell(\mathbb{R}^{2n-1})$. This defines a bounded Fredholm operator on Hardy space and we finally arrive at our desired geometric formula for the topological charge.

Theorem 3.4. *The topological charge of the complex soliton* T *is given by the characteristic class formula*

$$Q(\mathsf{T}) = \mathrm{ch}(\mu)\big[\,\mathrm{S}^{2n-1}\,\big]\,,$$

where

$$\mathrm{ch}(\mu) = \mu^*\left(\sum_{j=1}^{n}\frac{(-1)^j}{(j-1)!}\,\omega_{2j-1}\right)$$

and ω_i *are the standard generators of the rational cohomology* $\mathrm{H}^i(\mathrm{GL}(r,\mathbb{C}),\mathbb{Q})$.

Proof. Represent the partial isometry T in Bargmann quantization. Then the Toeplitz operator (3.19) is the image of T under the restriction map $\mathcal{F}_{\mathrm{B}}\otimes\Delta\to\mathrm{H}^2(\mathrm{S}^{2n-1},\mathrm{d}\Omega)\otimes\Delta$. This map is not unitary, but it is a bijection and consequently $\mathrm{index}(\mathsf{T}) = \mathrm{index}(\mathsf{T}_\mu)$. The result now follows from the Boutet de Monvel index theorem [15] applied to the Toeplitz operator T_μ and the fact that the sphere S^{2n-1} has trivial Todd class. □

Remark 3.5. For $n=1$, Theorem 3.4 reads $Q(\mathsf{T}) = \frac{1}{2\pi\,\mathrm{i}}\int_{\mathrm{S}^1}\mu^{-1}\,\mathrm{d}\mu$ and thus the topological charge of the noncommutative soliton coincides with the winding number of the function $\mu:\mathrm{S}^1\to\mathrm{S}^1$.

3.5. *Worldvolume construction*

We will now demonstrate how the soliton solution constructed in Section 3.3 has a natural interpretation in terms of D-branes. The construction of these solitons is intimately related to the Atiyah–Bott–Shapiro (ABS) construction $M_\sharp\,\mathrm{Spin}(\mathbb{V}_{2n})\to\mathrm{K}^\sharp(\mathbb{V}_{2n})$ of K-theory classes in terms of Clifford modules, whose generator is provided by Clifford multiplication (3.12). By Theorem 3.4, the topological charge of the noncommutative soliton coincides with the index of the classical ABS class of μ, whose winding number determines *D-brane charge* [91, 49, 67]. By elucidating this point we will link our solutions naturally with D-branes. Our ensuing worldvolume interpretation will thereby demonstrate the equivalence between the usual commutative and the noncommutative descriptions of D-branes, and will further provide a novel insight into the nature of the worldvolume geometries.

The basic idea behind the construction is to associate D-branes to algebras of "almost commuting" operators [46]. Let $\mathcal{B}(\mathcal{F}) \subset \mathrm{End}(\mathcal{F})$ be the C^*-algebra of bounded linear operators on the Hilbert space $\mathcal{F}$. The C^*-algebra $\mathcal{K} = \mathcal{K}(\mathcal{F})$ of compact operators on $\mathcal{F}$ is a closed ideal in $\mathcal{B}(\mathcal{F})$. The Toeplitz operators (3.17) generate a unique C^*-algebra called the *Toeplitz algebra* which we will denote by $\mathcal{A} \subset \mathcal{B}(\mathcal{F})$. To ease notation we write $X := \mathrm{S}^{2n-1}$. Let $C(X)$ be the commutative C^*-algebra of continuous complex-valued functions on X. In general, the map $C(X) \to \mathcal{A}$ defined by $f \mapsto \mathsf{T}_f$ is not an algebra homomorphism.

Proposition 3.3. *For each pair of functions $f, g \in C(X)$, the difference $\mathsf{T}_f\,\mathsf{T}_g - \mathsf{T}_{f\,g}$ in the Toeplitz algebra $\mathcal{A}$ is a compact operator on the Fock module $\mathcal{F}$.*

It follows that $[\mathsf{T}_f, \mathsf{T}_g] \in \mathcal{K}$ is compact for any $f, g \in C(X)$ [17], and compact operators are always regarded as "small" (being elements of a closed dense domain in $\mathcal{B}(\mathcal{F})$). Thus the Toeplitz algebra $\mathcal{A}$ is "almost commuting". We can identify operators which differ from one another only by a "small" perturbation by regarding them as elements of the *Calkin algebra* $\mathcal{Q}(\mathcal{F}) := \mathcal{B}(\mathcal{F}) \,/\, \mathcal{K}$ with the natural projection $\pi : \mathcal{B}(\mathcal{F}) \to \mathcal{Q}(\mathcal{F})$. The Calkin algebra is a unital C^*-algebra. In this way our explicit construction of static complex solitons in Section 3.3 leads naturally to the Brown-Douglas-Fillmore classification of essentially normal operators [17, 46].

Starting from the soliton configuration (3.19), the map $\mathsf{T}_\mu \mapsto \mu$ gives rise to a C^*-epimorphism $\beta : \mathcal{A} \to C(X) \otimes \mathbb{M}_r(\mathbb{C})$. The algebra $C(X) \otimes \mathbb{M}_r(\mathbb{C})$ is Morita equivalent to $C(X)$, and hence it will suffice to restrict our attention to the commutative algebra $C(X)$. It follows that the Toeplitz operators generate an *extension* of the commutative algebra of functions on X by compact operators, i.e. the noncommutative algebra $\mathcal{A}$ fits into a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{\ \beta\ } C(X) \longrightarrow 0\,. \tag{3.20}$$

Exactness of the sequence follows from Proposition 3.3 and the fact that $\mathsf{T}_f\,\mathsf{T}_g - \mathsf{T}_{f\,g} \in \ker(\beta)$ for any two functions $f, g \in C(X)$.

We can introduce a set of equivalence classes of extensions (3.20) as follows. Define a map $\tau : C(X) \to \mathcal{Q}(\mathcal{F})$ called the *Busby invariant* by $\tau(f) = \pi(\mathsf{T}_f)$. By Proposition 3.3, the Busby invariant is a unital C^*-monomorphism with $\mathcal{A} = \pi^{-1}(\mathrm{im}\,\tau)$. Any extension (3.20) can be uniquely characterized by a pair $(\mathcal{H}, \tau)$, where $\mathcal{H}$ is a separable Hilbert space and $\tau : C(X) \to \mathcal{Q}(\mathcal{H})$ is a unital C^*-monomorphism [17]. On the collection of pairs $(\mathcal{H}, \tau)$, there is a natural notion of unitary (or strong) equivalence and a natural direct sum operation [17, 11, 46]. The set $\mathrm{Ext}(C(X), \mathcal{K})$ of equivalence classes of extensions (3.20) is thus a semigroup. An extension is *trivial* if the exact sequence (3.20) splits, i.e. if the corresponding Busby invariant τ has a lift to all of $\mathcal{B}(\mathcal{H})$. The quotient of $\mathrm{Ext}(C(X),\mathcal{K})$ by trivial extensions is an abelian group which defines a dual homology theory to the K-theory of X. This is called the *analytic K-homology group* $\mathrm{K}_1^{\mathrm{a}}(X)$.

The constructions of this section bring us finally to our main result [11].

Theorem 3.5. *There is a one-to-one correspondence between static complex solitons* $\mathsf{T}\in\mathbb{V}_{2n}^{\theta}$ *with Toeplitz extension classes* (3.20) *in* $\mathrm{K}_1^{\mathrm{a}}(X)$ *and D-branes* (W,E,ζ) *on* X *with odd-dimensional worldvolumes* W.

Proof. Let $\Delta_W\to W$ be the spin^c bundle over the odd-dimensional spin^c manifold W. Let $\mathcal{H}=\mathrm{L}^2(W,\Delta_W\otimes E)$ be the Hilbert space of square-integrable sections of the twisted spin^c bundle over the worldvolume. Equip the Chan–Paton bundle $E\to W$ with a connection, and let W inherit the metric from X by pullback under the continuous map $\zeta:W\to X$. The corresponding twisted Dirac operator $\not{D}_E$ can be viewed as a closed unbounded operator $\not{D}_E:\mathcal{H}\to\mathcal{H}$. If the connection and metric are generic, then $\not{D}_E$ has no kernel and so the Hilbert space $\mathcal{H}$ admits an orthogonal decomposition

$$\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$$

into the positive/negative eigenspaces $\mathcal{H}_\pm$ of the Dirac operator $\not{D}_E$.

We can represent the commutative C^*-algebra $C(W)$ of worldvolume functions on $\mathcal{H}$ by multiplication operators M_f for $f\in C(W)$. Generically the operator M_f does not preserve the subspace $\mathcal{H}_+$, but as before its compression to a Toeplitz operator $\mathsf{T}_f=P_+\circ M_f:\mathcal{H}_+\to\mathcal{H}_+$ does, where now P_+ is the orthogonal projection $\mathcal{H}\to\mathcal{H}_+$. As before, for $f,g\in C(W)$ the difference $\mathsf{T}_f\,\mathsf{T}_g-\mathsf{T}_{f\,g}$ is a compact operator on $\mathcal{H}_+$ and so we get a Toeplitz extension

$$0\longrightarrow\mathcal{K}\longrightarrow\mathcal{A}_W\longrightarrow C(W)\longrightarrow 0$$

of the algebra of worldvolume functions, where $\mathcal{A}_W$ is the C^*-algebra generated by T_f for $f\in C(W)$. Let $\tau_W:C(W)\to\mathcal{Q}(\mathcal{H})$ be the corresponding Busby invariant. Then by using the pullback $\zeta^*:C(X)\to C(W)$ we can define a new Busby invariant $\widetilde{\tau}:=\tau_W\circ\zeta^*:C(X)\to\mathcal{Q}(\mathcal{H})$ which corresponds to an extension

$$0\longrightarrow\mathcal{K}\longrightarrow\widetilde{\mathcal{A}}\longrightarrow C(X)\longrightarrow 0$$

of the algebra of functions on all of X. The corresponding K-homology class in $\mathrm{K}_1^{\mathrm{a}}(X)$ is independent of the metric on X and of the choice of connection on E. $\square$

Remark 3.6. The proof of the converse of the result proven here is beyond the scope of these notes. The one-to-one correspondence between extension classes in $\mathrm{K}_1^{\mathrm{a}}(X)$ and D-branes requires taking the quotient of the set of all Baum–Douglas K-cycles (W, E, ζ) by a collection of equivalence relations [11, 74]. These relations make good physical sense and the corresponding equivalence classes $[W, E, \zeta]$ capture the novel dynamical processes of D-brane physics [46, 8, 74, 87, 58]. Note that in this correspondence the D-brane worldvolume W need not be a submanifold of X. The problem of finding spaces X for which the generators of K-homology can be constructed from cycles of X is related to the Hodge conjecture. A projective algebraic variety that satisfies both the requirements that cycles generate K-homology [74] and the hypothesis of the Hodge conjecture has certain restrictions on its cohomology. A Calabi–Yau threefold satisfies these conditions, and thus the K-cycles in physically viable string compactifications X correspond to D-branes whose worldvolumes are cycles of X.

Theorem 3.5 provides us with the proper perspective on the noncommutative solitons that we have constructed in this section. The solitons provide the K-homology version of the equivalence of D-brane charges in commutative and noncommutative field theories. This is asserted via the equivalence between the analytic index and the topological index via the Baum–Douglas K-cycle construction [11, 74]. The Toeplitz operators on Hardy space $\mathrm{H}^2(\mathrm{S}^{2n-1}, \mathrm{d}\Omega)$ determine an algebra $\mathcal{A}$ of endomorphisms providing a non-trivial extension (3.20) by compact operators. This defines an analytic K-homology class in $\mathrm{K}_1^{\mathrm{a}}(\mathrm{S}^{2n-1})$ which is the same as the class $[\not{D}]$ determined by the Dirac operator on S^{2n-1} [11]. In particular, Theorem 3.4 now follows from the ordinary Atiyah–Singer index theorem. We may associate this class with one in degree zero K-homology which is the appropriate receptacle for the classification of D-brane charges in Type IIB string theory [91, 67]. If we work in *relative* K-homology [74], then the connecting homomorphism in the six-term exact sequence for the pair $(\mathrm{B}^{2n}, \mathrm{S}^{2n-1} = \partial\mathrm{B}^{2n})$ yields an isomorphism

$$\partial : \mathrm{K}_0^{\mathrm{a}}\big(\mathrm{B}^{2n}, \mathrm{S}^{2n-1}\big) \xrightarrow{\approx} \mathrm{K}_1^{\mathrm{a}}\big(\mathrm{S}^{2n-1}\big)\,. \tag{3.21}$$

This determines an element of the compactly supported degree zero K-homology of $\mathbb{V}_{2n}$ associated to the noncommutative soliton. We have in this way provided a description of D-branes in terms of algebras $\mathcal{A}$ of "almost commuting" operators corresponding to "almost commutative" spaces which extend the hyperplane worldvolume $\mathbb{V}_{2n}$.

4. Gauge Theory on $\mathbb{V}^\theta_{2n}$

In the previous section we have arrived at a description of noncommutative solitons as D-branes in terms of K-cycles (W, E, ζ). In the correspondence it was essential to introduce a connection on the complex Chan–Paton vector bundle $E \to W$. It is natural to now construct worldvolume field theories using these connections. In the sequel we shall therefore focus our attention on *gauge theories* on noncommutative spaces and examine what their classical solutions can teach us. Worldvolume gauge theories are the essence of the novel dynamical properties of D-branes in string theory.

4.1. *Projective modules*

To construct gauge theory on $\mathbb{V}^\theta_{2n}$ we proceed in the usual way by introducing connections on projective modules over the noncommutative algebra. The natural class of projective modules are the collections of Fock modules $\mathcal{F}^q := \mathcal{F} \oplus \cdots \oplus \mathcal{F}$ (q times). There are also the trivial free modules of rank N given by N copies $\mathcal{H}^N$ of the algebra $\mathcal{H} := \mathbb{V}^\theta_{2n}$ itself. Let us now go through some general facts concerning projective modules over $\mathbb{V}^\theta_{2n}$ [51, 80].

Proposition 4.1. *Any finitely generated left projective module over the Moyal space $\mathbb{V}^\theta_{2n}$ is of the form $\mathcal{E}_{N,q} = \mathcal{H}^N \oplus \mathcal{F}^q$ for some $N, q \in \mathbb{N}_0$.*

Remark 4.1. The integer N corresponds to the rank, or zeroth Chern number $c_0(\mathcal{E}_{N,q})$, of the module. The integer q corresponds to a *topological charge* whose interpretation depends on dimension. For example, when $n = 1$ it is the *magnetic charge* or first Chern number $c_1(\mathcal{E}_{N,q})$, while when $n = 2$ it is the *instanton number* or second Chern number $c_2(\mathcal{E}_{N,q})$.

Corollary 4.1. *The K-theory of the Moyal $2n$-space is given by* $\mathrm{K}_0(\mathbb{V}^\theta_{2n}) = \mathbb{Z} \oplus \mathbb{Z}$ *with positive cone* $\mathrm{K}_0^+(\mathbb{V}^\theta_{2n}) = \mathbb{N} \oplus \mathbb{N}$.

Remark 4.2. Any two projective modules representing the same element of K-theory are isomorphic. The K-theory of $\mathbb{V}^\theta_{2n}$ is quite different from the (compactly supported) K-theory of the ordinary topologically trivial hyperplane $\mathbb{V}_{2n}$. It allows for non-trivial topological charges q and so resembles more closely the K-theory of the sphere S^{2n}. This feature will be responsible later on for the appearence of topologically non-trivial gauge field configurations which have no counterparts in ordinary gauge theory on $\mathbb{V}_{2n}$. However, the positive cone $\mathrm{K}_0^+(\mathbb{V}^\theta_{2n})$ is different from that of S^{2n} as one cannot have stable modules with negative charge $q < 0$ in the present case. This is due to a labelling problem, because there is no way to distinguish between the algebras corresponding to the noncommutativity parameters θ and $-\theta$ (see Remark 2.3). This property will manifest itself explicitly in the classical solutions that we shall construct and its origin will be elucidated in Section 5.5. Physically, it will imply that there is no way to produce vortices from anti-vortices on $\mathbb{V}^{\theta}_{2n}$ by simply changing the orientation of the hyperplane.

4.2. *Yang–Mills theory*

It is possible to define connections within the present class of noncommutative spaces in the usual spirit and formalism of noncommutative geometry [21]. However, the noncommutative space $\mathbb{V}^{\theta}_{2n}$ has enough symmetries so that a simple definition will suffice for our purposes. By a *connection* ∇ on a finitely-generated left projective $\mathbb{V}^{\theta}_{2n}$-module $\mathcal{E}$ we will mean a collection of anti-Hermitean $\mathbb{C}$-linear operators $\nabla_i : \mathcal{E} \to \mathcal{E}$, $i = 1, \dots, 2n$ satisfying the Leibniz rule

$$\nabla_i(f \cdot v) = \partial_i(f) \cdot v + f \cdot \nabla_i(v) \tag{4.1}$$

for all $i = 1, \dots, 2n$, $v \in \mathcal{E}$ and $f \in \mathbb{V}^{\theta}_{2n}$. The space of connections on $\mathcal{E}$ is denoted $\mathrm{Conn}(\mathcal{E})$.

Let us find the general form of a connection on a generic projective module as specified by Proposition 4.1. It is straightforward to show that only *trivial* gauge fields arise on the Fock module $\mathcal{F}$ [40], in accordance with the interpretation that $\mathcal{F}$ is like a single "point" on the noncommutative space (see Remark 2.2).

Proposition 4.2. *If ∇ is a connection on the Fock module $\mathcal{F}$, then*

$$\nabla_i = \Pi_m \circ \partial_i \circ \Pi_m + \alpha_i \, \mathbb{1} \,, \quad i = 1, \dots, 2n$$

for some $\alpha_i \in \mathbb{C}$ and fixed $m \in \mathbb{N}_0$.

Proof. From the Leibniz rule (4.1) and the inner derivation property (2.18) one has the identity

$$\nabla_i(f \cdot v) - f \cdot \nabla_i(v) = \mathrm{i} \sum_{j=1}^{2n} \left(\theta^{-1}\right)_{ij} \left[x^j \,, f\right] \cdot v$$

which may be rewritten as

$$\left[\nabla_i - \mathrm{i} \sum_{j=1}^{2n} (\theta^{-1})_{ij} \, x^j \,, f\right] \cdot v = 0$$

for all $f \in \mathbb{V}^\theta_{2n}$ and all $v \in \mathcal{F}$. It follows that the operator $\nabla_i - \mathrm{i}\sum_j (\theta^{-1})_{ij}\, x^j$ lives in the center of the algebra $\mathrm{End}_{\mathbb{V}^\theta_{2n}}(\mathcal{F}) \cong \mathbb{V}^\theta_{2n}$ and hence is proportional to the identity endomorphism $\mathbb{1}$ of $\mathcal{F}$. □

We can construct *non-trivial* gauge fields instead on the trivial module given by the algebra $\mathcal{H} = \mathbb{V}^\theta_{2n}$ itself. By the Leibniz rule (4.1) any connection $\nabla_i : \mathcal{H} \to \mathcal{H}$ can be written in the form

$$\nabla_i(v) = -\,\mathrm{i}\sum_{j=1}^{2n} \left(\theta^{-1}\right)_{ij} x^j \cdot v + D_i(v) \tag{4.2}$$

for $v \in \mathcal{H}$, where $D_i \in \mathrm{End}(\mathcal{H})$ is any anti-Hermitean operator on $\mathcal{H}$. We choose

$$D_i = \mathrm{i}\sum_{j=1}^{2n} \left(\theta^{-1}\right)_{ij} x^j + A_i\,, \tag{4.3}$$

where $A_i \in \mathrm{End}_{\mathbb{V}^\theta_{2n}}(\mathcal{H})$ are anti-Hermitean $\mathbb{V}^\theta_{2n}$-linear endomorphisms. As all Moyal spaces $\mathbb{V}^\theta_{2n}$, $\theta \in \mathbb{R} \setminus \{0\}$ are Morita equivalent (in fact isomorphic), A_i can be taken to be anti-Hermitean elements of the algebra $\mathbb{V}^\theta_{2n}$ itself. Then by the inner derivation property (2.18) one has

$$\nabla_i(v) = \partial_i(v) + A_i \cdot v\,. \tag{4.4}$$

As usual, we define the *curvature* of a connection to be a measure of the deviation of the mapping $\partial_i \mapsto \nabla_i$ from being a homomorphism of the Lie algebra (2.17) of automorphisms of $\mathbb{V}^\theta_{2n}$. This gives the collection of anti-Hermitean endomorphisms $F_{ij} \in \mathrm{End}_{\mathbb{V}^\theta_{2n}}(\mathcal{H}) \cong \mathbb{V}^\theta_{2n}$, $i,j = 1,\dots,2n$ defined by

$$F_{ij} = \left[\nabla_i\,,\,\nabla_j\right] = \left[D_i\,,\,D_j\right] - \mathrm{i}\left(\theta^{-1}\right)_{ij}\mathbb{1}\,. \tag{4.5}$$

The *Yang–Mills action functional* $\mathrm{YM} : \mathrm{Conn}(\mathcal{H}) \to [0,\infty)$ is defined by

$$\mathrm{YM}(\nabla) = -\frac{1}{4}\,\mathrm{Tr}\left[\sum_{i,j=1}^{2n}\left(F_{ij}\right)^2\right] = -\frac{1}{4}\,\mathrm{Tr}\left[\sum_{i,j=1}^{2n}\left([D_i,D_j] - \mathrm{i}\,(\theta^{-1})_{ij}\,\mathbb{1}\right)^2\right]. \tag{4.6}$$

The Yang–Mills functional (4.6) is invariant under the *gauge transformations*

$$A_i \longmapsto U\,\partial_i\big(U^{-1}\big) + U\,A_i\,U^{-1}\,, \quad i = 1,\dots,2n \tag{4.7}$$

with $U \in \mathrm{U}(\mathcal{H})$, which induce the unitary transformations

$$D_i \longmapsto U\,D_i\,U^{-1}\,, \quad F_{ij} \longmapsto U\,F_{ij}\,U^{-1}\,. \tag{4.8}$$

4.3. *Fluxons*

We can explicitly construct all exact solutions to Yang–Mills theory on the Moyal plane $\mathbb{V}_2^\theta$, and hence for the remainder of this section we will focus on this case. It is convenient to introduce formal complex combinations of the operators (4.3) using the basic elements (2.4) to write

$$D = -\frac{1}{\sqrt{2\,\theta}}\, a^\dagger + \frac{1}{2}\,(A_1 - \mathrm{i}\,A_2)\,, \quad \overline{D} = \frac{1}{\sqrt{2\,\theta}}\, a + \frac{1}{2}\,(A_1 + \mathrm{i}\,A_2)\,. \tag{4.9}$$

In terms of these operators the curvature (4.5) reads

$$F := F_{12} = 2\,\mathrm{i}\,\left(\big[\,\overline{D}\,,\,D\big] - \frac{1}{2\,\theta}\,\mathbb{1}\right) \tag{4.10}$$

and the Yang–Mills functional (4.6) becomes

$$\mathrm{YM}\big(\nabla\big) = \mathrm{YM}\big(D\,,\,\overline{D}\,\big) := 2\ \mathrm{Tr}\,\left(\big[\,\overline{D}\,,\,D\big] - \frac{1}{2\,\theta}\,\mathbb{1}\right)^2\,. \tag{4.11}$$

Applying the variational principle to the action (4.11) yields the *Yang–Mills equations* on $\mathbb{V}_2^\theta$ given by

$$\big[D\,,\,\big[\,\overline{D}\,,\,D\big]\big] = \big[\,\overline{D}\,,\,\big[\,\overline{D}\,,\,D\big]\big] = 0\,. \tag{4.12}$$

As previously, we are interested in special classes of solutions to these equations.

Definition 4.1. Let $q \in \mathbb{N}$. A *q-fluxon* on $\mathbb{V}_2^\theta$ is an anti-Hermitean solution $D, \overline{D} \in \mathrm{End}(\mathcal{H})$ of the Yang–Mills equations (4.12) which has topological charge $c_1(\mathcal{H}) = \mathrm{Tr}\,(F) = q$ and for which the Yang–Mills functional $\mathrm{YM}(D, \overline{D}\,)$ is well-defined and finite.

A fluxon solution is a soliton on the noncommutative plane carrying a magnetic charge or "flux" [71, 40]. Using the shift operator S introduced in Example 3.3, it is straightforward to explicitly construct all finite action solutions of the Yang–Mills equations on the Moyal plane $\mathbb{V}_2^\theta$ [40].

Theorem 4.1. *To each collection* $\lambda_0, \lambda_1, \dots, \lambda_{q-1} \in \mathbb{C}$ *of fixed complex numbers there bijectively corresponds a q-fluxon*

$$D^{(q)}_{\{\lambda_i\}} = \sum_{i=0}^{q-1} \lambda_i\ e_i{\otimes}e_i^* - \big(\mathsf{S}\big)^q\, c^\dagger\, \big(\mathsf{S}^\dagger\big)^q\,, \quad \overline{D}\,^{(q)}_{\{\lambda_i\}} = \sum_{i=0}^{q-1} \overline{\lambda}_i\ e_i{\otimes}e_i^* + \big(\mathsf{S}\big)^q\, c\, \big(\mathsf{S}^\dagger\big)^q$$

in $\mathrm{End}(\mathcal{H})$ *of action*

$$\mathrm{YM}\big(D^{(q)}_{\{\lambda_i\}}\,,\,\overline{D}\,^{(q)}_{\{\lambda_i\}}\big) = 2\pi\,\theta^{-1}\,q\,,$$

where c, $c^\dagger$ generate the irreducible representation of the Heisenberg algebra.

Proof. We need to find a pair of anti-Hermitean operators $D, \overline{D}$ on the free module $\mathcal{H}$ over $\mathbb{V}_2^\theta$ which obey the equations

$$\left[D\,,\,\overline{D}\right] = \frac{1}{2}\left(\frac{1}{\theta}\,\mathbb{1} + \mathrm{i}\,F\right), \quad \left[D\,,\,F\right] = \left[\,\overline{D}\,,\,F\right] = 0\,.$$

These equations imply that $D, \overline{D}, \frac{1}{2}\left(\frac{1}{\theta} + \mathrm{i}\,F\right)$ form a representation of the Heisenberg commutation relations, with the curvature F generating the center of the algebra. Under the action of these operators, the module $\mathcal{H}$ thereby decomposes as

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n$$

into irreducible representations $\mathcal{H}_n \cong \mathcal{F}$ of this Heisenberg algebra. Then for each n one has $F|_{\mathcal{H}_n} = f_n\,\mathbb{1}$ for some $f_n \in \mathrm{i}\,\mathbb{R}$. Set $d_n := \dim(\mathcal{H}_n)$. Then d_n is infinite unless $1 + \mathrm{i}\,\theta\,f_n = 0$. The finite action constraint requires $\mathrm{Tr}\,(F^2) < \infty$, where

$$\mathrm{Tr}\left(F^2\right) = \sum_n d_n\,f_n^2\,.$$

This is a sum of negative terms. If d_n is infinite for some n, then the only way to make this quantity well-defined is to have $f_n = 0$ in such a way that the regulated trace yields a finite product $d_n\,f_n^2$. On the other hand, if some $d_n \in \mathbb{N}$ then $f_n = -\frac{\mathrm{i}}{\theta}$, and the finite action condition implies that there are only finitely many such positive dimensions.

These facts imply that the fluxon solution is determined by a finite-dimensional linear subspace $V_q \subset \mathcal{H}$ which may be characterized as follows. Via a gauge transformation if necessary, we may assume that V_q is the linear span of the number basis vectors $e_0, e_1, \dots, e_{q-1}$. On V_q, the operators D and $\overline{D}$ commute, $[D, \overline{D}\,] = 0$, and without loss of generality may be taken to be diagonal operators with respect to the chosen basis of V_q so that

$$D\,\Big|_{V_q} = \sum_{i=0}^{q-1} \lambda_i\ e_i \otimes e_i^*\ , \quad \overline{D}\,\Big|_{V_q} = \sum_{i=0}^{q-1} \overline{\lambda}_i\ e_i \otimes e_i^*$$

for some fixed $\lambda_0, \lambda_1, \dots, \lambda_{q-1} \in \mathbb{C}$. On the orthogonal complement $\mathcal{H} \ominus V_q \cong \mathcal{H}$, there exists $N \in \mathbb{N}$ such that the operators are instead generically given by a reducible sum of N irreducible representations of the Heisenberg algebra as

$$D\,\Big|_{\mathcal{H}\ominus V_q} = \bigoplus_{k=0}^{N-1}\left(-c_{(k)}^\dagger\right), \quad \overline{D}\,\Big|_{\mathcal{H}\ominus V_q} = \bigoplus_{k=0}^{N-1}\left(c_{(k)}\right)$$

with $[c_{(k)}, c^\dagger_{(k)}] = 1$ for each $k = 0, 1, \dots, N-1$. By the Stone–von Neumann theorem, $c_{(k)} = c$ and $c^\dagger_{(k)} = c^\dagger$ for each k and hence

$$D \,\big|_{\mathcal{H}\ominus V_q} = -c^\dagger \otimes \mathbb{1}_N \,, \quad \overline{D} \,\big|_{\mathcal{H}\ominus V_q} = c \otimes \mathbb{1}_N \,.$$

This makes the Hilbert space $\mathcal{H} \ominus V_q$ into an N-fold Fock module $\mathcal{F}^N \cong \ell^2(\mathbb{N}_0^N)$ with number basis $e_n^{(k)}$, $n \in \mathbb{N}_0$, $k = 0, 1, \dots, N-1$ defined by the actions

$$\begin{aligned} c^\dagger \cdot e_n^{(k)} &= \sqrt{n+1}\; e_{n+1}^{(k)} \,, \\ c \cdot e_n^{(k)} &= \sqrt{n}\; e_{n-1}^{(k)} \,, \\ \mathbb{1}_N \cdot e_n^{(k)} &= e_n^{(k)} \,. \end{aligned}$$

Let us take $N = 1$. Let ${\mathsf{S}_q}^\dagger : \mathcal{H} \ominus V_q \to \mathcal{H}$ be a unitary isomorphism of $\mathbb{V}_2^\theta$-modules. Extend ${\mathsf{S}_q}^\dagger$ to all of $\mathcal{H}$ by setting it equal to 0 on V_q. Then as an operator on $\mathcal{H}$ it satisfies

$${\mathsf{S}_q}^\dagger\, \mathsf{S}_q = \mathbb{1} \,, \quad \mathsf{S}_q\, {\mathsf{S}_q}^\dagger = \mathbb{1} - \mathsf{P}_q \,,$$

where P_q is the orthogonal projection $\mathcal{H} \to V_q$. In other words, the endomorphism $\mathsf{S}_q \in \mathrm{End}(\mathcal{H})$ is a partial isometry of $\mathcal{H}$. With respect to the chosen number basis, we have $\mathsf{P}_q = \mathsf{P}_{(q)} = \sum_{i=0}^{q-1} e_i \otimes e_i^*$ and $\mathsf{S}_q = (\mathsf{S})^q$ where S is the shift endomorphism introduced in Example 3.3. The conclusion now follows from computing the corresponding curvature to get $F = \frac{\mathrm{i}}{\theta}\, \mathsf{P}_q$. □

Remark 4.3. The q-fluxon is labelled by the set of moduli $\lambda_i \in \mathbb{C}$, $i = 0, 1, \dots, q-1$ which describe the positions or separations of the *vortices* (carrying magnetic charge $q \in \mathbb{N}$) on $\mathbb{V}_2$ [40]. The explicit solution constructed above is of rank 1. Higher rank fluxons can be similarly constructed by choosing $N > 1$ in the proof of Theorem 4.1, with no qualitative change by the Hilbert hotel argument. Thus, in addition to their moduli, fluxons are labelled by K-theory charges $(N, q) \in \mathrm{K}_0^+(\mathbb{V}_2^\theta)$. Identifying the gauge equivalence classes of fluxon solutions now consists of quotienting by the discrete Weyl subgroup $S_q \subset \mathrm{U}(q) \subset \mathrm{U}(\mathcal{H})$ acting non-trivially on the subspace V_q above by permuting the fluxon positions λ_i.

Corollary 4.2. *The moduli space $\mathcal{G}_{N,q}(\mathbb{V}_2^\theta)$ of fluxons of K-theory charge (N, q) is the q-th symmetric product orbifold*

$$\mathcal{G}_{N,q}\big(\mathbb{V}_2^\theta\big) = \mathrm{Sym}^q\big(\mathbb{V}_2\big) \,.$$

Remark 4.4. The value of the Yang–Mills functional on a fluxon as in Theorem 4.1 diverges in the formal limit $\theta \to 0$, consistent with the lack of finite action topologically non-trivial field configurations in ordinary gauge theory on the plane $\mathbb{V}_2$. Note that the fluxon solution appears with only one sign of the topological charge q, consistent with the K-theory description of Section 4.1. These configurations are not global minima of the Yang–Mills functional in general and hence are *unstable*. In string theory they can be interpreted as describing q unstable D0-branes inside N D2-branes with worldvolume $\mathbb{V}_2$, in a background B-field and in the Seiberg–Witten limit [39, 40, 1].

5. Toroidal D-branes

In the sequel we will leave the setting of Moyal spaces and start looking at more complicated worldvolume geometries. As the simplest extension of our previous considerations, in this section we will still work with flat target spaces X but we will assume that the worldvolume is compactified on a two-dimensional torus T^2. This has the effect of bringing in non-trivial topological effects while still retaining the relative simplicity of flat worldvolume geometries. The presence of the constant B-field deforms the worldvolume to a noncommutative torus T^2_Θ which is the most studied and best understood example of a noncommutative space [21]. After a quick reminder of some pertinent aspects of the noncommutative geometry of this space, we will construct all (finite action) solutions of the corresponding Yang–Mills equations. We will then show that these solutions possess a remarkably intimate connection with the fluxon solutions constructed in the previous section.

5.1. *Solitons on the noncommutative torus*

The two-dimensional noncommutative torus T^2_Θ is the classic example of a noncommutative space and we will only briefly mention some facets of its geometry, primarily to set notation. Unless otherwise explicitly stated, we will fix an irrational number $\Theta \in (0,1) \cap (\mathbb{R} \setminus \mathbb{Q})$. We define T^2_Θ to be the associative unital $*$-algebra generated by a pair of unitaries U_1, U_2 obeying the single relation

$$U_1\, U_2 = \mathrm{e}^{\,2\pi\,\mathrm{i}\,\Theta}\; U_2\, U_1\,. \tag{5.1}$$

As with Moyal spaces, we will regard T^2_Θ as an appropriate "algebra of Schwartz functions" defined in terms of expansions in the generators U_1, U_2. There is again a natural trace $\mathrm{Tr}\, : \mathrm{T}^2_\Theta \to \mathbb{C}$ on the algebra which is positive and faithful, and hence a natural notion of integral. A collection of linear derivations $\partial_i : \mathrm{T}^2_\Theta \to \mathrm{T}^2_\Theta$ may be defined on generators by

$$\partial_i(U_j) = 2\pi\,\mathrm{i}\,\delta_{ij}\ U_i\,, \quad i,j=1,2 \tag{5.2}$$

and extended as automorphisms of T^2_Θ by linearity and the Leibniz rule. As in the Moyal case these derivations commute,

$$\big[\partial_i\,,\,\partial_j\big] = 0\,, \tag{5.3}$$

and they generate the action of the two-dimensional translation group of T^2 on the noncommutative algebra. There is also a Weyl–Wigner correspondence completely analogous to that of Section 2.3 which may be used to define T^2_Θ via deformation quantization [86], but we will not need this formalism here.

It is possible to now proceed in an analogous way to Section 3 with the construction of scalar field theories on T^2_Θ and their soliton solutions [62, 10, 50, 53]. The resulting configurations are similar to those of Moyal spaces, except that the non-trivial topology now leads to far richer structures. Explicit projector solitons are provided by the Powers–Rieffel projector on T^2_Θ [75]. Its Wigner function is not a localized solitonic "lump" like the Gaussian soliton field configurations on $\mathbb{V}^\theta_{2n}$. Rather, it is localized along one direction but yields stripe-like patterns in the other direction of T^2 [10, 55]. On the other hand, the homotopically equivalent Boca projector [13] more closely resembles the Gaussian projector solitons on $\mathbb{V}^\theta_{2n}$ [53, 55]. Partial isometry solitons on T^2_Θ can also be constructed as the "angular" operators appearing in the polar decompositions of closed bounded operators based on the Powers–Rieffel projection [55], in analogy to the angular dependence of the Wigner functions for the basic shift partial isometry S on $\mathbb{V}^\theta_{2n}$ (see Example 2.1). The explicit descriptions of these configurations are much more involved than in the Moyal case and are beyond the scope of these notes. Instead, we will now proceed to focus our attention to gauge theories on the noncommutative torus.

5.2. *Gauge theory*

We begin by describing the K-theory of the noncommutative torus [21].

Proposition 5.1. *The K-theory of the noncommutative torus is given by the ordered subgroup* $\mathrm{K}_0(\mathrm{T}^2_\Theta) = \mathbb{Z} + \mathbb{Z}\,\Theta$ *of* $\mathbb{R}$ *with positive cone* $\mathrm{K}_0^+(\mathrm{T}^2_\Theta) = \{(p,q)\in\mathbb{Z}^2 \mid p - q\,\Theta > 0\}$*.*

Proof. K-theory is stable under deformations of unital algebras so $\mathrm{K}_0(\mathrm{T}^2_\Theta) \cong \mathrm{K}_0(C(\mathrm{T}^2)) = \mathbb{Z}\oplus\mathbb{Z}$. For $(p,q)\in\mathbb{Z}^2$, let $\mathsf{P} = \mathsf{P}_{p,q} \in \mathbb{M}_n(\mathrm{T}^2_\Theta)$,

$n \in \mathbb{N}$ be a projector representing the corresponding Murray–von Neumann equivalence class. One has

$$\mathrm{Tr}\left(\mathsf{P}_{p,q}\right) = \mathrm{Tr}\left(\mathsf{P}_{p,q}\,\mathsf{P}_{p,q}{}^{\dagger}\right) = p - q\,\Theta$$

and hence the trace yields an isomorphism $\mathrm{Tr} : \mathrm{K}_0(\mathrm{T}^2_\Theta) \xrightarrow{\approx} \mathbb{Z} + \mathbb{Z}\,\Theta$, with the trivial identity projector $\mathbb{1}$ generating the first copy of $\mathbb{Z}$ and the Powers–Rieffel projector generating the second copy of $\mathbb{Z}$. The trace of $\mathsf{P}_{p,q}$ coincides with the Murray–von Neumann dimension of its image subspace. □

For $(p,q) \in \mathbb{Z}^2$, the space $\mathcal{E}_{p,q} = \mathsf{P}_{p,q}(\mathrm{T}^2_\Theta)^n$ is a finitely generated left projective module over T^2_Θ called a *Heisenberg module*.

Proposition 5.2. *Any finitely generated projective module over the noncommutative torus* T^2_Θ *which is not free is isomorphic to a Heisenberg module* $\mathcal{E}_{p,q}$.

Henceforth we will consider only Heisenberg modules over the noncommutative torus. The module $\mathcal{E}_{p,q}$ is a stable module labelled by the positive cone $(p,q) \in \mathrm{K}_0^+(\mathrm{T}^2_\Theta)$ of the K-theory group and has positive Murray–von Neumann dimension

$$\dim(\mathcal{E}_{p,q}) = \mathrm{Tr}\left(\mathsf{P}_{p,q}\right) = p - q\,\Theta > 0\,. \tag{5.4}$$

The integer

$$q = \frac{1}{2\pi\,\mathrm{i}}\,\mathrm{Tr}\,\left[\mathsf{P}_{p,q}\left(\partial_1(\mathsf{P}_{p,q})\,\partial_2(\mathsf{P}_{p,q}) - \partial_2(\mathsf{P}_{p,q})\,\partial_1(\mathsf{P}_{p,q})\right)\right] \tag{5.5}$$

is the Chern number $c_1(\mathcal{E}_{p,q})$ and will be referred to as the *topological charge* of the Heisenberg module. We define the *rank* N of $\mathcal{E}_{p,q}$ to be the positive integer

$$N = \gcd(p,q)\,. \tag{5.6}$$

We will always assume $q > 0$ in what follows.

We define connections ∇ on Heisenberg modules analogously to the Moyal case, i.e. as pairs of anti-Hermitean $\mathbb{C}$-linear operators $\nabla_1, \nabla_2 : \mathcal{E}_{p,q} \to \mathcal{E}_{p,q}$ obeying the usual Leibniz rule analogously to eq. (4.1). The space of connections on a Heisenberg module is denoted $\mathrm{Conn}(\mathcal{E}_{p,q})$. We can use the automorphisms ∂_i given by eq. (5.2) to define fixed fiducial connections and write $\nabla_i = \mathsf{P}_{p,q} \circ \partial_i \circ \mathsf{P}_{p,q} + A_i$ with $A_i \in \mathrm{End}_{\mathrm{T}^2_\Theta}(\mathcal{E}_{p,q})$. The curvature $F \in \mathrm{End}_{\mathrm{T}^2_\Theta}(\mathcal{E}_{p,q})$ of a connection ∇ is defined as usual by

$$F := \left[\nabla_1\,,\,\nabla_2\right], \tag{5.7}$$

and one may introduce the Yang–Mills functional $\mathrm{YM} : \mathrm{Conn}(\mathcal{E}_{p,q}) \to [0,\infty)$ on a *fixed* Heisenberg module as

$$\mathrm{YM}(\nabla) = -\frac{\tau}{2}\,\mathrm{Tr}\left([\nabla_1,\nabla_2]^2\right) = -\frac{\tau}{2}\,\mathrm{Tr}\left(F^2\right). \tag{5.8}$$

For later convenience we have introduced a constant parameter $\tau > 0$ which we identify as the (imaginary part of the) modulus of the torus T^2. It should accompany all trace operations in order to keep quantities consistently defined. The Yang–Mills functional (5.8) is invariant under the gauge transformations

$$\nabla_i \longmapsto U\,\nabla_i\,U^{-1}, \quad i = 1,2 \tag{5.9}$$

with $U \in \mathrm{U}(\mathcal{E}_{p,q})$.

Proposition 5.3. *Any Heisenberg module $\mathcal{E}_{p,q}$ admits a connection ∇^{c} of constant curvature*

$$F_{p,q} := \left[\nabla_1^{\mathrm{c}}\,,\,\nabla_2^{\mathrm{c}}\right] = \frac{2\pi\,\mathrm{i}}{\tau}\,\frac{q}{p - q\,\Theta}\,\mathsf{P}_{p,q}\,.$$

Remark 5.1. Since $\mathcal{E}_{p,q} = \mathsf{P}_{p,q}(\mathrm{T}^2_\Theta)^n$ for some $n \in \mathbb{N}$ and $(\mathsf{P}_{p,q})^2 = \mathsf{P}_{p,q}$, the projector $\mathsf{P}_{p,q}$ acts as the identity endomorphism on the Heisenberg module and hence the curvature $F_{p,q}$ is "constant". It now follows that the topological charge (5.5) can be expressed in the standard form

$$q = \frac{\tau}{2\pi\,\mathrm{i}}\,\mathrm{Tr}\left(F_{p,q}\right).$$

The equality of the two expressions for the topological charge is essentially the index theorem.

Using Proposition 5.3 we can obtain an explicit description of the Heisenberg module $\mathcal{E}_{p,q}$ as the separable Hilbert space

$$\mathcal{E}_{p,q} = \mathcal{F} \otimes \mathcal{W}_{p,q}\,. \tag{5.10}$$

The Fock module $\mathcal{F}$ is the irreducible representation of the Heisenberg commutation relation $[\nabla_1^{\mathrm{c}},\nabla_2^{\mathrm{c}}] = F_{p,q}$, while $\mathcal{W}_{p,q} \cong \mathbb{C}^q$ is the $q \times q$ representation of the Weyl algebra

$$\Gamma_1\,\Gamma_2 = \mathrm{e}^{\,2\pi\,\mathrm{i}\,p/q}\,\Gamma_2\,\Gamma_1 \tag{5.11}$$

whose unique irreducible module has rank $\frac{q}{\gcd(p,q)} = \frac{q}{N}$. The generators U_1, U_2 of the noncommutative torus then act on $\mathcal{E}_{p,q}$ as the operators

$$U_i = \exp\left(\frac{\sqrt{\tau}}{q}\,(p - q\,\Theta)\nabla_i^{\mathrm{c}}\right) \otimes \Gamma_i\,, \quad i = 1,2\,. \tag{5.12}$$

5.3. *Instantons*

Varying the Yang–Mills functional (5.8) in the usual way leads to the Yang–Mills equations

$$\big[\nabla_1\,,\,\big[\nabla_1\,,\,\nabla_2\big]\big]=\big[\nabla_2\,,\,\big[\nabla_1\,,\,\nabla_2\big]\big]=0 \tag{5.13}$$

on the Heisenberg module $\mathcal{E}_{p,q}$. As always, we are interested in the finite action solutions to these equations up to gauge equivalence defined by the transformations (5.9).

Definition 5.1. An *instanton* of K-theory charge $(p,q)\in\mathrm{K}_0^+(\mathrm{T}^2_\Theta)$ on T^2_Θ is a solution $\nabla\in\mathrm{Conn}(\mathcal{E}_{p,q})$ of the Yang–Mills equations (5.13) on the Heisenberg module $\mathcal{E}_{p,q}$ for which the Yang–Mills functional $\mathrm{YM}(\nabla)$ is well-defined and finite.

The explicit construction of such solutions is more involved than on the Moyal plane $\mathbb{V}^\theta_2$, because in the present case the derivations (5.2) are *outer* automorphisms of the noncommutative algebra.

An obvious solution to the Yang–Mills equations (5.13) is the constant curvature connection $\nabla=\nabla^{\rm c}$ on $\mathcal{E}_{p,q}$ of topological charge q. It has action

$$\mathrm{YM}(\nabla^{\rm c})=\frac{4\pi^2}{\tau}\,\frac{q^2}{p-q\,\Theta}\,, \tag{5.14}$$

and one can show that this is the absolute minimum value of the Yang–Mills functional on $\mathrm{Conn}(\mathcal{E}_{p,q})$ [22].

Proposition 5.4. $\mathrm{YM}(\nabla^{\rm c})=\inf\limits_{\nabla\in\mathrm{Conn}(\mathcal{E}_{p,q})}\mathrm{YM}(\nabla)\,.$

Remark 5.2. Proposition 5.4 implies that the constant curvature connections define *stable* vacuum states in noncommutative gauge theory. In the string theory setting, they correspond to $\frac12$-BPS configurations [51].

The remaining instanton solutions to Eqs. (5.13) are *unstable* and may be completely classified as follows [76, 68].

Definition 5.2. A *partition* of the K-theory charge $(p,q)\in\mathrm{K}_0^+(\mathrm{T}^2_\Theta)$ is a collection $\underline{(p,q)}=\{(p_k,q_k)\}$ of charges $(p_k,q_k)\in\mathrm{K}_0^+(\mathrm{T}^2_\Theta)$ for which

$$(p,q)=\sum_k\,(p_k,q_k)\,.$$

Theorem 5.1. *To each partition* $\underline{(p,q)} = \{(p_k, q_k)\}$ *of* $(p,q) \in \mathrm{K}_0^+(\mathrm{T}_\Theta^2)$ *with finitely many components* (p_k, q_k) *there bijectively corresponds an instanton* $\nabla = \nabla_{\underline{(p,q)}}$ *of K-theory charge* (p,q) *with action*

$$\mathrm{YM}\big(\,\nabla_{\underline{(p,q)}}\,\big) = \frac{4\pi^2}{\tau} \sum_k \frac{q_k^2}{p_k - q_k\,\Theta}\,.$$

Proof. The idea of the proof is similar to that of Theorem 4.1. An instanton ∇ satisfies the equations

$$\big[\nabla_1\,,\,\nabla_2\big] = F\,, \quad \big[\nabla_1\,,\,F\big] = \big[\nabla_2\,,\,F\big] = 0$$

which describes a Heisenberg algebra with generators ∇_1, ∇_2 and F giving the central element. Under the action of these operators, the Heisenberg module $\mathcal{E}_{p,q}$ thus decomposes into irreducible representations of this algebra to give

$$\mathcal{E}_{p,q} = \bigoplus_k \mathcal{E}_{p_k,q_k}$$

for some $(p_k, q_k) \in \mathbb{Z}^2$, with each submodule $\mathcal{E}_{p_k,q_k}$ a Heisenberg module over T_Θ^2, i.e. $p_k - q_k\,\Theta > 0$. By the Yang–Mills equations, a solution $\nabla : \mathcal{E}_{p_k,q_k} \to \mathcal{E}_{p_k,q_k}$ preserves the subspaces $\mathcal{E}_{p_k,q_k}$. As a consequence, $\nabla_{(k)}^{\mathrm{c}} := \nabla|_{\mathcal{E}_{p_k,q_k}}$ has constant curvature $F_{p_k,q_k} = F|_{\mathcal{E}_{p_k,q_k}}$. The Yang–Mills functional is additive,

$$\mathrm{YM}\big(\bigoplus_k \nabla_{(k)}\,\big) = \sum_k \mathrm{YM}\big(\nabla_{(k)}\big)\,,$$

and by Proposition 5.4 the constant curvature connection $\nabla_{(k)} = \nabla_{(k)}^{\mathrm{c}}$ is the global minimum of $\mathrm{YM}\big|_{\mathcal{E}_{p_k,q_k}}$ for each k.

Thus a critical point of the Yang–Mills functional on this direct sum decomposition is given by

$$\nabla = \nabla_{\underline{(p,q)}} := \bigoplus_k \nabla_{(k)}^{\mathrm{c}}\,,$$

and every classical solution of noncommutative Yang–Mills theory is characterized by module splittings in this way. The value of $\mathrm{YM}\big(\,\nabla_{\underline{(p,q)}}\,\big) = \sum_k \mathrm{YM}(\nabla_{(k)}^{\mathrm{c}})$ is given from Eq. (5.14). Since each term in this sum is positive, the sum is finite only if there are finitely many terms in the decomposition. Since the module decomposition is an orthogonal direct sum, one has $\dim(\mathcal{E}_{p,q}) = \sum_k \dim(\mathcal{E}_{p_k,q_k})$ which is equivalent to

$$p - q\,\Theta = \sum_k \big(p_k - q_k\,\Theta\big)\,.$$

Since Θ is an irrational number, this is equivalent to the partition conditions. $\square$

Remark 5.3. The conservation of topological charge

$$q = \sum_k q_k$$

arising from the partition condition characterizes Yang–Mills theory on a fixed Heisenberg module $\mathcal{E}_{p,q}$. If we extrapolate these solutions to the commutative case $\Theta = 0$ of ordinary gauge theory on T^2, then this theory is distinguished from "physical" Yang–Mills theory which would sum over all topological charges [68, 69]. In the noncommutative case only the fixed module definition of Yang–Mills theory is well-defined, as there is no way in this case to naturally "separate" the topological numbers p and q. Note that the global minimum of the Yang–Mills functional corresponds to the special case of the trivial partition $\underline{(p,q)} = \{(p,q)\}$ having only a single component.

Remark 5.4. Since $\underline{(p,q)}$ contains only finitely many components (although it can have an arbitrarily large number), one can pick out from the corresponding module decomposition the submodule of minimum dimension and thus order the partition components according to increasing Murray–von Neumann dimension [68]. The set of values of the Yang–Mills action on its critical point set (the set of all partitions or instantons) is discrete, and thus YM defines a Morse functional on the space of connections $\mathrm{Conn}(\mathcal{E}_{p,q})$ [76].

5.4. *Instanton moduli spaces*

It is straightforward to construct the moduli spaces of the solutions obtained in Section 5.3. We first consider the moduli space $\mathcal{G}_{(p,q)}(\mathrm{T}^2_\Theta)$ of constant curvature connections ∇^{c} on $\mathcal{E}_{p,q}$ corresponding to the global minima of the Yang–Mills functional [22, 68].

Theorem 5.2. *The moduli space $\mathcal{G}_{(p,q)}(\mathrm{T}^2_\Theta)$ of stable instantons on the Heisenberg module $\mathcal{E}_{p,q}$ is the Nth symmetric product orbifold*

$$\mathcal{G}_{(p,q)}\big(\mathrm{T}^2_\Theta\big) = \mathrm{Sym}^N\big(\tilde{\mathrm{T}}^2\big)$$

of the dual torus $\tilde{\mathrm{T}}^2$, where $N = \gcd(p,q)$ is the rank of $\mathcal{E}_{p,q}$.

Proof. Write the realization (5.10) of the Heisenberg module as

$$\mathcal{E}_{p,q} = \mathcal{F} \otimes \left(\mathcal{W}_{\zeta_1} \oplus \cdots \oplus \mathcal{W}_{\zeta_N}\right),$$

where, for each $i = 1,\dots,N$, the module $\mathcal{W}_{\zeta_i} \cong \mathbb{C}^{q/N}$ is the irreducible representation of the Weyl algebra (5.11) with central generator $\zeta_i \in \tilde{\mathrm{T}}^2$. The only gauge transformations (5.9) which act non-trivially on this decomposition live in the Weyl subgroup $S_N \subset \mathrm{U}(N) \subset \mathrm{U}(\mathcal{E}_{p,q})$ and act by permuting the various components $\mathcal{W}_{\zeta_i}$ of the direct sum. Dividing by this subgroup gives the desired moduli space. □

Remark 5.5. As in the case of fluxons, the stable instanton moduli space coincides with the quantum mechanical configuration space of a number of identical particles on $\tilde{\mathrm{T}}^2$. In this sense the gauge theory is "topological" [68, 70], in that it resembles more closely quantum mechanics rather than field theory. Notice that the number of instantons here is determined by the rank N of the gauge theory, unlike the fluxon number which is determined by the topological charge q (Corollary 4.2). In Section 5.5 we shall explain the connection between these two classes of gauge theory solitons. The noncommutative instanton moduli space coincides with the moduli space

$$\mathcal{G}_{(p,q)}\big(\mathrm{T}^2_\Theta\big) \cong \mathrm{Hom}\big(\pi_1(\mathrm{T}^2)\,,\,\mathrm{U}(N)\big)\,/\,\mathrm{U}(N)$$

of *flat* principal $\mathrm{U}(N)$-bundles over the torus which arises in ordinary gauge theory on T^2 [9]. In the noncommutative setting, the moduli space is completely determined by the symmetric product not only for flat connections but also for *all* constant curvature connections. In the commutative case, a constant curvature connection on a principal $\mathrm{U}(N)$-bundle over T^2 can be described as a flat connection on a non-trivial principal bundle over T^2 with structure group $\mathrm{U}(N)\,/\,\mathrm{U}(1) \cong \mathrm{SU}(N)\,/\,\mathbb{Z}_N$.

For the general case one has to be more careful because the distinct unstable instantons that we have constructed in Section 5.3 do not represent gauge equivalence classes [68]. From (5.10) and the reducibility of the corresponding Weyl algebra representation there is an isomorphism $\mathcal{E}_{m\,p,m\,q} \cong \oplus^m\,\mathcal{E}_{p,q}$ for any $m \in \mathbb{N}$. Both Heisenberg modules $\mathcal{E}_{p,q}$ and $\mathcal{E}_{m\,p,m\,q}$ have the *same* constant curvature $F_{p,q} = F_{m\,p,m\,q}$ and hence the corresponding instanton solutions should be identified through gauge invariance. This difficulty can be circumvented by modifying our previous definition of partition [68, 76].

Definition 5.3. An *instanton partition* of the K-theory charge $(p,q) \in \mathrm{K}_0^+(\mathrm{T}^2_\Theta)$ is a collection $\underline{\underline{(p,q)}} = \{(N_k,p_k,q_k)\}$ of triples of integers such that

1. $\underline{(p,q)} = \{(N_k\,p_k, N_k\,q_k)\}$ is a partition of (p,q);

2. $(p_k, q_k) \neq (p_l, q_l)$ for $k \neq l$; and

3. p_k and q_k are relatively prime for each k.

Given an arbitrary partition $\underline{(p,q)} = \{(p_k, q_k)\}$ of $(p,q) \in \mathrm{K}_0^+(\mathrm{T}^2_\Theta)$, we may write $(p_k, q_k) = N_k\,(p'_k, q'_k)$ for each k with $N_k := \gcd(p_k, q_k)$ and p'_k, q'_k coprime. Definition 5.3 then restricts to those partitions with *distinct* K-theory charges (p'_k, q'_k). They represent the distinct gauge equivalence classes of instanton solutions to the Yang–Mills equations [68, 76].

Theorem 5.3. *The moduli space* $\mathcal{G}_{\underline{\underline{(p,q)}}}(\mathrm{T}^2_\Theta)$ *of instantons on the Heisenberg module* $\mathcal{E}_{p,q}$ *corresponding to an instanton partition* $\underline{\underline{(p,q)}} = \{(N_k, p_k, q_k)\}$ *is given by*

$$\mathcal{G}_{\underline{\underline{(p,q)}}}(\mathrm{T}^2_\Theta) = \prod_k \mathrm{Sym}^{N_k}(\tilde{\mathrm{T}}^2)\,.$$

Proof. The partition $\{(N_k\,p_k, N_k\,q_k)\}$ modifies the module decompositions used in the proof of Theorem 5.1 to

$$\mathcal{E}_{p,q} = \bigoplus_k \mathcal{E}_{N_k\,p_k, N_k\,q_k}\ ,$$

where the constant curvatures of the submodules $\mathcal{E}_{N_k\,p_k, N_k\,q_k}$ are all distinct. Since gauge transformations preserve the constant curvature conditions, they also preserve each submodule $\mathcal{E}_{N_k\,p_k, N_k\,q_k}$ and so the instanton moduli space is given by the product

$$\mathcal{G}_{\underline{\underline{(p,q)}}}(\mathrm{T}^2_\Theta) = \prod_k \mathcal{G}_{(N_k\,p_k, N_k\,q_k)}(\mathrm{T}^2_\Theta)\,,$$

where $\mathcal{G}_{(N_k\,p_k, N_k\,q_k)}(\mathrm{T}^2_\Theta)$ is the moduli space of constant curvature connections on the Heisenberg module $\mathcal{E}_{N_k\,p_k, N_k\,q_k}$. The result now follows by Theorem 5.2. □

5.5. *Decompactification*

The classification of classical solutions which we have given is completely analogous to that which arises in *ordinary* gauge theory on a two-dimensional torus T^2. In the commutative case it is just the Atiyah–Bott bundle-splitting construction [9, 37, 68] and can be obtained in the present case by formally setting $\Theta = 0$ everywhere. The integers p and p_k then represent the true ranks of Hermitean vector bundles over T^2. The difference in the noncommutative case lies in the structure of the partitions that can contribute to finite action solutions [68]. Furthermore, in the commutative case if we were to *decompactify* the torus T^2 onto the plane $\mathbb{V}_2$ by formally sending the modulus $\tau \to \infty$, then the instanton solutions would "disappear", in the sense that their action vanishes in this limit. This follows by setting $\Theta = 0$ in Theorem 5.1, and it is consistent with the fact that there are no topologically non-trivial finite action gauge field configurations on the plane. Instead, as we will now show, the formal "decompactification" of the noncommutative torus T^2_Θ onto the Moyal plane $\mathbb{V}_2^\theta$ maps the instantons of this section in a very precise way onto the fluxons constructed in Section 4.3 [38]. This provides a rather remarkable limiting construction of the intricate fluxon solutions given by Theorem 4.1 in terms of the relatively simpler noncommutative torus instantons obtained through the Fock module realizations provided by Proposition 5.3.

We introduce the quantity

$$\theta := \frac{\tau\,\Theta}{2\pi} \tag{5.15}$$

which will turn out to be the noncommutativity parameter of the Moyal plane $\mathbb{V}_2^\theta$. We take the limits $\tau \to \infty$, $\Theta \to 0$ whilst keeping the combination (5.15) fixed. From the form of the Yang–Mills functional given by Theorem 5.1, the only finite action configurations which survive this limit are those partitions $\underline{(p,q)} = \underline{(0,-q)}_{\,0} := \{(0,-q_k)\}$ having $p_k = 0$ and $q_k, q > 0$ for all k (so that $q_k\,\Theta, q\,\Theta > 0$ by the partition constraints). The action of these instantons may be written in terms of the parameter (5.15) as

$$\mathrm{YM}\big(\nabla_{\underline{(0,-q)}_{\,0}}\big) = \frac{2\pi\,q}{\theta}\,. \tag{5.16}$$

By Theorem 4.1 this is the action of a fluxon of topological charge $q \in \mathbb{N}$.

To provide the explicit mapping of instantons onto fluxons, let us fix a finite action decompactification partition $\underline{(0,-q)}_{\,0}$. For $p = 0$ the Weyl algebra reads $\Gamma_1\,\Gamma_2 = \Gamma_2\,\Gamma_1$. With respect to the canonical orthonormal basis w_k, $k = 0,1,\dots,q-1$ of $\mathcal{W}_{0,-q}$, this equation is generically solved up to unitary isomorphism by operators of the form

$$\Gamma_i = \exp\left(-\frac{2\pi\,\mathrm{i}}{\sqrt{\tau}}\,\boldsymbol{\lambda}^i\right) := \sum_{k=0}^{q-1}\ \mathrm{e}^{\,-\frac{2\pi\,\mathrm{i}}{\sqrt{\tau}}\,\lambda_k^i}\ w_k \otimes w_k^* \tag{5.17}$$

with $\lambda_k^i \in \mathbb{R}$ for $i = 1,2$ and $k = 0,1,\dots,q-1$. The corresponding module splitting provided by Theorem 5.1 thereby yields an isomorphism

$$\mathcal{E}_{0,-q} = \bigoplus_{k=0}^{q-1}\,\mathcal{E}_{0,-q_k} \cong \mathcal{F}^q\,. \tag{5.18}$$

The global minimum of the Yang–Mills functional on (5.18) is the connection with constant curvature

$$F_{0,-q} = \tfrac{\mathrm{i}}{\theta}\ \mathsf{P}_{0,-q}\,. \tag{5.19}$$

The projector $\mathsf{P}_{0,-q}$ has rank $\mathrm{Tr}\,(\mathsf{P}_{0,-q}) = q\,\Theta$, so that $\frac{1}{\Theta}\,\mathsf{P}_{0,-q}$ has integer rank $q \in \mathbb{N}$. By representing the corresponding K-theory class by the Boca projection one can show that the image of $\frac{1}{\Theta}\,\mathsf{P}_{0,-q}$ is a q-dimensional subspace of the Fock module (5.18) in the decompactification limit, and thus the curvature (5.19) coincides with that of the q-fluxon solution of Theorem 4.1 [53, 55].

Proposition 5.5. *As endomorphisms of the Fock module $\mathcal{F}$ one has*

$$\lim_{\Theta\to 0}\ \tfrac{1}{\Theta}\,\mathsf{P}_{0,-q} = \mathsf{P}_{(q)} = \sum_{k=0}^{q-1} e_k \otimes e_k^*\,.$$

In the decompactification limit, we would like to map the noncommutative torus algebra $\mathbb{T}^2_\Theta$ onto the Moyal algebra $\mathbb{V}^\theta_2$ at the level of their representations as endomorphisms of the Fock module. At the level of generators this means that we would like to roughly identify U_i, acting on (5.18) via (5.12), with the exponential operators $\exp\big(\frac{2\pi\,\mathrm{i}}{\sqrt{\tau}}\,x^i\big)$ where the Moyal plane generators x^i obey $[x^1, x^2] = \mathrm{i}\,\theta\,\mathbb{1}$ via the Baker–Campbell–Hausdorff formula. The immediate problem which arises is that the x^i are defined on the trivial rank 1 module $\mathcal{H} = \mathbb{V}^\theta_2$, while

$$U_i = \exp\left(-\frac{2\pi}{\sqrt{\tau}}\,(\theta\,\nabla^{\mathrm{c}}_i \otimes \mathbb{1}_q + \mathrm{i}\,\mathbb{1}\otimes\boldsymbol{\lambda}^i)\right) \tag{5.20}$$

are defined on the Fock module $\mathcal{F}^q$. In order to make an identification of this type we need to ensure that all operators are defined on a common domain.

We first embed all the pertinent operators naturally into the projective $\mathbb{V}^\theta_2$-module $\mathcal{E}_{0,-q} \oplus \mathcal{H}$ via the definitions

$$\begin{aligned} \hat{\nabla}^{\mathrm{c}}_i &= \big(\nabla^{\mathrm{c}}_i \otimes \mathbb{1}_q\big)\ \oplus\ 0\,, \\ \hat{\boldsymbol{\lambda}}^i &= \big(\mathbb{1}\otimes\boldsymbol{\lambda}^i\big)\ \oplus\ 0\,, \\ \hat{x}^i &= \mathbf{0}_q\ \oplus\ x^i\,. \end{aligned} \tag{5.21}$$

Then we represent these endomorphisms on the free module $\mathcal{H}$ by finding their images under a natural unitary isomorphism of separable Hilbert spaces

$$\Xi_q\ :\ \mathcal{E}_{0,-q} \oplus \mathcal{H}\ \xrightarrow{\ \approx\ }\ \mathcal{H}\,. \tag{5.22}$$

To construct this isomorphism, let S be the shift endomorphism of the Fock module with $\ker(\mathsf{S})^q=\{0\}$ and $\ker(\mathsf{S}^\dagger)^q=\mathrm{im}(\mathsf{P}_{(q)})\cong\mathbb{C}^q$. Then the submodule $(\mathsf{S})^q\cdot\mathcal{H}$ is the orthogonal complement in $\mathcal{H}$ of $\mathcal{E}_{0,-q}\cong\mathsf{P}_{(q)}\cdot\mathcal{H}$. The isomorphism (5.22) is therefore given explicitly by

$$\Xi_q\left(\sum_{k=0}^{q-1}f\cdot e_k\ \oplus\ f'\right):=\mathsf{P}_{(q)}\cdot f+(\mathsf{S})^q\cdot f' \tag{5.23}$$

for $f,f'\in\mathcal{H}$. The inverse map is given for $f\in\mathcal{H}$ by

$$\Xi_q^{-1}(f)=\sum_{k=0}^{q-1}f\cdot e_k\ \oplus\ \big(\mathsf{S}^\dagger\big)^q\cdot f\,. \tag{5.24}$$

It is straightforward to work out the action of the isomorphism (5.22) on the operators (5.21) and as endomorphisms of $\mathcal{H}$ one finds

$$\begin{aligned}\Xi_q\,\hat{x}^i\,\Xi_q^{-1}&=\big(\mathsf{S}\big)^q\,x^i\,\big(\mathsf{S}^\dagger\big)^q\,,\\ \Xi_q\,\hat{\boldsymbol{\lambda}}^i\,\Xi_q^{-1}&=\sum_{k=0}^{q-1}\lambda_k^i\ e_k\otimes e_k^*\,.\end{aligned} \tag{5.25}$$

The desired identifications are now given by

$$U_i=\Xi_q\,\exp\left(-\frac{2\pi}{\sqrt{\tau}}\,\big(\theta\,\hat{\nabla}_i^{\mathrm{c}}+\mathrm{i}\,\hat{\boldsymbol{\lambda}}^i\big)\right)\,\Xi_q^{-1}:=\Xi_q\,\exp\left(\frac{2\pi\,\mathrm{i}}{\sqrt{\tau}}\,\hat{x}^i\right)\Xi_q^{-1} \tag{5.26}$$

for $i=1,2$. From (5.25) it then follows finally that the decompactification limit of the constant curvature connection of Yang–Mills theory on T^2_Θ thus leads to the operators

$$D_i:=\mathrm{i}\,\theta\,\Xi_q\,\hat{\nabla}_i^{\mathrm{c}}\,\Xi_q^{-1}=\sum_{k=0}^{q-1}\lambda_k^i\ e_k\otimes e_k^*+\big(\mathsf{S}\big)^q\,x^i\,\big(\mathsf{S}^\dagger\big)^q\,. \tag{5.27}$$

This coincides with the q-fluxon given by Theorem 4.1. Thus the decompactification of finite action instantons on the noncommutative torus provides a natural and clear way to describe the fluxon solutions of Yang–Mills gauge theory on $\mathbb{V}_2^\theta$. The construction of these solutions is very natural in the toroidal framework, and the torus instanton origin of fluxons explains many of their seemingly unusual properties in precise geometric ways [38]. For instance, the constraint $q>0$ on the sign of the fluxon charges can be traced back to the required positivity of the Murray–von Neumann dimension on the positive K-theory cone of stable Heisenberg modules, while the instability of fluxons is due to the instability of their instanton ancestors.

6. D-branes in Group Manifolds

In this final section we leave the setting of flat target spaces X and study some examples of D-branes in curved backgrounds. A particularly tractable class of examples is provided by the cases where $X = \mathrm{G}$ is a *group* manifold. While these spacetimes are not entirely realistic string backgrounds, they provide important solvable models which sometimes form subspaces of genuine target spaces. They possess enough symmetries so that a relatively complete classification of D-branes, and their noncommutative worldvolume geometries, may be readily obtained. We will describe the general quantization scheme for special classes of D-branes in these backgrounds, and how to construct the corresponding noncommutative worldvolume gauge theories. We then work out the simplest example of D-branes whose worldvolumes are fuzzy two-spheres in the group manifold of $\mathrm{G} = \mathrm{SU}(2)$ where everything can be made very explicit. A detailed review of these matters along with an exhaustive list of references can be found in [79].

6.1. *Symmetric D-branes*

Let G be a compact, simple, simply connected and connected Lie group possessing a bi-invariant metric. Let $\mathfrak{g}$ be the Lie algebra of G. The Lie bracket on $\mathfrak{g}$ is denoted $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and the metric on G induces an invariant inner product $\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. As in Section 1, a string in the target space $X = \mathrm{G}$ is a harmonic map $g : \Sigma \to \mathrm{G}$ on an oriented Riemann surface Σ. The special feature of group manifolds is that the string theory possesses an affine $\mathrm{G} \times \overline{\mathrm{G}}$ symmetry

$$g(z, \bar{z}) \longmapsto \Omega(z)\, g(z, \bar{z})\, \overline{\Omega}(\bar{z})^{-1}\,, \tag{6.1}$$

where $(z, \bar{z})$ are local complex coordinates on Σ and $\Omega, \overline{\Omega} : \Sigma \to \mathrm{G}$ are independent holomorphic and antiholomorphic maps.

Select a fixed boundary component of $\partial\Sigma$, and choose the local parametrization of the worldsheet Σ such that this component is located at $z = \bar{z}$. Then the symmetry (6.1) restricts to this boundary component as

$$g\big|_{\partial\Sigma} \longmapsto \Omega\, g\big|_{\partial\Sigma}\, \overline{\Omega}^{\,-1}\,. \tag{6.2}$$

A D-brane is now a boundary condition $g\big|_{\partial\Sigma} : \partial\Sigma \to W \subset \mathrm{G}$ which preserves enough of the affine $\mathrm{G} \times \overline{\mathrm{G}}$ symmetry as dictated by conformal invariance of the underlying boundary conformal field theory.

Definition 6.1. A *symmetric D-brane* in G is a boundary condition with

$$\overline{\Omega} = \omega \circ \Omega$$

on $\partial\Sigma$ for some isometric automorphism $\omega : \mathrm{G} \to \mathrm{G}$.

Symmetric D-branes preserve a maximal diagonal subgroup $\mathrm{G} \subset \mathrm{G} \times \overline{\mathrm{G}}$ of the affine symmetry. Their worldvolumes W are *twisted conjugacy classes*

$$W = \mathrm{C}_\omega(g) = \big\{ h\, g\, \omega(h)^{-1} \ \big| \ h \in \mathrm{G} \big\} \tag{6.3}$$

of elements $g \in \mathrm{G}$ [2, 28, 83, 27, 14].

Definition 6.2. Two symmetric D-branes $\mathrm{C}_\omega(g)$ and $\mathrm{C}_\omega(g')$ are *equivalent* if $g' = \mathrm{Ad}_h(g) := h\, g\, h^{-1}$ for some $h \in \mathrm{G}$.

Equivalent D-branes are described by twisted conjugacy classes which are simply translates of one another in the group manifold of G. To characterize the corresponding equivalence classes, let $\mathrm{Aut}(\mathrm{G})$ denote the group of isometric automorphisms of G. Let $\mathrm{Inn}(\mathrm{G}) \subset \mathrm{Aut}(\mathrm{G})$ be the invariant normal subgroup of inner automorphisms $g \mapsto \mathrm{Ad}_h(g)$. Then the factor group

$$\mathrm{Out}(\mathrm{G}) = \mathrm{Aut}(\mathrm{G}) \,/\, \mathrm{Inn}(\mathrm{G}) \tag{6.4}$$

consists of equivalence classes of metric-preserving outer automorphisms of G. To each element of the group (6.4) we can associate an equivalence class of symmetric D-branes foliating G, because every element of G belongs to one and only one twisted conjugacy class. The leaves of this foliation need not all have the same topology. Hence the D-brane foliation need not be a fibration.

Thus a generic worldvolume (6.3) representing an equivalence class of D-branes is described by taking $\omega \in \mathrm{Out}(\mathrm{G})$ to be an outer automorphism. The symmetric D-brane is then called an *ω-twisted D-brane*. When $\omega = \mathrm{id}_\mathrm{G}$ then $\mathrm{C}(g) := \mathrm{C}_{\mathrm{id}_\mathrm{G}}(g)$ is just an ordinary *conjugacy class* of the group G and the symmetric D-brane is called an *untwisted D-brane*. Generally, the twisted conjugacy class is diffeomorphic to a homogeneous space

$$\mathrm{C}_\omega(g) = \mathrm{G} \,/\, \mathrm{H}_\omega(g)\,, \tag{6.5}$$

where

$$\mathrm{H}_\omega(g) = \big\{ h \in \mathrm{G} \ \big| \ h\, g = g\, \omega(h) \big\} \tag{6.6}$$

is the isotropy subgroup of the element $g \in \mathrm{G}$.

Remark 6.1. It is possible to also construct D-branes with less symmetry, preserving a smaller subgroup than the diagonal $\mathrm{G} \subset \mathrm{G} \times \overline{\mathrm{G}}$. These are called *symmetry-breaking D-branes*. Generically, they are localized along *products* of twisted conjugacy classes of the Lie group G [73].

The supergravity fields on a symmetric D-brane are straightforward to construct [2, 83, 14]. For a given group element $g \in \mathrm{G}$, identify the tangent space $T_g\mathrm{G} \cong \mathfrak{g}$ to G at g with the Lie algebra of G. For tangent vectors $u, v \in \mathfrak{g}$ the invariant metric G is then given by

$$G(u,v) = \left\langle g^{-1}\, u \,,\, g^{-1}\, v \right\rangle , \tag{6.7}$$

while the invariant three-form H is

$$H(u,v,w) = -\left\langle g^{-1}\, u \,,\, \left[g^{-1}\, v \,,\, g^{-1}\, w \right] \right\rangle \tag{6.8}$$

for $u, v, w \in \mathfrak{g}$. We may introduce a B-field, with $H = \mathrm{d}B$, by the formula

$$B(u,v) = \left\langle g^{-1}\, u \,,\, \frac{1\!\!1 + \omega \circ \mathrm{Ad}_g}{1\!\!1 - \omega \circ \mathrm{Ad}_g} \left(g^{-1}\, v \right) \right\rangle \tag{6.9}$$

which is defined for $g^{-1}\, u \in \mathrm{im}(1\!\!1 - \omega \circ \mathrm{Ad}_g)$, i.e. for vectors u tangent to the twisted conjugacy class (6.3) containing $g \in \mathrm{G}$. We assume that these fields restrict non-degenerately to the twisted conjugacy classes.

6.2. *Untwisted D-branes*

Let us first describe the noncommutative worldvolume geometries in the somewhat more standard cases with $\omega = \mathrm{id}_{\mathrm{G}}$. The D-brane worldvolumes are the conjugacy classes $W = \mathrm{C}(g) = \{\mathrm{Ad}_h(g) \mid h \in \mathrm{G}\}$ which are diffeomorphic to the symmetric spaces $\mathrm{G}\,/\,\mathrm{H}(g)$, where $\mathrm{H}(g)$ is the stabilizer subgroup of the element $g \in \mathrm{G}$. Since G is compact and simple, every element is conjugate to some maximal torus T of G. Let us restrict to the set $\mathrm{G_r}$ of *regular* elements $g \in \mathrm{G}$, i.e. those elements which are conjugate to only one maximal torus. The set $\mathrm{G_r}$ is an open dense subset in G. Let $\mathrm{T_r} := \mathrm{T} \cap \mathrm{G_r}$. Then $\mathrm{C}(g) \cap \mathrm{T_r} \neq \emptyset$ and the intersections generate an orbit of the Weyl group $\mathrm{W} = \mathrm{N}(\mathrm{T_r})\,/\,\mathrm{T_r}$, where $\mathrm{N}(\mathrm{T_r})$ is the normalizer subgroup of the maximal torus. Hence there is a diffeomorphism realizing the conjugacy class as the flag manifold

$$\mathrm{C}(g) = \mathrm{G_r}\,/\,\mathrm{T_r} \ . \tag{6.10}$$

Since the Weyl group W further relates elements of $\mathrm{T_r}$, it follows that untwisted D-branes are parametrized by the Weyl chamber $\mathrm{T_r}\,/\,\mathrm{W}$.

Our main result here is that the untwisted D-branes in G in the semi-classical regime recover the Kirillov theory of coadjoint orbits [3], whose quantization gives all irreducible representations of the universal enveloping algebra $U(\mathfrak{g})$.

Theorem 6.1. *To each irreducible representation $\mathcal{V}$ of the Lie group* G *there bijectively corresponds an untwisted D-brane in the semi-classical limit such that the noncommutative algebra of functions on the quantized worldvolume $W_{\mathcal{V}}$ is given by*

$$\mathcal{A}(W_{\mathcal{V}}) \ = \ \mathrm{End}(\mathcal{V})\,.$$

Proof. The two-form B-field, defined on vectors tangent to the conjugacy class $\mathrm{C}(g)$, is given by the automorphism

$$B = \frac{\mathbb{1} + \mathrm{Ad}_g}{\mathbb{1} - \mathrm{Ad}_g}\,.$$

In the semiclassical limit, $H \to 0$ and the group manifold of G approaches flat space. We may then parametrize the conjugacy class by an element $X \in \mathfrak{g}$ such that $g \approx \mathbb{1} + X$ in the limit. The semiclassical B-field thereby becomes

$$B = -2\,(\mathrm{ad}_X)^{-1}\,,$$

which is just the standard Kirillov two-form on the Lie algebra $\mathfrak{g}$. The Seiberg–Witten bivector is given by

$$\theta = \frac{2}{B - G\,B^{-1}\,G} = \frac{1}{2}\,\big(\mathrm{Ad}_{g^{-1}} - \mathrm{Ad}_g\big)\,.$$

In the limit $H \to 0$ this bivector obeys the Jacobi identity and becomes

$$\theta = \mathrm{ad}_X\,,$$

which is the Kirillov–Kostant Poisson bivector. Casimir operators in $U(\mathfrak{g})$ are invariants of the conjugacy classes and hence may be used to label $\mathrm{C}(g)$. We may therefore associate to each conjugacy class $\mathrm{C}(g)$ whose second Casimir invariant is quantized in the requisite way an irreducible representation $\mathcal{V}_g$ of $U(\mathfrak{g})$. □

Remark 6.2. This theorem is consistent with the fact that conjugacy classes are the image under the exponential map of adjoint orbits, which are diffeomorphic to coadjoint orbits that are symplectic manifolds with respect to the natural Kirillov–Kostant–Souariu symplectic structure and hence are even-dimensional. Note that the invariant inner product $\langle -, - \rangle$ on $\mathfrak{g}$ plays a crucial role in this identification. It relates the D-brane worldvolume, which is an orbit of the *adjoint* action of the group G on itself, to the *coadjoint* action of G on its Lie algebra $\mathfrak{g}$. The quantization of the coadjoint orbits in turn gives the representations of G.

Remark 6.3. The proof of Theorem 6.1 shows that the semi-classical geometry of untwisted D-branes is very close to that of the cases studied in earlier sections with constant B-field [3]. Since G is compact, the algebra $\mathcal{A}(W_{\mathcal{V}})$ is finite-dimensional and so the noncommutative worldvolume geometry is now "fuzzy". This algebra carries a natural G-action on it which represents the isometry group of the noncommutative space. Note that not all conjugacy classes are admissible as D-brane worldvolumes. Only the *integer* conjugacy classes which are in one-to-one correspondence with the irreducible representations of G are allowed. The algebra of functions on the quantized conjugacy classes is then the corresponding endomorphism algebra of the representation.

Remark 6.4. In some instances Theorem 6.1 also extends to the case of non-compact and even non-semisimple Lie groups G, such as those corresponding to homogeneous plane wave backgrounds [42]. It is not clear how to generalize this result to symmetry-breaking D-branes whose classical worldvolumes are given as products of conjugacy classes [73].

6.3. *Example: D-branes in SU(2)*

Let us now study in detail the simplest example of this construction [3, 4, 79]. The rank 1 Lie group G $=$ SU(2) has no non-trivial Dynkin diagram automorphisms and hence all symmetric D-branes in SU(2) $\cong \mathrm{S}^3$ are untwisted. In the semi-classical limit $H \to 0$, the radius of the three-sphere S^3 grows and the group manifold "decompactifies" onto flat space $\mathbb{R}^3 = \mathrm{su}(2)$. The conjugacy classes are parametrized by the Weyl chamber $\mathrm{S}^1 \,/\, \mathbb{Z}_2$. Let $\vartheta \in [0, \pi]$ be the coordinate on this closed interval. Then ϑ parametrizes the azimuthal angle of S^3. For $\vartheta = 0, \pi$, the conjugacy classes are points corresponding to elements in the center $\mathbb{Z}_2 = \{-\mathbb{1}, \mathbb{1}\}$ of SU(2) placed at the north and south poles of the three-sphere. They correspond to D0-branes. For $\vartheta \in (0, \pi)$ the conjugacy classes are diffeomorphic to two-spheres $\mathrm{S}^2 \cong \mathrm{S}^3 \,/\, \mathrm{S}^1$, corresponding to D2-branes. The group manifold $\mathrm{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ has a standard foliation by two-spheres of increasing radius with two degenerate spheres S^2 placed at 0 and ∞. Because of the degeneration of the limiting spheres, the foliation is not a fibration (In particular this is *not* the Hopf fibration $\mathrm{S}^3 \to \mathrm{S}^2$ that we are describing here).

In this case $\theta = \mathrm{ad}_X$ is the standard Kirillov–Kostant Poisson bivector on the two-spheres in $\mathrm{su}(2) = \mathbb{R}^3$, and the worldvolume algebras of D-branes in SU(2) are the usual quantizations of these two-spheres via the coadjoint orbit method. By Theorem 6.1, quantizing functions on S^2 with the usual Poisson structure yields *fuzzy spheres* [61]

$$\mathcal{A}\big(\mathrm{S}^2_j\big) = \mathtt{M}_N(\mathbb{C}) \tag{6.11}$$

which are labelled by half-integers $j \in \frac{1}{2}\mathbb{N}_0$. The D-brane label j is proportional to the radius of its worldvolume S^2 and it represents the spin of the associated irreducible SU(2)-module $\mathcal{V}_j$ of dimension $N = 2j + 1$. The radii R_j of the corresponding integer conjugacy classes in SU(2) are quantized as $R_j^2 = j\,(j + 1)$ in order to match the corresponding second Casimir invariants.

The worldvolume algebra (6.11) has finite dimension $(2j + 1)^2$. It is thus a full matrix algebra which admits an action of SU(2) by conjugation with group elements in the N-dimensional representation of SU(2). Under this action, the SU(2)-module $\mathbb{M}_N(\mathbb{C})$ decomposes into a direct sum of irreducible representations $\mathcal{V}_J$ of dimension $2J + 1$, giving [61]

$$\mathcal{A}\big(\mathrm{S}_j^2\big) = \bigoplus_{J=0}^{N-1} \mathcal{V}_J\,. \tag{6.12}$$

Note that only integer values of the spin J appear in this direct sum. Let Y_a^J, $a = 1,\dots,2J + 1$ be a basis of the representation space $\mathcal{V}_J$. These elements are called *fuzzy spherical harmonics* and their multiplication rules can be worked out from the multiplication of $N \times N$ matrices to get [61]

$$Y_a^I\,Y_b^J = \sum_{K=0}^{\min(I+J,2j)}\ \sum_{c=1}^{2K+1} \begin{bmatrix} I & J & K \\ a & b & c \end{bmatrix} \begin{Bmatrix} I & J & K \\ j & j & j \end{Bmatrix} Y_c^K\,, \tag{6.13}$$

where the square brackets denote the Clebsch–Gordan coefficients of SU(2) and the curly brackets are the Wigner $6j$-symbols of $U(\mathrm{su}(2))$.

To construct field theories on the noncommutative worldvolumes (6.11), we will introduce derivations on $\mathcal{A}(\mathrm{S}_j^2)$ in analogy with the flat space case in Section 2.4 by finding automorphisms of the noncommutative algebra which represent isometries of the sphere S^2. Recall that the classical su(2) symmetry on the commutative algebra of functions $C(\mathrm{S}^2)$ is inherited from infinitesimal rotations in $\mathbb{R}^3$ as follows. Let $y = (y^i)_{i=1,2,3}$ denote local coordinates on $\mathbb{R}^3$. Let C^{ijk} be the structure constants of the Lie algebra su(2) in a suitable basis. Then

$$L_i = \sum_{j,k=1}^{3} C^{ijk}\,y^j\,\frac{\partial}{\partial y^k}\,, \quad i = 1,2,3 \tag{6.14}$$

generate an su(2) subalgebra in the Lie algebra of vector fields on $\mathbb{R}^3$. The classical functions Y_a^J above are then the usual spherical harmonics transforming under L_i in the representation $\mathcal{V}_J$ of su(2). Under the embedding $\mathrm{S}^2 \subset \mathbb{R}^3$ defined by $|y|^2 = 1$, this descends to an action of SU(2) on the algebra $C(\mathrm{S}^2)$.

Let us now write down the quantization of this su(2) symmetry. Let $f=\sum_{J,a} f_{Ja}\ Y^J_a$ be an element of the worldvolume algebra (6.12). In analogy with the flat space case, we then introduce derivatives through the adjoint representation of su(2) by

$$L_i(f):=\big[Y^1_i\,,\,f\big]\,, \quad i=1,2,3\,. \tag{6.15}$$

The automorphisms $L_i:\mathcal{A}(\mathrm{S}^2_j)\to\mathcal{A}(\mathrm{S}^2_j)$ generate the reducible action of SU(2) on the noncommutative worldvolume algebra under which it decomposes as in Eq. (6.12).

We are now ready to describe gauge theory on the noncommutative space (6.11). As always, the standard procedure for constructing connections and gauge theories in noncommutative geometry can be formally developed in this instance [61]. Here we will just write down the final result for a rank 1 gauge theory on the trivial projective module over the fuzzy sphere. We express an arbitrary connection $\nabla_i:\mathcal{A}(\mathrm{S}^2_j)\to\mathcal{A}(\mathrm{S}^2_j)$ in this case in the form

$$\nabla_i=L_i+A_i \tag{6.16}$$

with $A_i\in\mathrm{i}\,\mathrm{u}(N)$, $i=1,2,3$. Unlike the previous flat space actions, string theory considerations [4] dictate that the gauge theory action functional $S:\mathrm{i}\,\mathrm{u}(N)\to\mathbb{R}$ contains both Yang–Mills and Chern–Simons terms in the combination

$$S(A)=\ \mathrm{Tr}\,\left(\sum_{i,j=1}^3\left[\big(F_{ij}\big)^2+2\sum_{k=1}^3 C^{ijk}\ \mathrm{CS}_{ijk}(A)\right]\right)\,. \tag{6.17}$$

Here Tr denotes the usual $N\times N$ matrix trace on $\mathbb{M}_N(\mathbb{C})$, and

$$F_{ij}:=\big[\nabla_i\,,\,\nabla_j\big]=L_i(A_j)-L_j(A_i)+\big[A_i\,,\,A_j\big]-\mathrm{i}\sum_{k=1}^3 C^{ijk}\,A_k \tag{6.18}$$

is the curvature of the connection (6.16). The functional

$$\mathrm{CS}_{ijk}(A)=\ \mathrm{i}\,L_i(A_j)\,A_k+\frac{1}{3}\,A_i\,\big[A_j\,,\,A_k\big]+\frac{1}{2}\sum_{l=1}^3 C^{ijl}\,A_l\,A_k \tag{6.19}$$

is the *noncommutative Chern–Simons form* [4]. The action functional (6.17) is invariant under the (infinitesimal) gauge transformations

$$A_i\ \longmapsto\ A_i+L_i(\Lambda)-\mathrm{i}\,\big[A_i\,,\,\Lambda\big] \tag{6.20}$$

with $\Lambda\in\mathrm{u}(N)$.

Varying the functional (6.17) gives the equations of motion

$$\sum_{i=1}^{3} \big(L_i(F_{ij}) + \big[A_i\,,\,F_{ij}\big]\big) = 0\,, \quad j = 1,2,3 \tag{6.21}$$

expressing the usual fact that the curvature is covariantly constant. To solve these equations, introduce the shifted algebra elements

$$R_i := Y_i^1 + A_i \tag{6.22}$$

in $\mathbb{M}_N(\mathbb{C})$ to write them as

$$\sum_{i=1}^{3} \left[R_i,\ \big[R_i,\,R_j\big] - \sum_{k=1}^{3} C^{ijk}\,R_k\right] = 0\,. \tag{6.23}$$

There are then two classes of solutions. The first class arises from requiring that all three matrices (6.22) be mutually commuting in $\mathbb{M}_N(\mathbb{C})$, $[R_i, R_j] = 0$ for $i,j = 1,2,3$. They can therefore be simultaneously diagonalized and their simultaneous eigenvalues describe translates of N particles in the group target space $X = \mathrm{SU}(2)$. These are formally the same as the solutions found earlier in the cases of flat spaces [89].

A more interesting class of solutions with no flat space analog is provided by those configurations (6.22) which obey the commutation relations

$$\big[R_i\,,\,R_j\big] = \sum_{k=1}^{3} C^{ijk}\,R_k\,. \tag{6.24}$$

Such solutions have vanishing curvature $F_{ij} = 0$ and thus correspond to flat connections on the D-brane worldvolume. They determine N-dimensional unitary representations of su(2), i.e. homomorphisms

$$\pi_N\ :\ \mathrm{su}(2)\ \longrightarrow\ \mathrm{u}(N)\,. \tag{6.25}$$

Up to isomorphism, for any $n \in \mathbb{N}$ there is a unique irreducible representation of SU(2) of dimension n. Thus to any representation (6.25) we can assign an unordered *partition* $(n_i)_{i=1,\dots,r}$ of $N = n_1 + \cdots + n_r$, with n_i giving the dimensions of the irreducible submodules in π_N. These configurations are thus similar to the torus instantons that we constructed in Section 5.3. The partition characterizes the original representation uniquely up to gauge equivalence, and hence provides a simple classification for solutions of this type. It is also possible to formulate gauge theory on the fuzzy sphere in such a way that the classical solution set resembles more closely that of Yang–Mills theory on $\mathbb{V}_2^\theta$ and T_Θ^2, with intimate relationships between the seemingly distinct instanton configurations on the diverse noncommutative spaces [84, 85].

Remark 6.5. In the string theory setting, these solutions describe dynamical processes involving a stack of N particles corresponding to D0-branes, which are labelled by quantum mechanical instanton-type partition degrees of freedom. These D0-branes "decay" into a single D2-brane (for rank 1 gauge theory) with spherical worldvolume S^2 corresponding to the irreducible representation $\mathcal{V}_j$ of dimension $N = 2j + 1$ [4]. This condensation phenomenon is called the *dielectric effect* [65] and it is equivalent to vector bundle modification when D-branes are regarded as Baum–Douglas K-cycles in topological K-homology [46, 8, 74, 87].

6.4. *Twisted D-branes*

We close this final section with a tour beyond the SU(2) target space and untwisted D-branes. Let us consider the generic case of an ω-twisted D-brane corresponding to a non-trivial outer automorphism ω of the Lie group G. In this case, the dimension of a twisted conjugacy class $\mathrm{C}_\omega(g)$ is larger than the dimension of a regular conjugacy class. In particular, there are generically instances in which $\mathrm{C}_\omega(g)$ is odd-dimensional, and so in general the worldvolume W will not be a symplectic manifold. Nevertheless, there is a way to quantize these geometries that we shall now describe.

The main result here is that the ω-twisted D-branes in G are labelled by representations of the *invariant subgroup*

$$\mathrm{G}^\omega := \big\{ g \in \mathrm{G} \bigm| \omega(g) = g \big\} \tag{6.26}$$

which for $\omega \neq \mathrm{id}_{\mathrm{G}}$ is a proper subgroup of G. By conjugating to the maximal torus T, they are thus parametrized by the abelian subgroup $\mathrm{T}^\omega := \mathrm{G}^\omega \cap \mathrm{T}$ and there is a diffeomorphism

$$\mathrm{C}_\omega(g) = \mathrm{G} \,/\, \mathrm{T}^\omega \tag{6.27}$$

for any $g \in \mathrm{G}$. In the semi-classical regime, the quantization of twisted conjugacy classes, i.e. the noncommutative geometry of twisted D-branes, can be described as follows [5].

Theorem 6.2. *To each irreducible representation $\mathcal{V}^\omega$ of the invariant subgroup G^ω there bijectively corresponds an ω-twisted D-brane in the semi-classical limit such that the noncommutative algebra of functions on the quantized worldvolume $W_{\mathcal{V}^\omega}$ is given by*

$$\mathcal{A}\big(W_{\mathcal{V}^\omega}\big) \;=\; \big(C(\mathrm{G}) \otimes \mathrm{End}(\mathcal{V}^\omega)\big)^{\mathrm{G}^\omega} ,$$

where the superscript denotes the G^ω-invariant part and the G^ω-action $\mathrm{G}^\omega \times C(\mathrm{G}, \mathrm{End}\,\mathcal{V}^\omega) \to C(\mathrm{G}, \mathrm{End}\,\mathcal{V}^\omega)$ is defined by $(h, f(g)) \mapsto \mathcal{V}^\omega(h)\, f(g\,h)\, \mathcal{V}^\omega(h)^{-1}$ with $\mathcal{V}^\omega(h) \in \mathrm{GL}(\mathcal{V}^\omega)$ for all $h \in \mathrm{G}^\omega$.

Proof. Consider an open neighbourhood U of the identity element $\mathbb{1}$ of G. Let $g \in U$. Then the twisted conjugacy class of g can be represented as the fibration

$$\mathrm{C}_\omega(g) = \mathrm{G} \times_{\mathrm{G}^\omega} \mathrm{C}'(g)$$

over $\mathrm{G} / \mathrm{G}^\omega$ with fiber $\mathrm{C}'(g)$ which is a regular conjugacy class of the invariant subgroup G^ω. Here G^ω acts on G by right multiplication. As before in the untwisted case, in the semi-classical limit $H \to 0$ the conjugacy class $\mathrm{C}'(g)$ becomes small and approaches a coadjoint orbit of G^ω, while the Poisson manifold $\mathrm{G} / \mathrm{G}^\omega$ grows (approaching flat space) and its Poisson bivector scales down. Thus $\mathrm{C}'(g)$ becomes a noncommutative symplectic space while $\mathrm{G} / \mathrm{G}^\omega$ remains a classical space in the semi-classical regime. After quantization, we get a bundle with noncommutative fibers $\mathrm{End}(\mathcal{V}_g^\omega)$ and a classical base $\mathrm{G} / \mathrm{G}^\omega$. $\square$

Remark 6.6. $\mathcal{A}(W_{\mathcal{V}^\omega})$ is an associative matrix algebra of functions on the Lie group G. If $\mathcal{V}^\omega \cong \mathbb{C}$ is the trivial representation of G^ω, then the noncommutative algebra $\mathcal{A}(W_{\mathbb{C}})$ consists of functions on G which are simply invariant under right translations by elements of the invariant subgroup $\mathrm{G}^\omega \subset \mathrm{G}$. On the other hand, when $\omega = \mathrm{id}_{\mathrm{G}}$ is the trivial automorphism of G, one has $\mathrm{G}^\omega = \mathrm{G}$ and the noncommutative worldvolume algebra is $\mathrm{End}(\mathcal{V}^{\mathrm{id}_{\mathrm{G}}})$ consistently with Theorem 6.1.

Using the fact that the Lie group G is simple, simply-connected and compact, we can obtain an alternative realization of the noncommutative worldvolume algebra $\mathcal{A}(W_{\mathcal{V}^\omega})$ which makes its G-module structure more transparent.

Theorem 6.3. *All complexified vector bundles* $\mathrm{G} \times_{\mathrm{G}^\omega} (\mathcal{V}^\omega)^{\mathbb{C}} \longrightarrow \mathrm{G} / \mathrm{G}^\omega$ *are trivial.*

The proof of Theorem 6.3 is rather technical and can be found in [5].

Proposition 6.1. *There is a natural algebra isomorphism*

$$\mathcal{A}\big(W_{\mathcal{V}^\omega}\big) \cong \big\{ f \in C\big(\mathrm{G}\,,\,\mathrm{End}\,\mathcal{V}^\omega\big) \,\big|\, f(g\,h) = \mathcal{V}^\omega(h)^{-1}\, f(g)\, \mathcal{V}^\omega(h)\,, \\ g \in \mathrm{G}\,,\; h \in \mathrm{G}^\omega \big\}\,.$$

Proof. The vector space $C(\mathrm{G}) \otimes \mathrm{End}(\mathcal{V}^\omega) \cong C(\mathrm{G}, \mathrm{End}\,\mathcal{V}^\omega)$ of matrix-valued functions on G carries a natural $(\mathrm{G} \times \mathrm{G}^\omega)$-action $(\mathrm{G} \times \mathrm{G}^\omega) \times C(\mathrm{G}, \mathrm{End}\,\mathcal{V}^\omega) \to C(\mathrm{G}, \mathrm{End}\,\mathcal{V}^\omega)$ given by

$$\big((g,h)\,,\, f(g'\,)\big) \longmapsto \mathcal{V}^\omega(h)\, f\big(g^{-1}\,g'\,h\big)\, \mathcal{V}^\omega(h)^{-1}\,.$$

This leaves an action of G on the space $\mathcal{A}(W_{\mathcal{V}^\omega})$ of G^ω-invariants. The G-module $\mathcal{A}(W_{\mathcal{V}^\omega})$ can thereby be realized explicitly in terms of G^ω-equivariant functions on G as claimed. □

To construct gauge theory on the trivial rank 1 module over the algebra $\mathcal{A}(W_{\mathcal{V}^\omega})$, let T^a, $a = 1,\dots,\dim(\mathrm{G})$ be a basis of generators of the Lie algebra $\mathfrak{g}$ obeying

$$\big[T^a\,,\,T^b\big] = \sum_{c=1}^{\dim(\mathrm{G})} C^{abc}\,T^c\,. \tag{6.28}$$

Using the G-action on the algebra $\mathcal{A}(W_{\mathcal{V}^\omega})$ given by Proposition 6.1 and the exponential mapping $\exp:\mathfrak{g}\to\mathrm{G}$, we define Lie derivatives $L_a(f)$ of functions $f\in\mathcal{A}(W_{\mathcal{V}^\omega})$ by

$$\big(L_a(f)\big)(g) := \frac{\mathrm{d}}{\mathrm{d}t} f\big(\exp(-t\,T^a)\,g\big)\Big|_{t=0}\,, \quad a = 1,\dots,\dim(\mathrm{G}) \tag{6.29}$$

which as vector fields on $\mathfrak{g}$ obey the same Lie algebra relations (6.28). As in Section 6.3, the natural string-inspired gauge theory action functional $S_\omega:\mathcal{A}(W_{\mathcal{V}^\omega})\to\mathbb{R}$ reads [5]

$$S_\omega(A) = \int_{\mathrm{G}} \mathrm{d}\mu_{\mathrm{G}} \sum_{a,b=1}^{\dim(\mathrm{G})} \mathrm{Tr}\left(\big(F_{ab}\big)^2 + 2\sum_{c=1}^{\dim(\mathrm{G})} C^{abc}\,\mathrm{CS}_{abc}(A)\right) \tag{6.30}$$

with all objects defined in a completely analogous way to those of Section 6.3. Here $\mathrm{d}\mu_{\mathrm{G}}$ is the invariant Haar measure on the Lie group G. The classical solutions of this gauge theory again describe condensation processes on a configuration of D-branes which drive the entire system into another D-brane configuration [5].

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References

[1] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, "Unstable Solitons in Noncommutative Gauge Theory", J. High Energy Phys. **0104** (2001) 001 [`hep-th/0009142`].

[2] A. Yu. Alekseev and V. Schomerus, "D-Branes in the WZW Model", Phys. Rev. D **60** (1999) 061901 [hep-th/9812193].

[3] A. Yu. Alekseev, A. Recknagel and V. Schomerus, "Noncommutative World-volume Geometries: Branes on SU(2) and Fuzzy Spheres", J. High Energy Phys. **9909** (1999) 023 [hep-th/9908040].

[4] A. Yu. Alekseev, A. Recknagel and V. Schomerus, "Brane Dynamics in Background Fluxes and Noncommutative Geometry", J. High Energy Phys. **0005** (2000) 010 [hep-th/0003187].

[5] A. Yu. Alekseev, S. Fredenhagen, T. Quella and V. Schomerus, "Noncommutative Gauge Theory of Twisted D-Branes", Nucl. Phys. B **646** (2002) 127–157 [hep-th/0205123].

[6] J. Ambjørn, Y. M. Makeenko, J. Nishimura and R. J. Szabo, "Lattice Gauge Fields and Discrete Noncommutative Yang–Mills Theory", J. High Energy Phys. **0005** (2000) 023 [hep-th/0004147].

[7] F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari, "Noncommutative Geometry from Strings and Branes", J. High Energy Phys. **9902** (1999) 016 [hep-th/9810072].

[8] T. Asakawa, S. Sugimoto and S. Terashima, "D-Branes, Matrix Theory and K-Homology", J. High Energy Phys. **0203** (2002) 034 [hep-th/0108085].

[9] M. F. Atiyah and R. Bott, "The Yang–Mills Equations over Riemann Surfaces", Phil. Trans. Roy. Soc. Lond. A **308** (1982) 523–615.

[10] I. Bars, H. Kajiura, Y. Matsuo and T. Takayanagi, "Tachyon Condensation on a Noncommutative Torus", Phys. Rev. D **63** (2001) 086001 [hep-th/0010101].

[11] P. Baum and R. G. Douglas, "K-Homology and Index Theory", Proc. Symp. Pure Math. **38** (1982) 117–173.

[12] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, "Deformation Theory and Quantization 1: Deformations of Symplectic Structures", Ann. Phys. **111** (1978) 61.

[13] F.-P. Boca, "Projections in Rotation Algebras and Theta Functions", Commun. Math. Phys. **202** (1999) 325 [math.oa/9803134].

[14] P. Bordalo, S. Ribault and C. Schweigert, "Flux Stabilization in Compact Groups", J. High Energy Phys. **0110** (2001) 036 [hep-th/0108201].

[15] L. Boutet de Monvel, "On the Index of Toeplitz Operators of Several Complex Variables", Inv. Math. **50** (1979) 249.

[16] H. W. Braden and N. A. Nekrasov, "Instantons, Hilbert Schemes and Integrability", in *Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory*, eds. S. Pakuliak and G. von Gehlen (Kluwer, 2001) 35–54 [hep-th/0103204].

[17] L. G. Brown, R. G. Douglas and P. A. Fillmore, "Extensions of C^*-Algebras and K-Homology", Ann. Math. **105** (1977) 265.

[18] A. S. Cattaneo and G. Felder, "A Path Integral Approach to the Kontsevich Quantization Formula", Commun. Math. Phys. **212** (2000) 591–611 [math.qa/9902090].

[19] A. H. Chamseddine and J. Fröhlich, "Some Elements of Connes' Noncommutative Geometry and Spacetime Geometry", in *Yang Festschrift*,

eds. D. S. Liu and S.-T. Yau (International Press, 1995) 10–34 [hep-th/9307012].
[20] C. S. Chu and P. M. Ho, "Noncommutative Open String and D-Brane", Nucl. Phys. B **550** (1999) 151–168 [hep-th/9812219].
[21] A. Connes, *Noncommutative Geometry* (Academic Press, 1994).
[22] A. Connes and M. A. Rieffel, "Yang–Mills for Noncommutative Two-Tori", Contemp. Math. **62** (1987) 237–266.
[23] L. Cornalba and R. Schiappa, "Nonassociative Star Product Deformations for D-Brane Worldvolumes in Curved Backgrounds", Commun. Math. Phys. **225** (2002) 33–66 [hep-th/0101219].
[24] K. Dasgupta, S. Mukhi and G. Rajesh, "Noncommutative Tachyons", J. High Energy Phys. **0006** (2000) 022 [hep-th/0005006].
[25] M. R. Douglas, "Branes within Branes", in: *Strings, Branes and Dualities*, eds. L. Baulieu, P. Di Francesco, M. R. Douglas, V. A. Kazakov, M. Picco and P. Windey (Kluwer, 1999) 267–275 [hep-th/9512077].
[26] M. R. Douglas and N. A. Nekrasov, "Noncommutative Field Theory", Rev. Mod. Phys. **73** (2001) 977–1029 [hep-th/0106048].
[27] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, "The Geometry of WZW Branes", J. Geom. Phys. **34** (2000) 162–190 [hep-th/9909030].
[28] J. M. Figueroa-O'Farrill and S. Stanciu, "More D-Branes in the Nappi-Witten Background", J. High Energy Phys. **0001** (2000) 024 [hep-th/9909164].
[29] D. S. Freed and E. Witten, "Anomalies in String Theory with D-Branes", Asian J. Math. **3** (1999) 819 [hep-th/9907189].
[30] J. Fröhlich and K. Gawedzki, "Conformal Field Theory and Geometry of Strings", CRM Proc. Lecture Notes **7** (1994) 57–97 [hep-th/9310187].
[31] J. Fröhlich, O. Grandjean and A. Recknagel, "Supersymmetric Quantum Theory, Noncommutative Geometry and Gravitation", in: *Quantum Symmetries*, eds. A. Connes, K. Gawedzki and J. Zinn-Justin (North Holland, 1998) 221–385 [hep-th/9706132].
[32] A. Fujii, Y. Imaizumi and N. Ohta, "Supersymmetry, Spectrum and Fate of $\mathrm{D}0-\mathrm{D}p$ Systems with B-Field", Nucl. Phys. B **615** (2001) 61–81 [hep-th/0105079].
[33] S. Ghosh, "Energy Crisis or a New Soliton in the Noncommutative $\mathbb{C}P^1$ Model?", Nucl. Phys. B **670** (2003) 359–372 [hep-th/0306045].
[34] S. Ghosh, "Noncommutative Chern–Simons Soliton", Phys. Rev. D **70** (2004) 085007 [hep-th/0402029].
[35] R. Gopakumar, S. Minwalla and A. Strominger, "Noncommutative Solitons", J. High Energy Phys. **0005** (2000) 020 [hep-th/0003160].
[36] R. Gopakumar, M. Headrick and M. Spradlin, "On Noncommutative Multi-Solitons", Commun. Math. Phys. **233** (2003) 355–381 [hep-th/0103256].
[37] L. Griguolo, "The Instanton Contributions to Yang–Mills Theory on the Torus: Localization, Wilson Loops and the Perturbative Expansion", Nucl. Phys. B **547** (1999) 375–396 [hep-th/9811050].
[38] L. Griguolo, D. Seminara and R. J. Szabo, "Instantons, Fluxons and

Open Gauge String Theory", Adv. Theor. Math. Phys. **9** (2005) 775–860 [hep-th/0411277].
[39] D. J. Gross and N. A. Nekrasov, "Dynamics of Strings in Noncommutative Gauge Theory", J. High Energy Phys. **0010** (2000) 021 [hep-th/0007204].
[40] D. J. Gross and N. A. Nekrasov, "Solitons in Noncommutative Gauge Theory", J. High Energy Phys. **0103** (2001) 044 [hep-th/0010090].
[41] L. Hadasz, U. Lindström, M. Rocek and R. von Unge, "Noncommutative Multi-Solitons: Moduli Spaces, Quantization, Finite θ Effects and Stability", J. High Energy Phys. **0106** (2001) 040 [hep-th/0104017].
[42] S. Halliday and R. J. Szabo, "Isometric Embeddings and Noncommutative Branes in Homogeneous Gravitational Waves", Class. Quant. Grav. **22** (2005) 1945–1990 [hep-th/0502054].
[43] M. Hamanaka, "Noncommutative Solitons and D-Branes", hep-th/0303256.
[44] M. Hamanaka, Y. Imaizumi and N. Ohta, "Moduli Space and Scattering of D0-Branes in Noncommutative Super Yang–Mills Theory", Phys. Lett. B **529** (2002) 163–170 [hep-th/0112050].
[45] J. A. Harvey, "Komaba Lectures on Noncommutative Solitons and D-Branes", hep-th/0102076.
[46] J. A. Harvey and G. W. Moore, "Noncommutative Tachyons and K-Theory", J. Math. Phys. **42** (2001) 2765–2780 [hep-th/0009030].
[47] J. A. Harvey, P. Kraus and F. Larsen, "Exact Noncommutative Solitons", J. High Energy Phys. **0012** (2000) 024 [hep-th/0010060].
[48] J. A. Harvey, P. Kraus, F. Larsen and E. J. Martinec, "D-Branes and Strings as Noncommutative Solitons", J. High Energy Phys. **0007** (2000) 042 [hep-th/0005031].
[49] P. Hořava, "Type IIA D-Branes, K-Theory and Matrix Theory", Adv. Theor. Math. Phys. **2** (1999) 1373–1404 [hep-th/9812135].
[50] H. Kajiura, Y. Matsuo and T. Takayanagi, "Exact Tachyon Condensation on Noncommutative Torus", J. High Energy Phys. **0106** (2001) 041 [hep-th/0104143].
[51] A. Konechny and A. Schwarz, "Introduction to Matrix Theory and Noncommutative Geometry", Phys. Rept. **360** (2002) 353–465 [hep-th/0107251, hep-th/0012145].
[52] M. Kontsevich, "Deformation Quantization of Poisson Manifolds", Lett. Math. Phys. **66** (2003) 157–216 [q-alg/9709040].
[53] T. Krajewski and M. Schnabl, "Exact Solitons on Noncommutative Tori", J. High Energy Phys. **0108** (2001) 002 [hep-th/0104090].
[54] G. Landi, F. Lizzi and R. J. Szabo, "String Geometry and the Noncommutative Torus", Commun. Math. Phys. **206** (1999) 603–637 [hep-th/9806099].
[55] G. Landi, F. Lizzi and R. J. Szabo, "Matrix Quantum Mechanics and Soliton Regularization of Noncommutative Field Theory", Adv. Theor. Math. Phys. **8** (2004) 1–82 [hep-th/0401072].
[56] E. Langmann, "Interacting Fermions on Noncommutative Spaces: Exactly Solvable Quantum Field Theories in $2n+1$ Dimensions", Nucl. Phys. B **654** (2003) 404–426 [hep-th/0205287].
[57] E. Langmann, R. J. Szabo and K. Zarembo, "Exact Solution of Quantum

Field Theory on Noncommutative Phase Spaces", J. High Energy Phys. **0401** (2004) 017 [`hep-th/0308043`].
[58] O. Lechtenfeld, A. D. Popov and R. J. Szabo, "Noncommutative Instantons in Higher Dimensions, Vortices and Topological K-Cycles", J. High Energy Phys. **0312** (2003) 022 [`hep-th/0310267`].
[59] F. Lizzi and R. J. Szabo, "Duality Symmetries and Noncommutative Geometry of String Spacetimes", Commun. Math. Phys. **197** (1998) 667–712 [`hep-th/9707202`].
[60] F. Lizzi and R. J. Szabo, "Noncommutative Geometry and String Duality", J. High Energy Phys. Conf. Proc. corfu98/073 [`hep-th/9904064`].
[61] J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications* (Cambridge University Press, 1995).
[62] E. J. Martinec and G. W. Moore, "Noncommutative Solitons on Orbifolds", `hep-th/0101199`.
[63] Y. Matsuo, "Topological Charges of Noncommutative Soliton", Phys. Lett. B **499** (2001) 223–228 [`hep-th/0009002`].
[64] R. Minasian and G. W. Moore, "K-Theory and Ramond-Ramond Charge", J. High Energy Phys. **9711** (1997) 002 [`hep-th/9710230`].
[65] R. C. Myers, "Dielectric-Branes", J. High Energy Phys. **9912** (1999) 022 [`hep-th/9910053`].
[66] H. Nakajima, *Lectures on Hilbert Schemes of Points on Surfaces* (American Mathematical Society University Lecture Series, 1999).
[67] K. Olsen and R. J. Szabo, "Constructing D-Branes from K-Theory", Adv. Theor. Math. Phys. **4** (2000) 889–1025 [`hep-th/9907140`].
[68] L. D. Paniak and R. J. Szabo, "Instanton Expansion of Noncommutative Gauge Theory in Two Dimensions", Commun. Math. Phys. **243** (2003) 343–387 [`hep-th/0203166`].
[69] L. D. Paniak and R. J. Szabo, "Open Wilson Lines and Group Theory of Noncommutative Yang–Mills Theory in Two Dimensions", J. High Energy Phys. **0305** (2003) 029 [`hep-th/0302162`].
[70] L. D. Paniak and R. J. Szabo, "Lectures on Two-Dimensional Noncommutative Gauge Theory 2: Quantization", Lect. Notes Phys. **662** (2005) 205–237 [`hep-th/0304268`].
[71] A. P. Polychronakos, "Flux Tube Solutions in Noncommutative Gauge Theories", Phys. Lett. B **495** (2000) 407–412 [`hep-th/0007043`].
[72] A. D. Popov and R. J. Szabo, "Quiver Gauge Theory of Nonabelian Vortices and Noncommutative Instantons in Higher Dimensions", J. Math. Phys. **47** (2006) 012306 [`hep-th/0504025`].
[73] T. Quella, "On the Hierarchy of Symmetry Breaking D-Branes in Group Manifolds", J. High Energy Phys. **0212** (2002) 009 [`hep-th/0209157`].
[74] R. M. G. Reis and R. J. Szabo, "Geometric K-Homology of Flat D-Branes", Commun. Math. Phys. **266** (2006) 71–122 [`hep-th/0507043`].
[75] M. A. Rieffel, "C^*-Algebras Associated with Irrational Rotations", Pacific J. Math. **93** (1981) 415.
[76] M. A. Rieffel, "Critical Points of Yang–Mills for Noncommutative Two-Tori", J. Diff. Geom. **31** (1990) 535–546.

[77] F. A. Schaposnik, "Noncommutative Solitons and Instantons", Braz. J. Phys. **34** (2004) 1349–1357 [hep-th/0310202].
[78] V. Schomerus, "D-Branes and Deformation Quantization", J. High Energy Phys. **9906** (1999) 030 [hep-th/9903205].
[79] V. Schomerus, "Lectures on Branes in Curved Backgrounds", Class. Quant. Grav. **19** (2002) 5781–5847 [hep-th/0209241].
[80] A. Schwarz, "Gauge Theories on Noncommutative Euclidean Spaces", in: *Multiple Facets of Quantization and Supersymmetry*, eds. M. Olshanetsky et al., 794–803 [hep-th/0111174].
[81] N. Seiberg and E. Witten, "String Theory and Noncommutative Geometry", J. High Energy Phys. **9909** (1999) 032 [hep-th/9908142].
[82] A. Sen, "Some Issues in Noncommutative Tachyon Condensation", J. High Energy Phys. **0011** (2000) 035 [hep-th/0009038].
[83] S. Stanciu, "D-Branes in Group Manifolds", J. High Energy Phys. **0001** (2000) 025 [hep-th/9909163].
[84] H. Steinacker, "Quantized Gauge Theory on the Fuzzy Sphere as Random Matrix Model", Nucl. Phys. B **679** (2004) 66–98 [hep-th/0307075].
[85] H. Steinacker and R. J. Szabo, "Localization for Yang–Mills Theory on the Fuzzy Sphere", hep-th/0701041.
[86] R. J. Szabo, "Quantum Field Theory on Noncommutative Spaces", Phys. Rept. **378** (2003) 207–299 [hep-th/0109162].
[87] R. J. Szabo, "D-Branes, Tachyons and K-Homology", Mod. Phys. Lett. A **17** (2002) 2297–2316 [hep-th/0209210].
[88] R. Wimmer, "D0–D4 Brane Tachyon Condensation to a BPS State and its Excitation Spectrum in Noncommutative Super Yang–Mills Theory", J. High Energy Phys. **0505** (2005) 022 [hep-th/0502158].
[89] E. Witten, "Bound States of Strings and p-Branes", Nucl. Phys. B **460** (1996) 335–350 [hep-th/9510135].
[90] E. Witten, "D-Branes and K-Theory", J. High Energy Phys. **9812** (1998) 019 [hep-th/9810188].
[91] E. Witten, "Noncommutative Tachyons and String Field Theory", hep-th/0006071.
[92] E. Witten, "Overview of K-Theory Applied to Strings", Int. J. Mod. Phys. A **16** (2001) 693–706 [hep-th/0007175].